Counting Calabi-Yau black holes with (mock) modular forms

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References

- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207], to appear in Adv. Th. Math. Phys.
- *"Quantum geometry, stability and modularity"*, with S. Alexandrov, S. Feyzbakhsh, A. Klemm, T. Schimannek [arXiv:2301.08066]+ work in progress

Introduction

- The counting of BPS states in QFT or string models with extended supersymmetry has been a fertile arena for connections between physics and mathematics: algebraic/symplectic geometry, representation theory, automorphic forms...
- Connections to automorphic forms for reductive groups arise for models with 16 supercharges or more, where the moduli space of vacua is a symmetric space G/K, with no quantum corrections.
- In 4 dimensions, BPS states exist only in models with at least 8 supercharges, in which case the moduli space is no longer symmetric, though still constrained by SUSY.
- We shall be interested in counting BPS black holes in string models with $\mathcal{N} = 2$ SUSY in $D = 4$, primarily IIA/CY_3, which is under better mathematical control than IIB/CY_3 or $Het/K3 \times T^2$.

Introduction

	$IIA = M/S^1$	IIB	*Heterotic*
X	CY_3	CY_3	$K3 \times T^2$
$\mathcal{M}_V$	$\mathcal{M}_{Kahler}$	$\mathcal{M}_{\rm Complex}$	$\mathcal{M}_{\rm Narain}$
BPS states	$D6D4D2D0$	$D3$	$KK/F1/NS5/KK5$
Lattice Λ	$K(X)$	$H_3(X,\mathbb{Z})$	$\Gamma_N \oplus \Gamma_N^\vee$
Category $\mathcal{C}$	$D^b\mathrm{Coh}(X)$	$\mathrm{Fukaya}(X)$	?

- Under mirror symmetry, $IIA/X = IIB/\hat{X}$. When X is K3-fibered, $IIA/X = Het/K3 \times T^2$ for suitable choice of bundle on $K3 \times T^2$.
- BPS states correspond to stable objects of charge $\gamma \in \Lambda$ in the category of BPS states $\mathcal{C}$, counted by the BPS index $\Omega_\sigma(\gamma)$
- Upon compactification on a circle of radius R, BPS states in $D = 4$ induce $\mathcal{O}(\Omega(\gamma)e^{-R|Z(\gamma)|})$ corrections to the metric on the vector multiplet moduli space $\widetilde{\mathcal{M}}_V$ in $D = 3$.
- $\widetilde{\mathcal{M}}_V$ should admit an isometric action of $SL(2,\mathbb{Z})$ for M/T^2 or IIB/S^1, or $SL(3,\mathbb{Z})$ for Het/T^3, which puts constraints on $\Omega_\sigma(\gamma)$.

Mathematical preliminaries

- Let X a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}X$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes of coherent sheaves $\mathcal{E}^k$ on X,

$$E = (\cdots \stackrel{d^{-2}}{\to} \mathcal{E}^{-1} \stackrel{d^{-1}}{\to} \mathcal{E}^0 \stackrel{d^0}{\to} \mathcal{E}^1 \stackrel{d^1}{\to} \dots),$$

  with morphisms $d^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ such that $d^{k+1}d^k = 0$. Physically, $\mathcal{E}^k$ describe D6-branes for k even, or anti D6-branes for k odd, and d^k are open strings .

- $\mathcal{C}$ is graded by the Grothendieck group $K(\mathcal{C})$. Let $\Gamma \subset H^{\mathrm{even}}(X, \mathbb{Q})$ be the image of $K(\mathcal{C})$ under $E \mapsto \mathrm{ch}\, E = \sum_k (-1)^k \,\mathrm{ch}\, \mathcal{E}_k$. The lattice of electromagnetic charges Γ is equipped with the skew-symmetric (Dirac-Schwinger-Zwanziger) pairing

$$\langle E, E' \rangle = \chi(E', E) = \int_X (\mathrm{ch}\, E')^\vee \,\mathrm{ch}(E)\, \mathrm{Td}(TX) \in \mathbb{Z}$$

Bridgeland stability conditions

- Stability conditions are pairs $\sigma = (Z, \mathcal{A})$, where $Z : \Gamma \to \mathbb{C}$ is a linear map (the central charge) and $\mathcal{A} \subset \mathcal{C}$ is an Abelian subcategory (heart of bounded *t*-structure), subject to certain compatibility conditions. In particular, $\mathrm{Im} Z(E) \geq 0 \forall E \in \mathcal{A}$.
- Let $\mathcal{S} = \mathrm{Stab}(\mathcal{C})$ be the space of of stability conditions. If not empty, then it is a complex manifold of dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(X)$, locally parametrized by $Z(\gamma_i)$ with γ_i a basis of Γ.
- Stability conditions are known to exist only for a handful of CY threefolds, including the quintic in $\mathbb{P}^4$ *[Li'18]*. Their construction depends on the conjectural Bayer-Macrì-Toda (BMT) inequality. Weak stability conditions are much easier to construct.

Physical stability conditions

- Physics/Mirror symmetry conjecturally selects a subspace $\Pi \subset \mathrm{Stab}\,\mathcal{C}$, known as 'physical slice' or slice of Π-stability conditions, parametrized by complexified Kähler structure of X, or complex structure of $\widehat{X}$. Hence $\dim_{\mathbb{C}} \Pi = b_2(X) + 1 = b_3(\widehat{X})$.
- Along this slice, the central charge is given by the period

$$Z(\gamma) = \int_{\hat{\gamma}} \Omega_{3,0}$$

of the holomorphic 3-form on $\widehat{X}$ on a dual 3-cycle $\hat{\gamma} \in H_3(\hat{X}, \mathbb{Z})$.

- Near the large volume point in $\mathcal{M}_K(X)$, or MUM point in $\mathcal{M}_{cx}(\widehat{X})$,

$$Z(E) \sim -\int_X e^{-z^a H_a} \sqrt{Td(TX)}\, \mathrm{ch}(E)$$

where H_a is a basis of $H^2(X, \mathbb{Z})$, and $z^a = b^a + \mathrm{i} t^a$ are the complexified Kähler moduli.

Generalized Donaldson-Thomas invariants

- Given a (weak) stability condition $\sigma = (Z, \mathcal{A})$, an object $F \in \mathcal{A}$ is called σ-semi-stable if $\arg Z(F') \leq \arg Z(F)$ for every non-zero subobject $F' \subset F$ (where $0 < \arg Z \leq \pi$).
- Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semi-stable objects of class γ in $\mathcal{A}$. Following *[Joyce-Song'08]* one can associate the DT invariant $\bar{\Omega}_\sigma(\gamma) \in \mathbb{Q}$. When $\mathcal{M}_\sigma(\gamma)$ is a smooth projective variety, then $\bar{\Omega}_\sigma(\gamma) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma))$ is integer.
- Conjecturally, the invariants $\Omega_\sigma(\gamma) := \sum_{m|\gamma} \mu(m) \frac{\bar{\Omega}_\sigma(\gamma/m)}{m^2}$ are integer, and coincide with the physical BPS indices.
- Examples:
  1. $\Omega_\sigma(k[pt]) = -\chi_X$ for all $k \geq 1$ throughout the space of geometric stability conditions.
  2. For any $\beta \in H_2(X, \mathbb{Z})$, $\Omega_\sigma([\beta] + k[pt]) = GV_\beta^{(0)}$ for all $k \geq 0$ in the large volume limit.

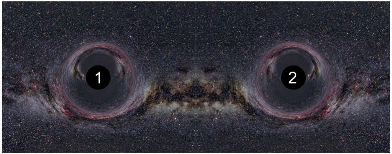
Wall-crossing

- The invariants $\bar{\Omega}_\sigma(\gamma)$ are locally constant on $\mathcal{S}$, but jump across walls of instability (or marginal stability), where the central charge $Z(\gamma)$ aligns with $Z(\gamma')$ where $\gamma' = \mathrm{ch}\, E'$ for a subobject $E' \subset E$. The jump is governed by a universal wall-crossing formula.

Joyce Song'08; Kontsevich Soibelman'08

- Physically, the jump corresponds to the (dis)appearance of multi-centered black hole bound states. In the simplest case,

$$\Delta\bar{\Omega}(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1,\gamma_2\rangle|\, \bar{\Omega}(\gamma_1)\, \bar{\Omega}(\gamma_2)$$



GV/DT/PT relation

- For a single D6-brane, the DT-invariant $DT(q,n) = \Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\sum_{Q,n} DT(Q,n)\, \mathrm{q}^n v^Q = M(-\mathrm{q})^{\chi_X} \prod_{Q,g,\ell} \left(1-(-\mathrm{q})^{g-\ell-1} v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\,\mathrm{GV}_Q^{(g)}}$$

where $M(\mathrm{q}) = \prod_{n\geq 1}(1-\mathrm{q}^n)^{-n}$ is the Mac-Mahon function.

Maulik Nekrasov Okounkov Pandharipande'06

- The topological string partition function is given by

$$\Psi_{\mathrm{top}}(z,\lambda) = M(-\mathrm{q})^{-\chi_X/2} Z_{DT}\ , \quad \mathrm{q} = e^{\mathrm{i}\lambda}, v = e^{2\pi \mathrm{i} z/\lambda}$$

can be computed by the direct integration method, assuming conifold gap conditions and Castelnuovo-type bounds $g \leq g_{\max}(Q)$ *[BCOV 93, Huang Klemm Quackenbush'06]*.

Rank 0 DT invariants from GV invariants

- Thm [Feyzbakhsh Thomas'20-22]: *Let (X, H) be any polarized CY3 satisfying the BMT conjecture (see below). Then all DT invariants for H-Gieseker stability are determined by rank 1 DT invariants, hence by GV invariants.*
- This relies on wall-crossing in a family of weak stability conditions parametrized by $(b, t) \in \mathbb{R} \times \mathbb{R}^+$, with degenerate central charge

$$Z^{\rm tilt}_{b,t}(E) = \tfrac{\rm i}{6} t^3 \,{\rm ch}_0 - \tfrac{1}{2} t^2 \,{\rm ch}_1^b - {\rm i} t \,{\rm ch}_2^b + 0 \,{\rm ch}_3^b$$

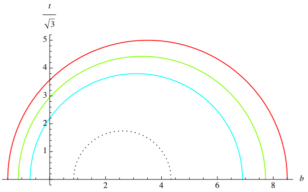
where ${\rm ch}_k^b = \int_X H^{3-k} e^{-bH} {\rm ch}(E)$. The BMT conjecture states that tilt-semistable objects exist only when $C_k := {\rm ch}_k^0$ satisfy

$$(C_1^2 - 2C_0 C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0 C_3 - C_1 C_2)b + (2C_2^2 - 3C_1 C_3) \geq 0$$

Bayer Macri Toda'11; Bayer Macri Stellari'16

Rank 0 DT invariants from GV invariants

- Walls for tilt stability are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



- The BMT inequality provides an empty chamber whenever the discriminant at $t = 0$ is positive:

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

$$\Updownarrow$$

$$\frac{8}{9\kappa}p^0q_1^3 - \frac{2}{3}\kappa q_0(p^1)^3 - (p^0q_0)^2 + \frac{1}{3}(p^1q_1)^2 - 2p^0p^1q_0q_1 \leq 0$$

hence when single centered black hole solutions are ruled out !

S-duality constraints on D4-D2-D0 indices

- For classes supported on an irreducible divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(X,\mathbb{Z})$, the generating series of rank 0 DT invariants

$$h_{p^a,q_a}(\tau) = \sum_n \bar{\Omega}_\star(0,p^a,q_a,n)\,\mathrm{q}^{n+\frac{1}{2}q_a\kappa^{ab}q_b+\frac{1}{2}p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

  should be a vector-valued, weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2(X) - 1$ and prescribed multiplier system.

- Here, $\bar{\Omega}_\star(0,p^a,q_a,n)$ is the index in the large volume attractor chamber

$$\bar{\Omega}_\star(\gamma) = \lim_{\lambda\to+\infty} \bar{\Omega}_{-\kappa^{ab}q_b+\mathrm{i}\lambda p^a}(\gamma)$$

  where κ^{ab} is the inverse of the quadratic form $\kappa_{ab} = \kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(X) - 1)$.

S-duality constraints on D4-D2-D0 indices

- By construction, $\Omega_\star(0, p^a, q_a, n)$ is invariant under tensoring with a line bundle $\mathcal{O}(m^a H_a)$ (aka spectral flow)

$$q_a \to q_a - \kappa_{ab} m^b \ , \quad n \mapsto n - m^a q_a + \tfrac{1}{2}\kappa_{ab} m^a m^b$$

  Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

- h_{p^a,q_a} transforms under the Weil representation of $\mathrm{Mp}(2,\mathbb{Z})$ associated to the lattice Λ, e.g.

$$h_{p^a,q_a}(-1/\tau) = \sum_{q'_a \in \Lambda^*/\Lambda} \frac{e^{-2\pi i \kappa^{ab} q_a q'_b + \frac{i\pi}{4}(b_2(X) + 2\chi(\mathcal{O}_\mathcal{D}) - 2)}}{\tau^{1+\frac{1}{2}b_2(X)} \sqrt{|\det(\kappa_{ab})|}} h_{p^a,q'_a}(\tau)$$

- Equivalently, $Z_p(\tau, v) = \sum_{q\in\Lambda^*/\Lambda} h_{p,q}(\tau)\Theta_q(\tau, v)$, where $\Theta_q(\tau, v)$ is the Siegel theta series for the indefinite lattice (Λ, κ_{ab}), transforms as a (non-holomorphic) Jacobi form.

Maldacena Strominger Witten'98, Cheng de Boer Dijkgraaf Manschot Verlinde'06

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor class $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for brevity) should be a vector-valued mock modular form of depth $n-1$.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(X)-1$. The completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$. Conditions for modularity were spelled out in *[Vignéras'78]*

- The relevant lattice for Θ_n and $\widehat{\Theta}_n$ is $\Lambda = H^2(X,\mathbb{Z})^{\oplus(n-1)}$, with signature $(r, d-r) = (n-1)(1, b_2(X)-1)$. The relevant Φ is a linear combination of generalized error functions $\mathcal{E}_{n-1}(\{C_i\}, x) := e^{\pi Q(x_+)} \star \prod_{i=1}^{n-1} \mathrm{sgn}(C_i, x)$ where $\star$ is the convolution product. *[Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016]*
- Similar theta series arise by integrating the r-form valued Kudla-Millson theta series on a suitable polyhedron in $Gr(r, d-r)$

Kudla Funke 2016-21

Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Abelian D4-D2-D0 invariants

- For $N=1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega_\star(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}\ , \quad q\in\mathbb{Z}/\kappa\mathbb{Z}$$

should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}[\kappa]$ with $\kappa = H^3$ [Gaiotto Strominger Yin'06]

- An overcomplete basis is given for κ even by

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

where $D = \mathrm{q}\partial_\mathrm{q} - \frac{w}{12}E_2$, is the Serre derivative and $4a+6b+2\ell-2\kappa-\frac{c_2}{2}+\frac{1}{2} = -\frac{3}{2}$.

- For κ odd, the same works with $\vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} (-1)^{\kappa k}\, k\mathrm{q}^{\frac{1}{2}\kappa k^2}$.

Rank 0 DT invariants from GV invariants

- For a D4-D2-D0 charge $\gamma = (0, r, q, n)$ close enough to the (usual) Bogomolov-Gieseker bound, [Toda'13, Feyzbakhsh'22]

$$\bar{\Omega}_{r,q}(n) = \sum_{r_i, Q_i, n_i} (-1)^{\langle \gamma_1, \gamma_2 \rangle} \mathrm{DT}(Q_1, n_1)\, \mathrm{PT}(Q_2, n_2)$$

  where $\mathrm{DT}(Q_1, n_1), \mathrm{PT}(Q_2, n_2)$ counts BPS states with charge $\gamma_1 = (1, 0, -Q_1, -n_1), \gamma_2 = (-1, 0, Q_2, -n_2)$, respectively

- Alternatively, one can study wall crossing for $\gamma = (-1, 0, q, n)$. For (q, n) large enough, there is an empty chamber and a single wall corresponding to $\overline{D6} \to \overline{D6} + D4$ contributes to $PT(q, n)$:

$$PT(q, n) = (-1)^{\langle \overline{D6(1)}, \gamma_{D4} \rangle + 1} \langle \overline{D6(1)}, \gamma_{D4} \rangle\, \bar{\Omega}(\gamma_{D4})$$

  with $\overline{D6(1)} := \mathcal{O}_X(H)[1]$ and $\gamma_{D4} = (0, 1, q, n)$ [Feyzbakhsh'22].

Modular predictions for D4-D2-D0

- Using this idea, we can compute all polar terms and many non-polar ones, and verify modular invariance. E.g. for X_5:

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \Big(\underline{5 - 800\mathrm{q} + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + 75474060100\mathrm{q}^4 + 28096675153255\mathrm{q}^5 + 3756542229485475\mathrm{q}^6 + 277591744202815875\mathrm{q}^7 + 13610985014709888750\mathrm{q}^8 + \dots\Big),$$

$$h_{1,\pm 1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625\mathrm{q}} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + 3102750380125\mathrm{q}^4 + 577727215123000\mathrm{q}^5 + \dots\Big)$$

$$h_{1,\pm 2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0\mathrm{q}} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + 953712511250\mathrm{q}^4 + 217571250023750\mathrm{q}^5 + 22258695264509625\mathrm{q}^6 + \dots\Big)$$

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

Mock modularity for non-Abelian D4-D2-D0 indices

- For D4-D2-D0 indices with $N = 2$ units of D4-brane charge, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_{q}(\tau,\bar{\tau}) = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa}\, k^2},$$

and $\beta(x) = 2|x|^{-1/2} e^{-\pi x} - 2\pi \mathrm{Erfc}(\sqrt{\pi |x|})$.

- The series $\Theta^{(\kappa)}_{q}$ is convergent but not modular invariant.

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

  will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.

- For $\kappa = 1$, the series $\Theta_q^{(1)}$ is the one appearing in the modular completion of the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]* (or rank 2 Vafa-Witten invariants on $\mathbb{P}^2$)

$$\begin{aligned} H_0(\tau) &= -\tfrac{1}{12} + \tfrac{1}{2}\mathrm{q} + \mathrm{q}^2 + \tfrac{4}{3}\mathrm{q}^3 + \tfrac{3}{2}\mathrm{q}^4 + \dots \\ H_1(\tau) &= \mathrm{q}^{\frac{3}{4}} \left(\tfrac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots \right) \end{aligned}$$

  Thus we can choose $g_q^{(1)} = H_q(\tau)$.

Mock modularity for non-Abelian D4-D2-D0 indices

X	χ_X	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$
\begin{aligned}
h_{2,\mu} =& \frac{5397523 E_4^{12} + 70149738 E_4^9 E_6^2 - 12112656 E_4^6 E_6^4 - 61127530 E_4^3 E_6^6 - 2307075 E_6^8}{46438023168 \eta^{100}} \vartheta_\mu^{(1,2)} \\
&+ \frac{-10826123 E_4^{10} E_6 - 14574207 E_4^7 E_6^3 + 20196255 E_4^4 E_6^5 + 5204075 E_4 E_6^7}{1934917632 \eta^{100}} D\vartheta_\mu^{(1,2)} \\
&+ (-1)^{\mu+1} H_{\mu+1}(\tau)\, h_1(\tau)^2
\end{aligned}
$$

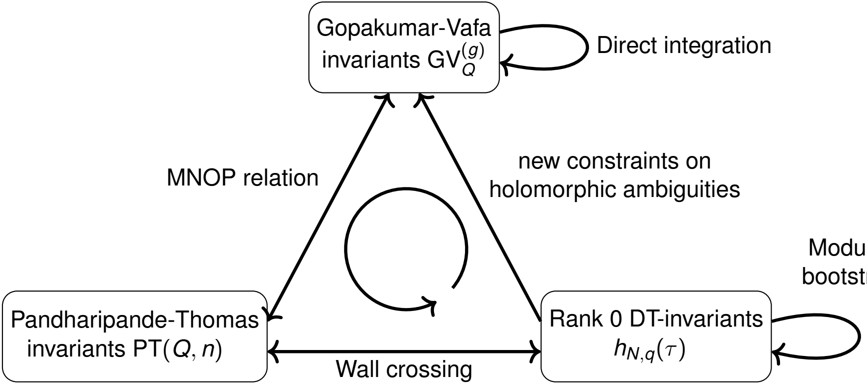
with $h_1 = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\, \eta^{35}} = \mathrm{q}^{-\frac{35}{24}} (\underline{3 - 575\mathrm{q}} + \dots)$, leading to integer DT invariants

$$
\begin{aligned}
h_{2,0}^{(\mathrm{int})} =& \mathrm{q}^{-\frac{19}{6}} \left(\underline{7 - 1728\mathrm{q} + 203778\mathrm{q}^2 - 13717632\mathrm{q}^3} - 23922034036\mathrm{q}^4 + \dots \right) \\
h_{2,1}^{(\mathrm{int})} =& \mathrm{q}^{-\frac{35}{12}} \left(\underline{-6 + 1430\mathrm{q} - 1086092\mathrm{q}^2} + 208065204\mathrm{q}^3 + \dots \right)
\end{aligned}
$$

- The extension to other one-parameter models is in progress.

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



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Quantum geometry from stability and modularity

X	χ_X	κ	type	g_{integ}	g_{mod}	g_{avail}
$X_5(1^5)$	-200	5	F	53	69	64
$X_6(1^4,2)$	-204	3	F	48	63	48
$X_8(1^4,4)$	-296	2	F	60	80	60
$X_{10}(1^3,2,5)$	-288	1	F	50	91	65
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	21	21
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	64	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	42

Thanks for your attention !