Modularity of BPS indices on Calabi-Yau threefolds

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Geometry and Automorphy of Supersymmetric Partitions
KIPMU 14/2/2023

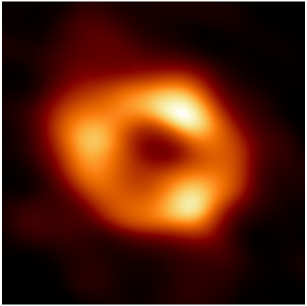
References

- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- *"Quantum geometry, stability and modularity"*, with S. Alexandrov, S. Feyzbakhsh, A. Klemm, T. Schimannek [arXiv:2301.08066]

Introduction

- A driving force in high energy theoretical physics has been the quest for a microscopic explanation of the entropy of black holes. Providing a derivation of the Bekenstein-Hawking formula is a benchmark test of any theory of quantum gravity.

$$S_{BH} = \frac{A}{4G_N}$$

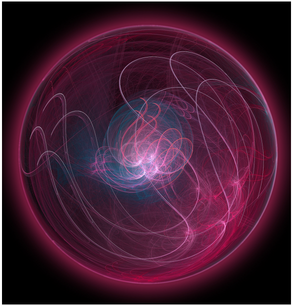


$$S_{BH} \stackrel{?}{=} \log \Omega$$

Sgr A, Event Horizon Telescope 2022*

Black hole microstates as wrapped D-branes

- Back in 1996, Strominger and Vafa argued that String Theory passes this test with flying colors, at least in the context of BPS black holes in vacua with extended SUSY: black hole micro-states can be understood as bound states of D-branes wrapped on the internal manifold, and sometimes can be counted efficiently.



Calabi-Yau black hole, courtesy F. Le Guen

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold $\mathfrak{X}$, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.

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- The problem becomes a question in enumerative geometry: for fixed $\gamma \in K(\mathfrak{X})$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ for a Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.

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- Physical arguments predict that suitable generating series of rank 0 DT invariants (counting D4-D2-D0 bound states) should have specific modular properties. This gives very good control on their asymptotic growth, and allows to check whether $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

Simplest example: Abelian three-fold

- For $\mathfrak{X} = T^6$, $\Omega_z(\gamma)$ depends only on a certain quartic polynomial $I_4(\gamma)$ in the charges, and is moduli independent. It is given by the Fourier coefficient $c(I_4(\gamma)+1)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq 0} c(n)\, \mathrm{q}^{n-1} = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

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- Recall that $f(\tau) := \sum_{n\geq 0} c(n)\mathrm{q}^{n-\Delta}$ (with $\mathrm{q} = e^{2\pi i\tau}$, $\mathrm{Im}\tau > 0$) is a modular form of weight w if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau) \quad \Rightarrow \quad c(n) \overset{n\to\infty}{\sim} \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$$

in agreement with $S_{BH}(\gamma) = \frac{1}{4}A(\gamma)$.

Wall-crossing and mock modularity

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- Second, the generating series of rank 0 DT invariants in the large volume attractor chamber, denoted by $\Omega_\star(\gamma)$, are generally not modular but rather mock modular of higher depth.
- A (depth one) mock modular form of weight w transforms inhomogeneously under $\Gamma \subset SL(2,\mathbb{Z})$ (or $Mp(2,\mathbb{Z})$ if $w \in \mathbb{Z}+\frac{1}{2}$)

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar{\rho})}(\tau+\rho)^{-w}\mathrm{d}\rho\right]$$

where $g(\tau)$ is an ordinary modular form of weight $2-w$, known as the shadow.

Wall-crossing and mock modularity

- Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{i\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho$$

transforms like a modular form of weight w, and satisfies the holomorphic anomaly equation

$$\tau_2^w \partial_{\bar\tau} \widehat{f}(\tau,\bar\tau) \propto \overline{g(\tau)}$$

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- The Fourier coefficients still grow as $c(n) \sim \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$ but subleading corrections are markedly different.

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4. Test modularity for compact CY threefolds with $b_2(\mathfrak{X}) = 1$, using recent results of S. Feyzbakhsh and R. Thomas
5. Obtain new constraints on higher genus GW/GV invariants from modularity of rank 0 DT invariants

- Let $\mathfrak{X}$ a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes of coherent sheaves $\mathcal{E}^k$ on $\mathfrak{X}$,

$$E = (\cdots \stackrel{d^{-2}}{\to} \mathcal{E}^{-1} \stackrel{d^{-1}}{\to} \mathcal{E}^0 \stackrel{d^0}{\to} \mathcal{E}^1 \stackrel{d^1}{\to} \dots),$$

with morphisms $d^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ such that $d^{k+1}d^k = 0$. Physically, $\mathcal{E}^k$ describe D6-branes for k even, or anti D6-branes for k odd, and d^k are open strings .

Mathematical preliminaries

- Let $\mathfrak{X}$ a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes of coherent sheaves $\mathcal{E}^k$ on $\mathfrak{X}$,

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- $\mathcal{C}$ is graded by the Grothendieck group $K(\mathcal{C})$. Let $\Gamma \subset H^{\mathrm{even}}(\mathfrak{X}, \mathbb{Q})$ be the image of $K(\mathcal{C})$ under $E \mapsto \mathrm{ch}\, E = \sum_k (-1)^k \,\mathrm{ch}\, \mathcal{E}_k$. The lattice of electromagnetic charges Γ is equipped with the skew-symmetric (Dirac-Schwinger-Zwanziger) pairing

$$\langle E, E' \rangle = \chi(E', E) = \int_{\mathfrak{X}} (\mathrm{ch}\, E')^\vee \,\mathrm{ch}(E)\, \mathrm{Td}(T\mathfrak{X}) \in \mathbb{Z}$$

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  3. For any non-zero $E \in \mathcal{A}$, $(i)\, \mathrm{Im} Z(E) \geq 0$ and $(ii)\, \mathrm{Im} Z(E) = 0 \Rightarrow \mathrm{Re} Z(E) < 0$. Relax (ii) for weak stability conditions.

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- If $\mathcal{S}$ is not empty, then it is a complex manifold of dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(\mathfrak{X})$, locally parametrized by $Z(\gamma_i)$ with γ_i a basis of Γ.
- Stability conditions are known to exist only for a handful of CY threefolds, including the quintic in $\mathbb{P}^4$ *[Li'18]*. Their construction depends on the conjectural Bayer-Macrì-Toda (BMT) inequality. Weak stability conditions are much easier to construct.

Physical stability conditions

- Physics/Mirror symmetry conjecturally selects a subspace $\Pi \subset \mathrm{Stab}\,\mathcal{C}$, known as 'physical slice' or slice of Π-stability conditions, parametrized by complexified Kähler structure of $\mathfrak{X}$, or complex structure of $\widehat{\mathfrak{X}}$. Hence $\dim_{\mathbb{C}} \Pi = b_2(\mathfrak{X}) + 1 = b_3(\widehat{\mathfrak{X}})$.

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- Along this slice, the central charge is given by the period

$$Z(\gamma) = \int_{\hat{\gamma}} \Omega_{3,0}$$

of the holomorphic 3-form on $\widehat{\mathfrak{X}}$ on a dual 3-cycle $\hat{\gamma} \in H_3(\hat{\mathfrak{X}}, \mathbb{Z})$.

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- Near the large volume point in $\mathcal{M}_K(\mathfrak{X})$, or MUM point in $\mathcal{M}_{cx}(\widehat{\mathfrak{X}})$,

$$Z(E) \sim -\int_{\mathfrak{X}} e^{-z^a H_a} \sqrt{Td(T\mathfrak{X})}\, \mathrm{ch}(E)$$

where H_a is a basis of $H^2(\mathfrak{X}, \mathbb{Z})$, and $z^a = b^a + \mathrm{i} t^a$ are the complexified Kähler moduli.

Generalized Donaldson-Thomas invariants

- Given a (weak) stability condition $\sigma = (Z, \mathcal{A})$, an object $F \in \mathcal{A}$ is called σ-semi-stable if $\arg Z(F') \leq \arg Z(F)$ for every non-zero subobject $F' \subset F$ (where $0 < \arg Z \leq \pi$).

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- Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semi-stable objects of class γ in $\mathcal{A}$. Following *[Joyce-Song'08]* one can associate the DT invariant $\bar\Omega_\sigma(\gamma) \in \mathbb{Q}$. When γ is primitive and $\mathcal{M}_\sigma(\gamma)$ is a smooth projective variety, then $\bar\Omega_\sigma(\gamma) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma))$.

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- Conjecturally, the generalized DT invariant defined by

$$\Omega_\sigma(\gamma) = \sum_{m|\gamma} \frac{\mu(m)}{m^2} \bar\Omega_\sigma(\gamma/m)$$

is integer for any γ, and coincides with the physical BPS index along the slice $\Pi \subset \mathrm{Stab}\,\mathcal{C}$.

Wall-crossing

- The invariants $\bar{\Omega}_\sigma(\gamma)$ are locally constant on $\mathcal{S}$, but jump across walls of instability (or marginal stability), where the central charge $Z(\gamma)$ aligns with $Z(\gamma')$ where $\gamma' = \mathrm{ch}\, E'$ for a subobject $E' \subset E$. The jump is governed by a universal wall-crossing formula.

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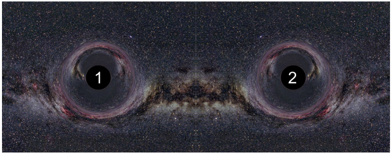
- Physically, the jump corresponds to the (dis)appearance of multi-centered black hole bound states. In the simplest case,

$$\Delta\bar{\Omega}(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1,\gamma_2\rangle|\, \bar{\Omega}(\gamma_1)\, \bar{\Omega}(\gamma_2)$$



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  2. $\widetilde{\mathcal{M}}_V$ parametrizes the Kähler structure of $\mathfrak{X}$ + radius R + Ramond gauge fields in $H^{\rm odd}(\mathfrak{X})$
- Both factors carry a quaternion-Käler metric. $\mathcal{M}_H$ is largely irrelevant for this talk, but note that $\mathcal{M}_H$ and $\widetilde{\mathcal{M}}_V$ get exchanged under mirror symmetry.

S-duality constraints on DT invariants

- Near $R \to \infty$, $\widetilde{\mathcal{M}}_V$ is a torus bundle over $\mathbb{R}^+ \times \mathcal{M}_K$ with semi-flat QK metric, but the QK metric receives $\mathcal{O}(e^{-R|Z(\gamma)|})$ corrections from Euclidean black holes winding around S^1.

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- These corrections are determined from the DT invariants $\Omega_z(\gamma)$ by a twistorial construction à la Gaiotto-Moore-Neitzke *[Alexandrov BP Saueressig Vandoren'08]*
- Since type IIA/$S^1(R)$ is the same as M-theory on $T^2(\tau)$, $\widetilde{\mathcal{M}}_V$ must have an isometric action of $SL(2,\mathbb{Z})$. This strongly constrains the DT invariants in the large volume limit *[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Persson, Rocek, Saueressig, Theis, Vandoren '06-19]*

S-duality constraints on BPS indices

Requiring that $\widetilde{\mathcal{M}}_V$ admits an isometric action of $SL(2,\mathbb{Z})$ near large volume, one can show that DT invariants $\Omega_z(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ satisfy

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- For classes supported on an irreducible divisor $\mathcal{D}$ of class $p^a \gamma_a \in \Lambda = H_4(\mathfrak{X}, \mathbb{Z})$, the generating series of rank 0 DT invariants

$$h_{p^a,q_a}(\tau) := \sum_n \bar{\Omega}_\star(0, p^a, q_a, n)\, \mathrm{q}^{n + \frac{1}{2} q_a \kappa^{ab} q_b - \frac{1}{2} p^a q_a - \frac{\chi(\mathcal{D})}{24}}$$

should be a vector-valued, weakly holomorphic modular form of weight $w = -\frac{1}{2} b_2(\mathfrak{X}) - 1$ and prescribed multiplier system.

S-duality constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \bar{\Omega}_\star(0,p^a,q_a,n)\,\mathrm{q}^{n+\frac{1}{2}q_a\kappa^{ab}q_b+\frac{1}{2}p^aq_a-\frac{\chi(\mathcal{D})}{24}}$$

- Here, $\bar{\Omega}_\star(0,p^a,q_a,n)$ is the index in the large volume attractor chamber

$$\bar{\Omega}_\star(\gamma) = \lim_{\lambda\to+\infty} \bar{\Omega}_{-\kappa^{ab}q_b+\mathrm{i}\lambda p^a}(\gamma)$$

where κ^{ab} is the inverse of the quadratic form $\kappa_{ab}=\kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(\mathfrak{X})-1)$.

S-duality constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \bar{\Omega}_\star(0, p^a, q_a, n)\, \mathrm{q}^{n+\frac12 q_a \kappa^{ab} q_b + \frac12 p^a q_a - \frac{\chi(\mathcal{D})}{24}}$$

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- At least for CY threefolds with $b_2(\mathfrak{X}) = 1$, $\bar{\Omega}_\star(\gamma)$ coincides with the DT invariant counting Gieseker semi-stable sheaves.

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$$h_{p^a,q_a}(\tau) = \sum_n \bar\Omega_\star(0,p^a,q_a,n)\,\mathrm{q}^{n+\frac12 q_a\kappa^{ab}q_b+\frac12 p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

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- At least for CY threefolds with $b_2(\mathfrak{X})=1$, $\bar\Omega_\star(\gamma)$ coincides with the DT invariant counting Gieseker semi-stable sheaves.
- The classical Bogolomov-Gieseker inequality guarantees that n is bounded from below, $n \geq -\frac12 q_a\kappa^{ab}q_b - \frac12 p^a q_a$.

S-duality constraints on D4-D2-D0 indices

- By construction, $\Omega_\star(0, p^a, q_a, n)$ is invariant under tensoring with a line bundle $\mathcal{O}(\epsilon^a H_a)$ (aka spectral flow)

$$q_a \to q_a - \kappa_{ab}\epsilon^b \ , \quad n \mapsto n - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

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  Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

- h_{p^a,q_a} transforms under the Weil representation of $\mathrm{Mp}(2,\mathbb{Z})$ associated to the lattice Λ, e.g.

$$h_{p^a,q_a}(-1/\tau) = \sum_{q'_a\in\Lambda^*/\Lambda} \frac{e^{-2\pi\mathrm{i}\kappa^{ab}q_a q'_b + \frac{\mathrm{i}\pi}{4}(b_2(\mathfrak{X})+2\chi(\mathcal{O}_\mathcal{D})-2)}}{\tau^{1+\frac{1}{2}b_2(\mathfrak{X})}\sqrt{|\det(\kappa_{ab})|}} h_{p^a,q'_a}(\tau)$$

D4-D2-D0 indices from elliptic genus

- Summing over all D2-brane charges and using spectral flow invariance, one gets

$$\begin{aligned} Z_p(\tau, v) &:= \sum_{q\in\Lambda,n} \bar{\Omega}_\star(0, p^a, q_a, n)\, \mathrm{q}^{n+\frac{1}{2}q_a\kappa^{ab}q_b} e^{2\pi\mathrm{i}q_a v^a} \\ &= \sum_{q\in\Lambda^*/\Lambda} h_{p,q}(\tau)\Theta_q(\tau, v) \end{aligned}$$

where $\Theta_q(\tau, v)$ is the (non-holomorphic) Siegel theta series for the indefinite lattice (Λ, κ_{ab}). S-duality then requires that Z_p should transform as a (skew-holomorphic) Jacobi form.

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- The Jacobi form Z_p can be interpreted as the elliptic genus of the $(0, 4)$ superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for brevity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

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- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(\mathfrak{X})-1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Crash course on indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2} Q(x)} \in L_1(\Lambda \otimes \mathbb{R})$.

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- The relevant lattice $\Lambda = H^2(\mathfrak{X}, \mathbb{Z})^{\oplus(n-1)}$ has signature $(r, d-r) = (n-1)(1, b_2(\mathfrak{X})-1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.

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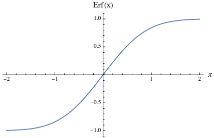
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$



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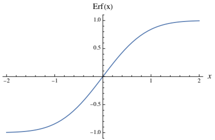

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- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n \geq 3$ involves generalized error functions $\mathcal{E}_{n-1}(\{C_i\}, x)$, obtained as a convolution of $e^{\pi Q(x_+)}$ with $\prod_{i=1}^{n-1}(C_i, x)$. *[Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016]*

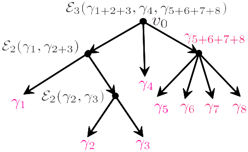
Explicity modular completions

- The series $\widehat{\Theta}_n$ appearing in the holomorphic anomaly equation

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

have a kernel given by a sum over rooted trees,

$$\widehat{\Phi}_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1}\mathcal{E}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v$$



For each vertex with n descendants, $\mathcal{E}_v = \mathcal{E}_{n-1}(\{C_i\},x)$ with suitable arguments.

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$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

are not modular, but their anomaly cancels against that of h_p:

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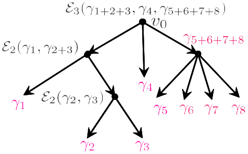

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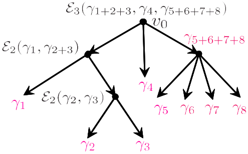

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- NB: these formulae hold for generating series of refined invariants, otherwise derivatives of error functions appear.

Alexandrov Manschot BP 18-19

Mock modularity for Vafa-Witten invariants

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- Since $b_2^+(S) = 1$, Vafa-Witten invariants for rank $N > 1$ have non-trivial dependence on the Kähler form $J = z^a H_a$.
- The large volume attractor point corresponds to the canonical polarization $J \propto c_1(S)$. Denote by $\bar\Omega_\star(N, \mu, n)$ the corresponding DT invariants.

Mock modularity for local CY

- We predict that the generating series

$$h_{N,\mu}(\tau) = \sum_n \bar{\Omega}_\star(N,\mu,n)\, \mathrm{q}^{n-\frac{N-1}{2N}\mu^2-N\frac{\chi(S)}{24}} \,, \quad \mu \in \mathbb{Z}/N\mathbb{Z}$$

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- For $N=1$, the moduli space reduces to the Hilbert scheme of n points on S, and the generating series is manifestly modular *[Goettsche'90]*,

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- For $N>1$, one expects non-holomorphic contributions from the boundary of the space of flat connections where the holonomy becomes reducible *[Vafa Witten 94; Dabholkar Putrov Witten '20]*.

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- For $S = \mathbb{P}^2$, rank 2 Vafa-Witten invariants are related to Hurwitz class numbers, counting equivalence classes of binary quadratic forms *[Klyachko'91, Yoshioka'94]*

$$h_{2,\mu}(\tau) = \frac{3H_\mu(\tau)}{\eta^6} \qquad \begin{cases} H_0(\tau) = & -\frac{1}{12} + \frac{1}{2}\mathrm{q} + \mathrm{q}^2 + \frac{4}{3}\mathrm{q}^3 + \frac{3}{2}\mathrm{q}^4 + \dots \\ H_1(\tau) = & \mathrm{q}^{\frac{3}{4}}\left(\frac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots\right) \end{cases}$$

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- This is one of the simplest examples of depth 1 mock modular forms, with completion *[Hirzebruch Zagier'75-76]*

$$\widehat{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) + \frac{3(1+\mathrm{i})}{8\pi(\eta^3)^2} \int_{-\bar\tau}^{\mathrm{i}\infty} \frac{\sum_{m\in\mathbb{Z}+\frac{\mu}{2}} e^{2\mathrm{i}\pi m^2 u}\mathrm{d}u}{(\tau+u)^{3/2}}$$

consistent with our general prescription (the integral can be expressed in terms of Erfc)

Mock modularity for Vafa-Witten invariants

- For any del Pezzo surface S and any rank N, the VW invariants can be obtained by a sequence of blow ups and wall-crossings. The generating series is expressed in terms of generalized Appell-Lerch sums *[Yoshioka'95-96, Manschot'10-14]*

$$\sum_{k\in\mathbb{Z}^r} \frac{\mathrm{q}^{\frac12 \mathcal{Q}(k)}}{\prod_{i=1}^{N-1}(1-e^{2\pi\mathrm{i}u_i}\mathrm{q}^{(C_i,k)})}$$

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- It would be nice to interpret $h_{N,\mu}$ as the graded character $\mathrm{Tr} q^{L_0 - \frac{c}{24}}$ for some VOA acting on the cohomology of the moduli stack of semistable coherent sheaves on S. Mock modularity would then follow if the VOA is quasi-lisse *[Arakawa Kawasetsu'16]*

Modularity for one-modulus compact CY

- Let $\mathfrak{X}$ be a compact CY threefold with $H^2(\mathfrak{X},\mathbb{Z})=\mathbb{Z}H$. Can we compute rank 0 DT invariants $\bar\Omega_\star(0,NH,q,n)$ and test modularity ?

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- The central charge $Z_z(\gamma)$ is expressed in terms of hypergeometric functions. GV invariants $\mathrm{GV}^{(g)}_Q$ can be computed up to high genus by direct integration *[Huang Klemm Quackenbush'06]*, and are related to rank 1 DT invariants by *[Maulik Nekrasov Okounkov Pandharipande'06]*.

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- I will concentrate on $N = 1$, and discuss $N = 2$ if time permits.

Gaiotto Strominger Yin '06-07; Alexandrov Gaddam Manschot BP'22

Modularity for one-modulus compact CY

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	$c_2(T\mathfrak{X})$	$\chi(\mathcal{O}_{\mathcal{D}})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Abelian D4-D2-D0 invariants

- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega_\star(0, H, q, n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}\ , \quad q \in \mathbb{Z}/\kappa\mathbb{Z}$$

should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}[\kappa]$ with $\kappa = H^3$.

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- An overcomplete basis is given for κ even by

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where $D = \mathrm{q}\partial_\mathrm{q} - \frac{w}{12}E_2$, is the Serre derivative (Alternatively, one may use Rankin-Cohen brackets).

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- For κ odd, the same works with $\vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} (-1)^{\kappa k}\, k^2 \mathrm{q}^{\frac{1}{2}\kappa k^2}$.

A naive Ansatz for the polar terms

- $h_{1,q}$ is uniquely determined by the polar terms $n < \frac{\chi(\mathcal{D})}{24} - \frac{q^2}{2\kappa} - \frac{q}{2}$, but the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar terms !

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- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*
- Earlier studies *[Gaiotto Strominger Yin'06]* suggest that only bound states of the form $(D6 + qD2 + nD0, \overline{D6(-1)})$ contribute to polar coeffs:

$$\Omega(0,1,q,n) = (-1)^{\chi(\mathcal{O}_\mathcal{D})-q-n+1} \left(\chi(\mathcal{O}_\mathcal{D}) - q - n\right) DT(q,n)$$

where $DT(q,n)$ counts ideal sheaves with $\mathrm{ch}_2 = q$ and $\mathrm{ch}_3 = n$ *[Alexandrov Gaddam Manschot BP'22]*

GV/DT/PT relation

- For a single D6-brane, the DT-invariant $DT(q,n) = \Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\sum_{Q,n} DT(Q,n)\, \mathrm{q}^n v^Q = M(-\mathrm{q})^{\chi_{\mathfrak{X}}} \prod_{Q,g,\ell} \left(1-(-\mathrm{q})^{g-\ell-1} v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\mathrm{GV}_Q^{(g)}}$$

where $M(\mathrm{q}) = \prod_{n\geq 1}(1-\mathrm{q}^n)^{-n}$ is the Mac-Mahon function.

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- Pandharipande-Thomas invariants $PT(Q,n)$ counting stable pairs $E = (\mathcal{O}_{\mathfrak{X}} \stackrel{s}{\to} F)$ with $[F]=Q$ and $\chi(F)=n$ satisfy a similar relation without the Mac-Mahon factor $M(-\mathrm{q})^{\chi_{\mathfrak{X}}}$.

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- The topological string partition function is given by

$$\Psi_{\mathrm{top}}(z,\lambda)=M(-\mathrm{q})^{-\chi_{\mathfrak{X}}/2} Z_{DT}\ , \quad \mathrm{q}=e^{\mathrm{i}\lambda}, v=e^{2\pi\mathrm{i}z/\lambda}$$

can be computed by the direct integration method.

Modular predictions for D4-D2-D0 indices (naive)

- Remarkably, there exists a vv modular form with integer Fourier coefficients matching these polar terms for almost all CICY (except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$), reproducing earlier results *[Gaiotto Strominger Yin]* by for X_5, X_6, X_8, X_{10} and $X_{3,3}$

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- $X_5 = \mathbb{P}^4[5]$:

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \left(\underline{5 - 800\mathrm{q} + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + \dots \right)$$

$$h_{1,\pm 1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625\mathrm{q}} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + \dots \right)$$

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- $X_{10} = \mathbb{P}^4_{5,2,1,1,1}[10]$:

$$h_{1,0} \stackrel{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576\, \eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 576\mathrm{q}} + 271704\mathrm{q}^2 + \cdots \right)$$

Rank 0 DT invariants from GV invariants

- Our Ansatz for polar terms was just an educated guess. Fortunately, recent progress in Donaldson-Thomas theory allows to compute D4-D2-D0 indices in a rigorous fashion, and compare with modular predictions.

Bayer Macri Toda'11; Toda'11; Feyzbakhsh Thomas'20-22

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- The key idea is to consider a family of weak stability conditions on the boundary of Stab $\mathcal{C}$, called tilt stability, with degenerate central charge

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i} t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

with $\mathrm{ch}_k^b(E) = \int_{\mathfrak{X}} H^{3-k} e^{-bH} \mathrm{ch}(E)$. The heart $\mathcal{A}_b$ is generated by length-two complexes $\mathcal{E} \xrightarrow{d} \mathcal{F}$ with $\mathrm{ch}_1^b(\mathcal{E}) > 0, \mathrm{ch}_1^b(\mathcal{F}) \leq 0$.

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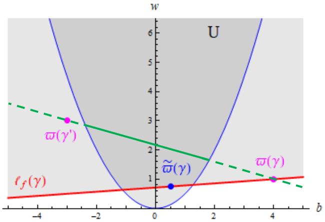
- Tilt stability agrees with physical stability at large volume, but the chamber structure is much simpler: walls are straight lines in the plane spanned by $(b, w = \frac{1}{2}b^2 + \frac{1}{6}t^2)$, with $w > \frac{1}{2}b^2$.



$$\nu_{b,w}(E) = \frac{\mathrm{ch}_2 .H - w\, \mathrm{ch}_0 .H^3}{\mathrm{ch}_1 .H^2 - b\, \mathrm{ch}_0 .H^3}$$

$$\varpi(E) = \left(\frac{\mathrm{ch}_1 .H^2}{\mathrm{ch}_0 .H^3}, \frac{\mathrm{ch}_2 .H}{\mathrm{ch}_0 .H^3} \right)$$

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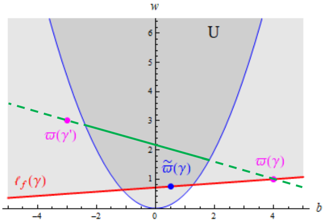

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- Importantly, for any $\nu_{b,w}$-semistable object E there is a conjectural inequality on Chern classes $C_i := \int_{\mathfrak{X}} \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)w + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from GV invariants

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.

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- In particular for (q,n) large enough, the PT invariant counting tilt-stable objects of class $(-1,0,q,n)$ is given by *[Feyzbakhsh'22]*

$$PT(q,n) = (-1)^{\langle \overline{D6(1)},\gamma\rangle+1} \langle \overline{D6(1)},\gamma\rangle\, \Omega(\gamma)$$

with $\overline{D6(1)} := \mathcal{O}_{\mathfrak{X}}(H)[1]$ and $\gamma = (0,H,q,n)$. Using spectral flow invariance, one finds for all $m \geq m_0(q,n)$

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)},\gamma\rangle+1}}{\langle \overline{D6(1-m)},\gamma\rangle} PT(q',n')} \qquad \begin{cases} q' = q+\kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

Modular predictions for D4-D2-D0 (rigorous)

- Using an extension of this idea, we have computed most of the polar terms, and many non-polar ones, for all models except $X_{3,2,2}, X_{2,2,2,2}$. In all cases, modularity holds with flying colors !

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- Note that *[Toda'13, Feyzbakhsh'22]* also prove a version of our $D6 - \overline{D6}$ ansatz, but under very restrictive conditions which are only satisfied by the most polar terms.

Mock modularity for non-Abelian D4-D2-D0 indices

- Let us consider D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa} k^2},$$

and $\beta(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$.

Mock modularity for non-Abelian D4-D2-D0 indices

- Let us consider D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

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- For $\kappa = 1$, the series $\Theta^{(1)}_q$ is the one appearing in the modular completion of rank 2 Vafa-Witten invariants on $\mathbb{P}^2$!

Mock modularity for non-Abelian D4-D2-D0 indices

- The series $\Theta_q^{(\kappa)}$ is convergent but not modular invariant. Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

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- To construct $g_q^{(\kappa)}$, notice that for κ prime, $\Theta_q^{(\kappa)}$ is obtained from $\Theta_q^{(1)}$ by acting with the Hecke-type operator *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

$$(\mathcal{T}_\kappa[\phi])_q(\tau) = \frac{1}{\kappa} \sum_{\substack{a,d>0\\ ad=\kappa}} \left(\tfrac{\kappa}{d}\right)^{w+\frac12} \delta_\kappa(q,d) \sum_{b=0}^{d-1} e^{-\pi \mathrm{i}\, \frac{b}{a}\, q^2}\, \phi_{dq}\left(\tfrac{a\tau+b}{d}\right),$$

with $q \in \Lambda^*/\Lambda(\kappa)$ and $\delta_\kappa(q,d) = 1$ if $q \in \Lambda^*/\Lambda(d)$ and 0 otherwise.

Mock modularity for non-Abelian D4-D2-D0 indices

- For $\kappa = 1$, a candidate for $g_q^{(1)}$ is well-known: the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}q + q^2 + \frac{4}{3}q^3 + \frac{3}{2}q^4 + \dots$$
$$H_1(\tau) = q^{\frac{3}{4}}\left(\frac{1}{3} + q + q^2 + 2q^3 + q^4 + \dots\right)$$

For any κ, we can thus choose $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.

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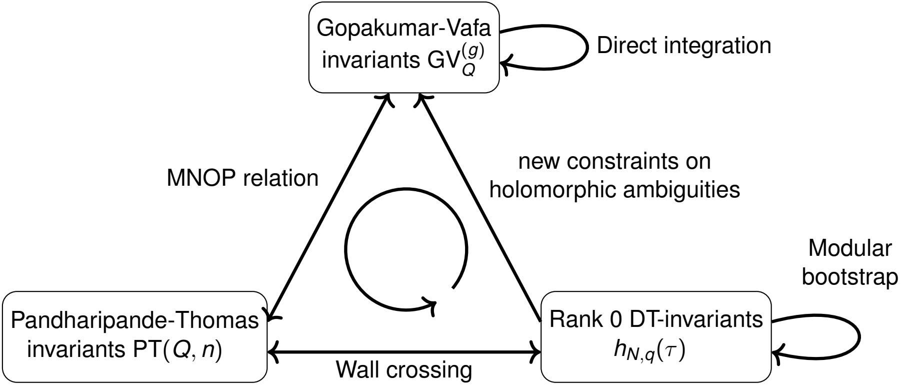
- The vv modular form $\widetilde{h}_{2,q}$ is uniquely specified by its polar terms but those must satisfy constraints for such a form to exist, and integrality is not guaranteed !
- Explicit formulae by S. Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus seem prohibitive so far.

Mock modularity for non-Abelian D4-D2-D0 indices

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	type	$g_{\rm integ}$	$g_{\rm mod}$	$g_{\rm avail}$
$X_5(1^5)$	-200	5	F	53	69	60
$X_6(1^4,2)$	-204	3	F	48	63	48
$X_8(1^4,4)$	-296	2	F	60	80	48
$X_{10}(1^3,2,5)$	-288	1	F	50	70	47
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	21	21
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$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	64	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	42

Conclusion

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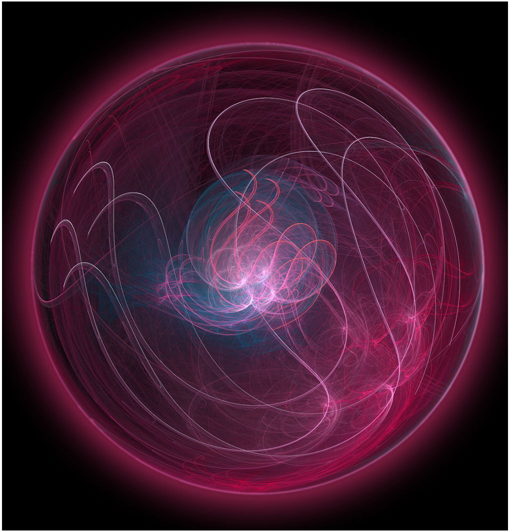
- Using modularity and GV/DT/PT relations, we can not only compute D4D2-D0 indices, but also push $\Psi_{\rm top}$ to higher genus !
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

Thanks for your attention !



A new explicit formula (S. Feyzbakhsh'23)

- Let $(\mathfrak{X}, H)$ be a smooth polarised CY threefold with $\mathrm{Pic}(\mathfrak{X}) = \mathbb{Z}.H$ satisfying the BMT conjecture.

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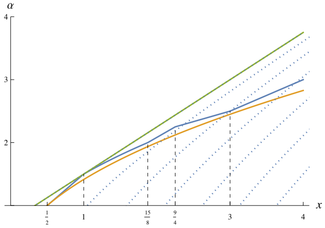
- Fix $m \in \mathbb{Z}, \beta \in H_2(\mathfrak{X}, \mathbb{Z})$ and define $x = \frac{\beta.H}{H^3}, \quad \alpha = -\frac{3m}{2\beta.H}$

$$f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1 \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8} \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4} \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3 \\ \frac{1}{2}x + 1 & \text{if } 3 \leq x \end{cases}$$



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Theorem (wall-crossing for class $(-1, 0, \beta, -m)$):

- If $f(x) < \alpha$ then the stable pair invariant $\mathrm{PT}_{m,\beta}$ equals

$$\sum_{(m', \beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} \, \mathrm{PT}_{m',\beta'} \; \Omega\left(0,\; H,\; \frac{H^2}{2} - \beta' + \beta\,,\; \frac{H^3}{6} + m' - m - \beta'.H\right),$$

where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(\mathfrak{X}).H$.

Corollary (Castelnuovo bound): $\mathrm{PT}_{m,\beta} = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

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where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(\mathfrak{X}).H$.

- The sum runs over $(m',\ \beta') \in H_0(\mathfrak{X}, \mathbb{Z}) \oplus H_2(\mathfrak{X}, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq \ m' \ \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

In particular, $\beta'.H < \beta.H$.

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