BPS Modularity on Calabi-Yau threefolds

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Workshop "New connections in number theory and physics"
Newton Institute, 26/08/2022

References

- "*"Indefinite theta series and generalized error functions"*, with S. Alexandrov, S. Banerjee, J. Manschot, Selecta Math. 24 (2018) 3927 [arXiv:1606.05495]
- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, in progress.

Precision counting of $\mathcal{N} = 8$ BPS black holes

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- The simplest case arises in type II strings compactified on a torus T^d. The manifest $SL(d, \mathbb{Z})$ symmetry is enhanced to T-duality $O(d, d, \mathbb{Z})$ and even U-duality $E_{d+1}(\mathbb{Z})$, constraining the moduli dependence of higher-derivative interactions in the low energy effective action.

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- BPS states consist of D-branes, NS5-branes and KK-monopoles wrapped on subtori, and bound states thereof. They induce instanton corrections in one dimension lower, breaking the continuous $E_{d+1}(\mathbb{R})$ symmetry to an arithmetic subgroup.

Precision counting of $\mathcal{N}=8$ BPS black holes

- The index $\Omega(\gamma)$ counting BPS bound states with charge γ is independent of the moduli and, for the most interesting case of $1/8$-BPS states on T^6, given by the Fourier coefficient $c(I_4(\gamma))$ of a weak modular form, growing as $e^{\pi\sqrt{n}} \sim e^{S_{BH}(\gamma)}$ *[Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005]*

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} c(n)\, q^n = \frac{1}{q} + 2 + 8q^3 + 12q^4 + 39q^7 + 56q^8 + \dots$$

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- These indices can be read off from non-perturbative corrections to the $D^6\mathcal{R}^4$ coupling *[BP'15, Bossard Kleinschmidt BP'20]*.
- Mathematically, the index $\Omega(\gamma)$ is given by a reduced Donaldson-Thomas invariant for the category of coherent sheaves on $X = T^6$. These invariants were computed rigorously for some choices of γ, though modularity and $E_7(\mathbb{Z})$-duality remains mysterious. *[Bryan Oberdieck Pandharipande Yin'15]*.

Precision counting of $\mathcal{N}=4$ BPS black holes

- The case of type II strings compactified on $K3\times T^2$ (or suitable orbifolds thereof) is richer. The duality group is enhanced from $O(3,19,\mathbb{Z})\times SL(2,\mathbb{Z})$ to $O(6,22,\mathbb{Z})\times SL(2,\mathbb{Z})$, or even $O(8,24,\mathbb{Z})$ after reducing on a circle.
- The index $\Omega_z(\gamma)$ counting 1/4-BPS states is given by a Fourier coefficient of a meromorphic Siegel modular form, with a moduli-dependent integration contour. The index jumps when the contour crosses a pole, reflecting the (dis)appearance of two-centered bound states *[Dijkgraaf Verlinde Verlinde'96; Cheng Verlinde'07]*

Precision counting of $\mathcal{N} = 4$ BPS black holes

- In the attractor chamber where no bound states contribute, the index is given by a Fourier coefficient of a mock modular form *[Dabholkar Murthy Zagier '12]*. Its modular completion gives access to the asymptotics of $\Omega(\gamma) \sim e^{S_{BH}(\gamma)}$ as $|\gamma| \to \infty$.
- The index and integration contour can also be read off from non-perturbative corrections to $D^2\mathcal{F}^4$ couplings, given by some genus-two theta lift *[Bossard Cosnier-Horeau BP'16-18]*.
- The prediction is confirmed by rigorous computations of reduced DT invariants for some γ *[Bryan Oberdieck'18]*.

Precision counting of Calabi-Yau black holes

- For type IIA strings compactified on a CY threefold X of generic $SU(3)$ holonomy, the moduli space is no longer a symmetric space. The duality group reduces to the monodromy group $\Gamma \subset Sp(2b_2+2,\mathbb{Z}) \times Sp(2b_3,\mathbb{Z})$.

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- Further reducing on a circle and viewing type IIA/S^1 as M/T^2, one expects the two-derivative action and BPS spectrum to be constrained by $SL(2,\mathbb{Z}) \times \Gamma \ltimes H_{2b_2+2} \times H_{2b_3}$ where H_n is a discrete Heisenberg group of large gauge transformations.

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- The main complication is that BPS bound states can involve an arbitrary number of constituents, leading to complicated dependence of the BPS index / DT invariant $\Omega_z(\gamma)$ on the Kähler moduli. Jumps across walls of marginal stability are governed by a universal wall-crossing formula *[Kontsevich Soibelman'08]*.

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- Can one determine $\Omega_z(\gamma)$ exactly for some range of γ and z ?

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S-duality constraints on BPS indices

By requiring that the moduli space metric admits an isometric action of $SL(2,\mathbb{Z})$ near the large volume point, one can show *[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Rocek, Saueressig, Theis, Vandoren '06-19]*:

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- For $\gamma = (0,p^a,q_a,n)$ supported on an ample divisor $\mathcal{D}$ of class $p^a\gamma_a$, the generating series

$$h_{p^a,q_a}(\tau) = \sum_n \Omega_\star(\gamma) q^{n-\frac{1}{2}q_a k^{ab} q_b}$$

should be a vector-valued weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2 - 1$ and prescribed multiplier system.

Modular constraints on D4-D2-D0 indices

- Here, $\Omega_\star(\gamma)$ is the index in the large volume attractor chamber

$$z^a_\star(\gamma) = \lim_{\lambda\to+\infty} \left(-q^a + \mathrm{i}\lambda p^a\right), \qquad \begin{cases} q^a = \kappa^{ab} q_b \\ \kappa_{ab} = \kappa_{abc} p^c \end{cases}$$

invariant under spectral flow (tensoring with line bundle on $\mathcal{D}$)

$$q_a \to q_a - \kappa_{ab}\epsilon^b, \quad n \mapsto n - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

Thus, $q_a \in \Lambda^*/\Lambda$, taking $|\det(\kappa_{ab})|$ possible values.

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- This modularity constraint also follows from the fact that $Z_p = \sum_{q\in\Lambda^*/\Lambda} h_{p,q}(\tau)\Theta_q(\tau, v)$ coincides with the elliptic genus of the $(0,4)$ superconformal field theory obtained by wrapping an M5-brane on $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

Modular constraints on D4-D2-D0 indices

- A vector-valued weak modular form of negative weight is uniquely determined by the polar coefficients $\Omega(0, p, q, n)$ with $n - \frac{1}{2}q_a\kappa^{ab}q_b < 0$, which are themselves constrained by modularity.

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- Provided the leading polar coefficient is non-zero, the Hardy-Ramanujan-Rademacher expansion gives

$$\log \Omega_\star(\gamma) \sim 2\pi \sqrt{\frac{n}{6} \kappa_{abc} p^a p^b p^c} + \mathcal{O}(\log n)$$

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- I will discuss later how to compute polar indices in some simple CY3 manifolds. For now, let me continue with the general story.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i$, the generating series h_p (omitting q index for simplicity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n - 1$ and same weight/multiplier system.

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- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{n=2}^{\infty} \sum_{p=\sum_{i=1}^{n} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(S)-1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{n=2}^{\infty} \sum_{p=\sum_{i=1}^{n} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right) e^{-\mathrm{i}\pi\tau Q(k)}$$

where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\mathbb{R}^d)$.

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  - $\left[\partial_x^2 + 2\pi(x\partial x - \lambda)\right]\Phi = 0$ [*]
- The operator $\partial_{\bar\tau}$ acts by sending $\Phi \to (x\partial_x - \lambda)\Phi$. Thus ϑ is holomorphic if Φ is homogeneous. But unless $r = 0$, $f(x)$ will fail to be integrable !

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.

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- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), B(C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{B(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{B(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

satisfies [*] with $\lambda = 0$. As $|x| \to \infty$,

$$\widehat{\Phi}(x) \to \mathrm{sgn} B(C, x) - \mathrm{sgn} B(C', x)$$

The holomorphic theta series ϑ_Φ and its modular completion $\vartheta_{\widehat{\Phi}}$ are key for understanding Ramanujan mock theta functions.

Indefinite theta series

- For $r > 1$, one can construct solutions of $[*]$ which asymptote to $\prod_i \mathrm{sgn}[B(C_i, x)]$ as $|x| \to \infty$: the generalized error functions

$$E_r(C_1, \ldots C_r; x) = \int_{\langle C_1, \ldots, C_r \rangle} \mathrm{d}x' \, e^{-\pi Q(x_+ - x')} \prod_i \mathrm{sgn}[B(C_i, x')]$$

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- Taking suitable linear combinations of $E_r(C_1, \dots C_r; x)$, one can construct a kernel Φ which leads to a convergent, modular (but non-holomorphic) theta series ϑ.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

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- More geometrically, ϑ arises by integrating the form-valued Kudla-Milsson theta series on a suitable polyhedron in $Gr(r, d - r)$

Kudla Funke 2016-17

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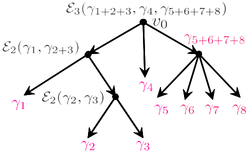
Kudla Funke 2016-17

- For applications to BPS indices, $(r, d-r) = (n-1)(1, b_2(X)-1)$.

Explicity modular completions

- The series $\hat{\Theta}_n$ appearing in the holomorphic anomaly are exactly of that type, with kernel given by a sum over planar trees,

$$\widehat{\Phi}_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1} \mathcal{E}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v$$



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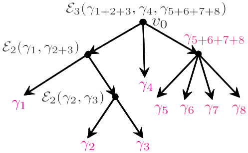

- The series Θ_n appearing in the modular completion are not modular, but their anomaly cancels against the anomaly of h_p:

$$\Phi_n = \mathrm{Sym} \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T - 1} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)}$$

where $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ with $\mathcal{E}_v^{(0)}(x) = \lim_{\lambda \to \infty} \mathcal{E}_v(\lambda x)$.

- The series $\hat{\Theta}_n$ appearing in the holomorphic anomaly are exactly of that type, with kernel given by a sum over planar trees,

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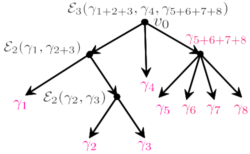

- The series Θ_n appearing in the modular completion are not modular, but their anomaly cancels against the anomaly of h_p:

$$\Phi_n = \mathrm{Sym} \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T - 1} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \backslash \{v_0\}} \mathcal{E}_v^{(0)}$$

where $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ with $\mathcal{E}_v^{(0)}(x) = \lim_{\lambda \to \infty} \mathcal{E}_v(\lambda x)$.

- NB: these formulae hold for generating series of refined invariants, otherwise derivatives of error functions appear.

Simplifications in one-divisor case

- On a threefold with $b_4(X) = 1$, the D4-brane charge $p^a = N p_0^a$ is a multiple of the class p_0 of the primitive divisor $\mathcal{D}$, which we assume to be ample, with self-intersection $\kappa := [\mathcal{D}]^3 = |\Lambda^*/\Lambda|$. The modular completion involves a sum over partitions $N = \sum_{i=1}^{n} N_i$.

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- Remarkably, only partitions of length two contribute to the holomorphic anomaly. In terms of the 'elliptic genus' $Z_N = \sqrt{\frac{\kappa}{N}} \sum_q \widehat{h}_{N,q}(\tau,\bar\tau)\Theta_q(\bar\tau, v)$, this reduces to

$$\mathcal{D}_{\bar\tau} Z_N = \frac{\sqrt{2\tau_2}}{32\pi\mathrm{i}} \sum_{N=N_1+N_2} N_1 N_2 Z_{N_1} Z_{N_2}$$

Minahan Nemeschansky Vafa Warner'98; Alexandrov Manschot BP'19

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- In contrast, the modular completion involves a sum over partitions of arbitrary length.

Outline

Mock modularity for local CY I

- A class of (non-compact) CY threefolds with $b_4(X) = 1$ is obtained by taking the total space $X = K_S$ of the canonical bundle over a complex Fano surface S.
- The BPS index $\Omega_z(\gamma)$ for $\gamma = (0, N, \mu, n)$ coincides with the Vafa-Witten invariant, given by (up to sign) by the Euler number of the moduli space $\mathcal{M}_{N,\mu,n}$ of semi-stable sheaves of rank N on S.
- Since $b_2^+(S) = 1$, the Vafa-Witten invariants depend on the Kähler form J on S. The large volume attractor point corresponds to the canonical polarization $J \propto c_1(S)$.
- The generating series

$$h_{N,\mu} = \sum_n \bar{\Omega}_\star(0, N, \mu, n)\, q^{n - \frac{N-1}{2N}\mu^2 - N\frac{\chi(S)}{24}}$$

is invariant under $\mu \mapsto \mu + N$, and should transform as a vv mock modular form of weight $w = -1 - \frac{b_2(S)}{2}$ and depth $N - 1$.

Mock modularity for local CY II

- Similarly, the generating series $h_{N,\mu}(\tau, z)$ of refined Vafa-Witten invariants

$$\Omega_\star(\gamma, y) = \sum_p (-y)^{p-\dim_{\mathbb{C}} \mathcal{M}} b_p(\mathcal{M})$$

with $y = e^{2\pi i z}$ (or rather its rational counterpart) is expected to transform as a vector-valued mock Jacobi form of weight $w = -\frac{b_2(S)}{2}$, index $m = -\frac{1}{6}K_S^2(N^3 - N)$ and depth $N - 1$

Goettsche Kool 18; Alexandrov BP Manschot 19

- For $N = 1$, the generating series is manifestly modular *[Goettsche'90]*,

$$h_{1,\mu}(\tau, z) = \frac{i}{\theta_1(\tau, 2z)\eta^{b_2(S)-1}} \overset{z\to 0}{\to} \frac{1}{4\pi i z} \frac{1}{\eta^{b_2(S)+2}}$$

Mock modularity for local CY III

- For $S=\mathbb{P}^2$, rank 2 Vafa-Witten invariants are related to Hurwitz class numbers *[Klyachko'91, Yoshioka'94]*

$$h_{2,\mu}(\tau)=\frac{3H_\alpha(\tau)}{\eta^6}\qquad\begin{cases}H_0(\tau)= & -\frac{1}{12}+\frac{1}{2}q+q^2+\frac{4}{3}q^3+\frac{3}{2}q^4+\dots\\ H_1(\tau)= & q^{\frac34}\left(\frac13+q+q^2+2q^3+q^4+\dots\right)\end{cases}$$

which is probably the simplest example of depth 1 mock modular form, with completion *[Hirzebruch Zagier'75-76]*

$$\widehat{h}_{0,2}(\tau)=h_{0,2}(\tau)-\frac{3\mathrm{i}}{4\sqrt{2\pi}\eta^6}\int_{-\bar\tau}^{\mathrm{i}\infty}\frac{\sum_{m\in\mathbb{Z}}e^{2\mathrm{i}\pi m^2 u}\mathrm{d}u}{[-\mathrm{i}(\tau+u)]^{3/2}}$$

consistent with our general prescription.

- From the point of view of twisted $\mathcal{N}=4$ Yang-Mills theory on S, the non-holomorphic contribution arises from the boundary of the space of flat connections where the holonomy becomes reducible

Vafa Witten 94; Dabholkar Putrov Witten '20

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Mock modularity for local CY IV

- For $S = \mathbb{P}^2, \mathbb{F}_0$ or any other del Pezzo surface, the VW invariants can be obtained in principle for any rank N by a sequence of blow ups and wall-crossings. Alternatively, one can relate them to DT invariants for a suitable quiver associated to an exceptional collection on S.
- Using our general prescription, one can easily obtain the modular completion of the generating series. Moreover, with some ingenuity one can produce explicit solutions for all N, which (conjecturally) provide VW invariants for any del Pezzo surface and any rank *[Alexandrov'20]*.
- Having the modular completion, one can apply Rademacher's circle method to extract the asymptotics of VW invariants as the instanton number n goes to infinity *[Bringmann Manschot'13, Bringmann Nazaroglu'18]*

Outline

Modularity for one-modulus compact CY

- We now return to the case of D4-D2-D0 indices on compact CY3, and specialize to one-parameter models, $b_2(X) = b_4(X) = 1$ with $p = N[\mathcal{D}]$ where $\mathcal{D}$ is an ample divisor with $[\mathcal{D}]^3 = \kappa$.

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- For $N = 1$, the generating series

$$h_{1,\mu} = \sum_{n\in\mathbb{Z}} \Omega(0,1,\mu,n)\, q^{n+\frac{\mu^2}{2\kappa}+\frac{\mu}{2}-\frac{\chi(\mathcal{D})}{24}}$$

with $\mu \in \mathbb{Z}/\kappa\mathbb{Z}$ should transform as a vector-valued modular form of weight $-\frac{3}{2}$ (in a suitable Weil representation).

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with $\mu \in \mathbb{Z}/\kappa\mathbb{Z}$ should transform as a vector-valued modular form of weight $-\frac{3}{2}$ (in a suitable Weil representation).
- Thus $h_{1,\mu}$ is uniquely determined by the polar coefficients $\Omega(0, 1, \mu, n < \frac{\chi(\mathcal{D})}{24} - \frac{\mu^2}{2\kappa} - \frac{\mu}{2}$. However, the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar coefficients ! *[Gaiotto Strominger Yin '06-07; Manschot Moore'07]*

Modularity for one-modulus compact CY

CICY	$\chi(X)$	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Modularity for one-modulus compact CY

- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*. For a single D6-brane, the rank 1 DT-invariant $DT(q_a, n) = \Omega(1, 0, q_a, n)$ can be computed from Gopakumar-Vafa invariants via the GV/DT relation

$$\Psi_{\rm top}(X^I) = [M(-e^{2\pi i X^0})]^{\chi/2} \sum_{q_a,n} DT(q_a, n)\, e^{2\pi i(q_a X^a + n X^0)}$$

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- Assuming that all polar coefficients come from two-centered bound states of a $D6 - qD2 - nD0$ and $\overline{D6}$ with -1 unit of flux, we predict *[Alexandrov Gaddam Manschot BP'22, Collinucci Wyder'09]*

$$\Omega(0, 1, q, n) = (-1)^{\chi(\mathcal{O}_\mathcal{D}) - q - n + 1} (\chi(\mathcal{O}_\mathcal{D}) - q - n)\, DT(q, n)$$

with $DT(0, 0) = 1$ (Recall $\Delta\Omega = (-1)^{\langle\gamma_1, \gamma_2\rangle + 1} |\langle\gamma_1, \gamma_2\rangle|\, \Omega(\gamma_1)\Omega(\gamma_2)$)

Modularity for one-modulus compact CY

- An overcomplete basis of vector-valued weakly holomorphic modular forms with desired multiplier system for any N is given by $E_4^a E_6^b D^\ell(\vartheta^{(N,\kappa)})_q/\eta^{4\kappa r^3+rc_2}$ with $4a+6b+2l-2\kappa N^3-\frac{1}{2}Nc_2=-2$, where $D=q\partial_q-\frac{w}{12}E_2$ is the Serre derivative, and

$$\vartheta_q^{(N,\kappa)}(\tau)=\sum_{k\in\mathbb{Z}+\frac{q}{\kappa N}+\frac{N}{2}}(-1)^{\kappa N^2 k}\, e^{\mathrm{i}\pi\kappa k^2\tau+2\pi\mathrm{i}\kappa Nkz}|_{z=0}$$

for κ even, or its z-derivative at $z=0$ for κ odd.

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  for κ even, or its z-derivative at $z=0$ for κ odd.
- Remarkably, there exists a modular form with integer Fourier coefficients matching these polar terms for all models – except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$:-(

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- In particular, it satisfies the modular constraint for $X_{3,3}$ and $X_{4,4}$, and reproduces earlier results by Gaiotto and Yin for X_5, X_6, X_8, X_{10} and $X_{3,3}$:-)

Modularity for one-modulus compact CY

- X_5:

$$h_{1,0} = q^{-\frac{55}{24}} \left(\underline{5 - 800q + 58500q^2} + 5817125q^3 + \ldots \right)$$

$$h_{1,1} = q^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 + \ldots \right)$$

$$h_{1,2} = q^{-\frac{55}{24}+\frac{2}{5}} \left(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + \ldots \right)$$

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- X_6:

$$h_{1,0} = q^{-\frac{15}{8}} \left(\underline{-4 + 612q} - 40392q^2 + 146464860q^3 + \dots \right)$$
$$h_{1,1} = q^{-\frac{15}{8}+\frac{2}{3}} \left(\underline{0 - 15768q} + 7621020q^2 + 10739279916q^3 + \dots \right)$$

Modularity for one-modulus compact CY

- X_8:

$$h_{1,0} = q^{-\frac{46}{24}} \left(\underline{-4 + 888q} - 86140q^2 + 132940136q^3 + \ldots \right),$$

$$h_{1,1} = q^{-\frac{46}{24}+\frac{3}{4}} \left(\underline{0 - 59008q} + 8615168q^2 + 21430302976q^3 + \ldots \right).$$

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- X_{10}:

$$h_{1,0} \overset{?}{=} q^{-\frac{35}{24}} \left(\underline{3 - 576q} + 271704q^2 + 206401533q^3 + \cdots \right)$$

Alas, mathematical results *[Feyzbakhsh and Thomas'21-22]* give instead

$$h_{1,0} \overset{!}{=} q^{-\frac{35}{24}} \left(\underline{3 - 575q} + 271955q^2 + 206406410q^3 + \cdots \right)$$
$$= \frac{203E_4^4 + 445E_4E_6^2}{216\,\eta^{35}}$$

off by one from our Ansatz, as suggested by *[van Herck Wyder'09]*

Rank 0 DT invariants from GV invariants I

- By exploiting wall-crossing and vanishing theorems (in particular a Bogomolov-Gieseker-type inequality on Chern classes of stable coherent sheaves), *[Feyzbakhsh Thomas'20-22]* show that rank 0 DT invariants (counting D4-D2-D0 bound states) can be expressed in terms of rank 1 DT/PT invariants, in turn related to GV invariants.
- Specifically, for $\gamma = (0, 1, q, n)$ and (q, n) 'large enough',

$$PT(q,n) = (-1)^{\chi(\mathcal{O}_X(H),\gamma)+1} \chi(\mathcal{O}_X(H),\gamma)\, \Omega(\gamma)$$

  Using spectral flow invariance, one obtains for m large enough

$$\Omega(\gamma) = \frac{(-1)^{1+\chi(\mathcal{O}_X(1-m),\gamma)}}{\chi(\mathcal{O}_X(1-m),\gamma)} PT(\mu', n') \qquad \begin{cases} q' = q + \kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

Rank 0 DT invariants from GV invariants

- For polar degeneracies, (q', n') lies close to Castelnuovo bound $n' \geq -\frac{(q')^2}{2\kappa} - \frac{q'}{2}$, so $PT(q', n')$ is a linear combination of GV invariants $N^{(g)}_{q'}$ and near-maximal genus. The latter can be computed recursively by integrating the holomorphic anomaly equations for $\Psi_{\rm top}$ *[Huang Klemm Quackenbush'06]*

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- Using this idea (and some improvements), we have computed most of the polar terms (and some non-polar ones) for all models except $X_{4,2}, X_{4,3}, X_{3,2,2}, X_{2,2,2,2}$ – for those the required degree is currently out of reach.

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- We find that our $D6 - \overline{D6}$ ansatz is correct for $X_5, X_6, X_8, X_{3,3}, X_{4,4}$, $X_{6,6}$ but misses some $\mathcal{O}(1)$ contributions for $X_{10}, X_{6,2}, X_{6,4}$. Their physical interpretation is currently unknown.

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- Note that *[Feyzbakhsh'22]* also proves an analogue of our $D6 - \overline{D6}$ ansatz, but under very restrictive conditions satisfied only by the most polar terms.

Mock modularity for one-modulus compact CY

- Finally, let us discuss D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, the generating series $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vector-valued modular form of weight $-\frac{3}{2}$, with modular completion

$$\widehat{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta_{\frac{3}{2}}\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa} k^2},$$

with $\beta_{\frac{3}{2}}(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$, such that

$$\partial_{\bar\tau}\Theta^{(\kappa)}_q = \frac{(-1)^q\sqrt{\kappa}}{16\pi i \tau_2^{3/2}} \sum_{k\in 2\kappa\mathbb{Z}+q} e^{\frac{\pi i \bar\tau}{2\kappa} k^2}$$

Mock modularity for one-modulus compact CY

- The series $\Theta_q^{(\kappa)}$ is convergent but not modular invariant. Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

will be an ordinary weakly holomorphic vv modular form, uniquely determined by its polar part.

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- To construct $g_q^{(\kappa)}$, notice that for κ prime, $\Theta_q^{(\kappa)}$ is obtained from $\Theta_q^{(1)}$ by acting with the Hecke-type operator *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

$$(\mathcal{T}_\kappa[\phi])_q(\tau) = \frac{1}{\kappa} \sum_{\substack{a,d>0\\ ad=\kappa}} \left(\tfrac{\kappa}{d}\right)^{w+\frac{1}{2}D} \delta_\kappa(q,d) \sum_{b=0}^{d-1} e^{-\pi \mathrm{i}\,\frac{b}{a}\,q^2}\, \phi_{dq}\left(\tfrac{a\tau+b}{d}\right),$$

with $q \in \Lambda^*/\Lambda(\kappa)$ and $\delta_\kappa(q,d) = 1$ if $q \in \Lambda^*/\Lambda(d)$ and 0 otherwise.

Mock modularity for one-modulus compact CY

- For $\kappa = 1$, the series $\Theta_q^{(1)}$ is the one appearing in the modular completion of rank 2 VW invariants on $\mathbb{P}^2$! Thus $g_q^{(1)}$ can be chosen to be the generating series of Hurwitz class numbers H_q, and upgraded to $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.

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- For κ not prime, the action of $\mathcal{T}_\kappa$ on $\Theta_q^{(1)}$ is more complicated, e.g.

$$\begin{aligned}(\mathcal{T}_4[\Theta^{(1)}])_q &= 2\Theta_q^{(4)} + \delta_q^{(2)}(\Theta_q^{(4)} + \Theta_{q+4}^{(4)}) \\ (\mathcal{T}_6[\Theta^{(1)}])_q &= 4\Theta_q^{(6)} - 2\delta_{q+1}^{(2)}(\Theta_q^{(6)} - \Theta_{q+6}^{(6)})\end{aligned}$$

When κ is a prime power, one can disentangle these terms, but the cases $\kappa = 6$ or 12 remain to be understood.

Mock modularity for one-modulus compact CY

- For $\kappa = 1$, the series $\Theta_q^{(1)}$ is the one appearing in the modular completion of rank 2 VW invariants on $\mathbb{P}^2$! Thus $g_q^{(1)}$ can be chosen to be the generating series of Hurwitz class numbers H_q, and upgraded to $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.
- For κ not prime, the action of $\mathcal{T}_\kappa$ on $\Theta_q^{(1)}$ is more complicated, e.g.

$$\begin{aligned}(\mathcal{T}_4[\Theta^{(1)}])_q &= 2\Theta_q^{(4)} + \delta_q^{(2)}(\Theta_q^{(4)} + \Theta_{q+4}^{(4)}) \\ (\mathcal{T}_6[\Theta^{(1)}])_q &= 4\Theta_q^{(6)} - 2\delta_{q+1}^{(2)}(\Theta_q^{(6)} - \Theta_{q+6}^{(6)})\end{aligned}$$

When κ is a prime power, one can disentangle these terms, but the cases $\kappa = 6$ or 12 remain to be understood.
- The vv modular form $\widetilde{h}_{2,q}$ is uniquely specified by its polar terms (n_2 of them in the table below), but those must satisfy constraints for such a form to exist (C_2 of them), and integrality is not guaranteed !

Mock modularity for one-modulus compact CY

CICY	χ	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for one-modulus compact CY

- Mathematical results by Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus is prohibitive so far.

Mock modularity for one-modulus compact CY

- Mathematical results by Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus is prohibitive so far.
- Our $D6-\overline{D6}$ ansatz has a natural generalization for any D4-brane charge, allowing N units of flux on the $\overline{D6}$-brane:

$$\Omega(0,N,q,n) \overset{?}{=} (-1)^{\chi(\mathcal{O}_{N\mathcal{D}})-Nq-n+1}(\chi(\mathcal{O}_{N\mathcal{D}})-Nq-n)\, DT(q,n)$$

but the resulting polar terms are not compatible with mock-modularity or integrality...

Conclusion I

- The existence of an isometric action of S-duality on the vector-multiplet moduli space in $D = 3$, leads to strong modularity constraints on rank 0 DT invariants in the large volume limit.
- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p is a mock modular form of depth $n - 1$, with an explicit shadow. From the knowledge of polar coefficients, one can in principle reconstruct all invariants. But computing those is hard !
- A mathematical understanding of the origin of modularity and a better understanding of the physical origin of the non-holomorphic contributions, would be highly desirable.
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

Thanks for your attention !