Modular bootstrap for BPS indices on Calabi-Yau threefolds

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Seminar on Conformal Field Theory and related topics
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References

- "*"Indefinite theta series and generalized error functions"*, with S. Alexandrov, S. Banerjee, J. Manschot, Selecta Math. 24 (2018) 3927 [arXiv:1606.05495]
- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, in progress.

Introduction

- A driving force in high energy theoretical physics has been the quest for a microscopic explanation of the entropy of black holes. Providing a derivation of the Bekenstein-Hawking formula $S_{BH} = \frac{1}{4G_N} A$ is a benchmark test of any theory of quantum gravity.
- Back in 96, Strominger and Vafa argued that String Theory passes this test, at least in the context of BPS black holes in vacua with extended supersymmetry: at weak coupling, BPS states are bound states of D-branes (along with fundamental strings and NS-branes) wrapped on calibrated cycles of the internal manifold.
- In some corner of moduli space, D-brane bound states can be understood as excitations of an effective black string, supporting a (0,4) superconformal field theory. BPS indices counting such states are encoded in the elliptic genus, and their asymptotic growth at large charge is governed by modularity.

Precision counting of BPS black holes

- In the context of type IIA strings compactified on a Calabi-Yau three-fold $\mathfrak{Y}$, bound states of D6-D4-D2-D0-branes are best understood as stable objects in the derived category $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{Y}$.
- The problem becomes a question in enumerative geometry: for fixed charge $\gamma \in H_{\mathrm{even}}(\mathfrak{Y}, \mathbb{Q})$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting (with signs) stable objects in $\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- While the modular invariance of suitable generating series of $\Omega_z(\gamma)$ is clear from physics, it is a very non trivial prediction about DT invariants, which often can only be verified a posteriori.
- Importantly, $\Omega_z(\gamma)$ is robust under complex structure deformations but depends on the Kähler moduli $z \in \mathcal{M}_K(\mathfrak{Y})$, and it is not clear a priori in which chamber should modularity hold.

Precision counting of $\mathcal{N}=8$ BPS black holes

- For $\mathfrak{Y}=T^6$, the index $\Omega(\gamma)$ counting 1/8-BPS states depends only on the $E_{7(7)}$ quartic invariant $n=I_4(\gamma)$, and is moduli independent. It is given by the Fourier coefficient $c(n)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} c(n)\, \mathrm{q}^n = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

Bryan Oberdieck Pandharipande Yin'15

- The Harder-Ramanujan-Hardy formula gives $c(n) \sim e^{\pi\sqrt{n}}$ as $n\to\infty$, in agreement with $S_{BH}(\gamma)=\frac{1}{4}A(\gamma)$.
- The full Rademacher expansion can be derived by localization in supergravity *[Dabholkar Gomes Murthy'10, Iliesiu Turiaci Murthy'22]*

Precision counting of $\mathcal{N}=4$ BPS black holes

- For $\mathfrak{Y}=K_3\times T_2$ (and orbifolds thereof preserving $\mathcal{N}=4$ SUSY), the BPS index counting 1/4-BPS states with charge $\gamma=(Q,P)$ is a Fourier coefficient of a meromorphic Siegel modular form,

$$\Omega_z(\gamma)=\oint_{\mathcal{C}(\gamma,z)}\frac{e^{2\pi \mathrm{i}\mathrm{Tr}(\tau\cdot\gamma\otimes\gamma)}}{\Phi_{k-2}(\tau)}\,,\qquad \gamma\otimes\gamma=\begin{pmatrix}Q^2 & Q\cdot P\\ Q\cdot P & P^2\end{pmatrix}$$

Dijkgraaf Verlinde Verlinde '96; David Jatkar Sen '05-06; ...

- The result is manifestly invariant under $SL(2,\mathbb{Z})\times O(\Gamma_e)$ acting o n both $\gamma\in\Gamma_e\oplus\Gamma_e^*$ and $z\in\mathcal{M}_4=\frac{SL(2)}{U(1)}\times\frac{O(6,2k-2)}{O(6)\times O(2k-2)}$.
- The integration contour $\mathcal{C}(\gamma,z)$ depends on γ and on $z\in\mathcal{M}_4$. For large $|\gamma|$, a saddle-point computation gives $\log\Omega_z\sim\frac{1}{4}A(\gamma)$.

Wall-crossing for $\mathcal{N} = 4$ BPS black holes

- When z crosses real codimension-1 walls

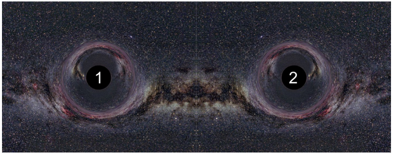
$$W(\gamma_1, \gamma_2) = \{z \in \mathcal{M}_4, M(\gamma_1 + \gamma_2) = M(\gamma_1) + M(\gamma_2)\}$$

where γ_1, γ_2 are 1/2-BPS charge vectors, the contour $\mathcal{C}(\gamma, z)$ crosses a pole of $1/\Phi(\tau)$, so that the index $\Omega_z(\gamma)$ jumps according to the primitive wall-crossing formula

$$\Delta\Omega(\gamma_1 + \gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1, \gamma_2\rangle| \, \Omega(\gamma_1) \, \Omega(\gamma_2)$$

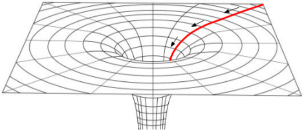
Denef Moore '07; Cheng, Verlinde '07; Sen '07-08

corresponding to contributions of bound states of two 1/2-BPS black holes.



Attractor indices and mock modular forms

- One may extract the contributions of single-centered black holes by evaluating $\Omega(\gamma, z)$ at the attractor point z_γ, where two-centered bound states are not allowed to form.



$$r^2 \frac{\mathrm{d}z^a}{\mathrm{d}r} = g^{ab}\partial_b M^2(\gamma, z)$$

- The attractor indices $\Omega_*(\gamma) = \Omega_{z_\gamma}(\gamma)$ turn out to be Fourier coefficients of a (vector-valued) mock modular form $h(\tau)$ of weight $w = \frac{3}{2} - k$ *[Dabholkar Murthy Zagier '12]*.
- More precisely, there exists a modular form $g(\tau)$ of dual weight $2 - w$ such that $\widehat{h}(\tau, \bar\tau) := h(\tau) + \int_{-\bar\tau}^{\mathrm{i}\infty} \overline{g(-\bar\rho)}(\tau + \rho)^{-w}\mathrm{d}\rho$ transforms like a modular form of weight w. The completion satisfies the holomorphic anomaly equation $\tau_2^w \partial_{\bar\tau}\widehat{h}(\tau, \bar\tau) \propto \overline{g(\tau)}$

Precision counting of Calabi-Yau black holes

- When $\mathfrak{Y}$ is a CY threefold of generic $SU(3)$ holonomy, the moduli space is no longer a symmetric space. Instead, it factorizes into a product $\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$
    1. $\mathcal{M}_V$ parametrizes the Kähler structure of $\mathfrak{Y}$, and receives worldsheet instanton corrections weighted by GW/GV invariants
    2. $\mathcal{M}_H$ parametrizes the dilaton + complex structure of $\mathfrak{Y}$+ Ramond gauge fields, and receives D-instanton corrections (largely irrelevant for this talk)
- The BPS indices $\Omega_z(\gamma)$ are independent of $\mathcal{M}_H$, but have a complicated chamber structure on $\mathcal{M}_V$, due to the possibility of BPS bound states with an arbitrary number of constituents. The full wall-crossing formula for $\Delta\Omega(N_1\gamma_1 + N_2\gamma_2)$ is needed *[Kontsevich Soibelman'08, Joyce Song'08]*.

Precision counting of Calabi-Yau black holes

- Upon reducing on a circle, $\mathcal{M}_H$ goes along for the ride but $\mathcal{M}_V$ extends to a larger quaternion-Kähler space $\widetilde{\mathcal{M}}_V$ parametrizing the radius R, Kähler moduli and Ramond gauge fields along S_1.
- At large R, $\widetilde{\mathcal{M}}_V$ is a flat torus bundle over $\mathbb{R}^+ \times \mathcal{M}_V$, but it receives $\mathcal{O}(e^{-RM(\gamma)})$ corrections from Euclidean black holes winding around S^1, weighted by the same DT invariants $\Omega_z(\gamma)$ counting black holes in $D = 4$.
- Since type IIA/S^1 is the same as M-theory on T^2, $\widetilde{\mathcal{M}}_V$ must have an isometric action of $SL(2,\mathbb{Z})$. This enforces modularity constraints on DT invariants. *[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Rocek, Saueressig, Theis, Vandoren '06-19]*
- By mirror symmetry, $\widetilde{\mathcal{M}}_V$ is also the hypermultiplet moduli space in type IIB on $\hat{\mathfrak{Y}}$, invariant under usual $SL(2,\mathbb{Z})$ S-duality.

S-duality constraints on BPS indices

Requiring that $\widetilde{\mathcal{M}}_V$ admits an isometric action of $SL(2,\mathbb{Z})$ near large volume, large radius limit, one can show

- For n D0-branes, $\Omega_z(0,0,0,n) = -\chi_{\mathfrak{Y}}$ (independent of n)
- For D2-branes supported on a curve of class $q_a\gamma^a \in \Lambda^* = H_2(\mathfrak{Y},\mathbb{Z})$, $\Omega_z(0,0,q_a,n) = N^{(0)}_{q_a}$ is given by the genus-zero GV invariant (independent of n)
- For D4-branes supported on an ample divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(\mathfrak{Y},\mathbb{Z})$, the generating series

$$h_{p^a,q_a}(\tau) := \sum_n \Omega_\star(0,p^a,q_a,n)\, \mathrm{q}^{n-\frac{1}{2}q_a\kappa^{ab}q_b}$$

  should be a vector-valued weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2(\mathfrak{Y}) - 1$ and prescribed multiplier system.

Modular constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \Omega_\star(0, p^a, q_a, n)\, \mathrm{q}^{n-\frac{1}{2}q_a\kappa^{ab}q_b}$$

- Here, κ^{ab} is the inverse of the quadratic form $\kappa_{ab} = \kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(\mathfrak{Y}) - 1)$, and $\Omega_\star(\gamma)$ is the index in the large volume attractor chamber

$$z^a_\star(\gamma) = \lim_{\lambda\to+\infty} \left(-\kappa^{ab}q_b + \mathrm{i}\lambda p^a\right)$$

- In particular, $\Omega_\star(0, p^a, q_a, n)$ is invariant under spectral flow (tensoring with line bundle on $\mathcal{D}$)

$$q_a \to q_a - \kappa_{ab}\epsilon^b\ , \quad n \mapsto n - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

Thus, q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

D4-D2-D0 indices from elliptic genus

- Equivalently, summing over all D2-brane charges and using spectral flow invariance, one gets

$$
\begin{aligned}
Z_p(\tau, v) &:= \sum_{q \in \Lambda, n} \Omega_\star(0, p^a, q_a, n)\, \mathrm{q}^{n - \frac{1}{2} q_a \kappa^{ab} q_b} e^{2\pi \mathrm{i} q_a v^a} \\
&= \sum_{q \in \Lambda^* / \Lambda} h_{p,q}(\tau) \Theta_q(\tau, v)
\end{aligned}
$$

  where $\Theta_q(\tau, v)$ is the (non-holomorphic) Siegel theta series for the indefinite lattice (Λ, κ_{ab}). S-duality then requires that Z_p should transform as a (non-holomorphic) Jacobi form.
- The Jacobi form Z_p can be interpreted as the elliptic genus of the $(0, 4)$ superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

Modular constraints on D4-D2-D0 indices

- A weak modular form $h(\tau) = \sum_n c(n) \mathrm{q}^{n+\Delta}$ of weight $w < 0$ is uniquely determined by polar terms with $n + \Delta < 0$. The existence of cusp forms in dual weight $2 - w$ may impose constraints on polar coefficients *[Bantay Gannon'07, Manschot Moore'07]*
- Provided the leading polar coefficient is non-zero, the Hardy-Ramanujan-Cardy formula gives

$$\log \Omega_\star(\gamma) \sim 4\pi \sqrt{|\Delta| n} \sim 2\pi \sqrt{\frac{n}{6} \kappa_{abc} p^a p^b p^c}$$

in precise agreement with the Bekenstein-Hawking entropy.
- I will discuss later how to compute polar indices in some simple CY3 manifolds. For now, let me continue with the general story.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for simplicity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2} b_2(\mathfrak{Y}) - 1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau} \widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda\otimes\mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda\otimes\mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The operator $\partial_{\bar\tau}$ acts by sending $\Phi \to (x\partial_x - \lambda)\Phi$. Thus ϑ is holomorphic if Φ is homogeneous. But unless $r = 0$, $f(x)$ will fail to be integrable !

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

satisfies [*] with $\lambda = 0$. As $|x| \to \infty$,

$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

The holomorphic theta series ϑ_Φ and its modular completion $\vartheta_{\widehat{\Phi}}$ are key for understanding Ramanujan mock theta functions.

Indefinite theta series

- For $r > 1$, one can construct solutions of [$*$] which asymptote to $\prod_i \mathrm{sgn}(C_i, x)$ as $|x| \to \infty$: the generalized error functions

$$E_r(C_1, \dots C_r; x) = \int_{\langle C_1,\dots,C_r \rangle} \mathrm{d}x' \, e^{-\pi Q(x_+ - x')} \prod_i \mathrm{sgn}(C_i, x')$$

  where x_+ is the projection of x on the positive plane $\langle C_1, \dots, C_r \rangle$.

- Taking suitable linear combinations of $E_r(C_1, \dots C_r; x)$, one can construct a kernel Φ which leads to a convergent, modular (but non-holomorphic) theta series $\vartheta_{\Phi,q}$.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

- More geometrically, ϑ arises by integrating the r-form valued Kudla-Millson theta series on a suitable polyhedron in $Gr(r, d-r)$

Kudla Funke 2016-17

- For applications to BPS indices, $(r, d-r) = (n-1)(1, b_2(\mathfrak{Y}) - 1)$.

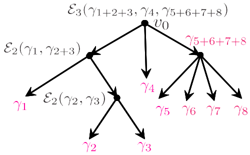
Explicity modular completions

- The series $\widehat{\Theta}_n$ appearing in the holomorphic anomaly equation

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

are exactly of that type, with kernel given by a sum over rooted trees,

$$\widehat{\Phi}_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1}\mathcal{E}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v$$



Explicity modular completions

- The series Θ_n appearing in the modular completion

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

  are not modular, but their anomaly cancels against the anomaly of h_p:

$$\Phi_n = \mathrm{Sym} \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1} \mathcal{E}_{v_0}^{(+)} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v^{(0)}$$

  where $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ with $\mathcal{E}_v^{(0)}(x) = \lim_{\lambda\to\infty} \mathcal{E}_v(\lambda x)$.

- NB: these formulae hold for generating series of refined invariants, otherwise derivatives of error functions appear

Alexandrov Manschot BP 18-19

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Modularity for one-modulus compact CY

- We now specialize to compact CY threefolds with $b_2(\mathfrak{Y}) = 1$ and $p = N[\mathcal{D}]$ where $\mathcal{D}$ is an ample divisor with $[\mathcal{D}]^3 := \kappa$.
  [Gaiotto Strominger Yin '06-07; Alexandrov Gaddam Manschot BP'22]
- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n \in \mathbb{Z}} \Omega(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}$$

  should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}/\kappa\mathbb{Z}$
- $h_{1,q}$ is uniquely determined by the polar terms $n < \frac{\chi(\mathcal{D})}{24} - \frac{q^2}{2\kappa} - \frac{q}{2}$, but the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar terms !

Modularity for one-modulus compact CY

CICY	$\chi(\mathfrak{Y})$	κ	$c_2(T\mathfrak{Y})$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Computing the polar terms

- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes. For the most polar terms, only states with $[D6]=\pm 1$ ought to contribute *[Denef Moore'07]*.
- For a single D6-brane, the DT-invariant $DT(q,n)=\Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\begin{aligned}\Psi_{\rm top} =& M(-\mathrm{p})^{\chi_{\mathfrak{Y}}/2}\sum_{q,n} DT(q,n)\,\mathrm{p}^n v^q \\ =& M(-\mathrm{p})^{\chi_{\mathfrak{Y}}}\prod_{q,g,\ell}\left(1-(-\mathrm{p})^{g-\ell-1}v^q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}N_q^{(g)}}\end{aligned}$$

Maulik Nekrasov Okounkov Pandharipande'06

- Earlier studies by Gaiotto Strominger Yin suggest that only bound states of the form $(D6-qD2-nD0,\overline{D6(-1)})$ contribute. If so:

$$\boxed{\Omega(0,1,q,n)=(-1)^{\chi(\mathcal{O}_\mathcal{D})-q-n+1}(\chi(\mathcal{O}_\mathcal{D})-q-n)\,DT(q,n)\,DT(0,0)}$$

- While this prescription seems ad hoc, it reproduces all cases before considered ! *[Alexandrov Gaddam Manschot BP'22]*

Modularity for one-modulus compact CY

- A basis of vector-valued weakly holomorphic modular forms with desired multiplier system is given by where

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k \in \mathbb{Z} + \frac{q}{\kappa} + \frac{1}{2}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

  (with extra insertion of $(-1)^{\kappa k} k^2$ for κ odd) and $D = \mathrm{q}\partial_\mathrm{q} - \frac{w}{12} E_2$, is the Serre derivative (Alternatively, use Rankin-Cohen brackets).
- Remarkably, there exists a modular form with integer Fourier coefficients matching these polar terms for all models ☺
  – except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$ ☹
- In particular, our ansatz for polar terms satisfies the modular constraint for $X_{3,3}$ and $X_{4,4}$, and reproduces earlier results by *[Gaiotto Yin]* for X_5, X_6, X_8, X_{10} and $X_{3,3}$ ☺

Modularity for one-modulus compact CY

- X_5 (Quintic in $\mathbb{P}^4$):

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \left(\underline{5 - 800q + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + \dots \right)$$

$$h_{1,1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625q} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + \dots \right)$$

$$h_{1,2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \left(\underline{0 + 0q} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + \dots \right)$$

- X_6 (Sextic in $W\mathbb{P}^{2,1,1,1,1}$):

$$h_{1,0} = \mathrm{q}^{-\frac{15}{8}} \left(\underline{-4 + 612\mathrm{q}} - 40392\mathrm{q}^2 + 146464860q^3 + \dots \right)$$

$$h_{1,1} = \mathrm{q}^{-\frac{15}{8}+\frac{2}{3}} \left(\underline{0 - 15768\mathrm{q}} + 7621020\mathrm{q}^2 + 10739279916\mathrm{q}^3 + \dots \right)$$

- X_{10} (Decantic in $W\mathbb{P}^{5,2,1,1,1}$):

$$h_{1,0} \overset{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576\,\eta^{35}}$$

$$= \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 576q} + 271704\mathrm{q}^2 + 206401533\mathrm{q}^3 + \cdots \right)$$

Rank 0 DT invariants from GV invariants

- Our Ansatz for polar terms was just an educated guess. Fortunately, recent progress in Donaldson-Thomas theory allows to compute D4-D2-D0 indices in a rigorous fashion, and compare with modular predictions.

Bayer Macri Toda'11; Toda'11; Feyzbakhsh Thomas'20-22

- The key idea is to consider a (non-physical) slice in the space of Bridgeland stability conditions, called tilt stability, with degenerate central charge

$$Z(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i}\, t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

with $\mathrm{ch}_k^b = \int_{\mathfrak{Y}} H^{3-k} e^{-bH} \mathrm{ch}$, and heart $\mathcal{A}$ given by length-two complexes of coherent sheaves $\mathcal{E} \to \mathcal{F}$ with $\mu(\mathcal{E}) \leq b, \mu(\mathcal{F}) > b$.

Rank 0 DT invariants from GV invariants

- Along this slice, walls of marginal stability are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.
- Most importantly, there is a conjectural bound on ch_3 for any tilt-stable object, *[Bayer Macri Toda'11]*

$$\mathrm{ch}_2^b = \frac{1}{6} t^2 \,\mathrm{ch}_0^b \quad \Rightarrow \quad \mathrm{ch}_3^b \leq \frac{t^2}{18} \mathrm{ch}_1^b$$

  The BMT bound is known to hold for $X_5, X_6, X_8, X_{4,2}$ *[Li'19,Koseki'20]*.
- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.

Rank 0 DT invariants from GV invariants

- In particular for $\gamma = (0, 1, q, n)$ and (q, n) large enough, the PT invariant counting states with charge $(-1, 0, q, n)$ is given by

$$PT(q,n) = (-1)^{\langle \overline{D6(1)},\gamma\rangle+1} \langle \overline{D6(1)},\gamma\rangle\, \Omega(\gamma)$$

Using spectral flow invariance, one obtains for m large enough

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)},\gamma\rangle+1}}{\langle \overline{D6(1-m)},\gamma\rangle} PT(q',n')} \qquad \begin{cases} q' = q + \kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

- PT invariants can be computed from the topological string partition function using the GV/PT relation

$$\sum_{q,n} PT(q,n)\, \mathrm{p}^n v^q = \prod_{q,g,\ell} \left(1 - (-\mathrm{p})^{g-\ell-1} v^q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} N_q^{(g)}}$$

- GV invariants can be computed recursively by integrating the holomorphic anomaly equations for $\Psi_{\rm top}$ *[Huang Klemm Quackenbush'06]*

Rank 0 DT invariants from GV invariants

- Using this idea, we have computed most of the polar terms (and many non-polar ones) for all models except $X_{3,2,2}, X_{2,2,2,2}$ – for those the required GV invariants are currently out of reach.

Alexandrov, Feyzbakhsh, Klemm,, BP, Schimannek, to appear

- We find that our educated guess is correct for $X_5, X_6, X_8, X_{3,3}, X_{4,4}, X_{6,6}$ ☺ , but (as anticipated by *[van Herck Wyder'09]*) misses some $\mathcal{O}(1)$ contributions for $X_{10}, X_{6,2}, X_{6,4}, X_{4,3}$ ☹ E.g. for X_{10},

$$h_{1,0} = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\,\eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 575q} + 271955\mathrm{q}^2 + \cdots \right)$$

In all cases, modularity holds with flying colors ! ☼ ♫ ☺

- Note that *[Toda'13, Feyzbakhsh'22]* also prove a version of our $D6 - \overline{D6}$ ansatz, but under very restrictive conditions which are only satisfied by the most polar terms.

Mock modularity for one-modulus compact CY

- Finally, let us discuss D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where
$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k \in 2\kappa\mathbb{Z}+q} |k|\, \beta\!\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa} k^2},$$

- For $\kappa = 1$, the series $\Theta^{(1)}_q$ is the one appearing in the modular completion of rank 2 Vafa-Witten invariants on $\mathbb{P}^2$!

Mock modularity for one-modulus compact CY

- The series $\Theta_q^{(\kappa)}$ is convergent but not modular invariant. *Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form.* Then

$$\widetilde{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

  will be an ordinary weakly holomorphic vv modular form, hence uniquely determined by its polar part.

- To construct $g_q^{(\kappa)}$, notice that for κ prime, $\Theta_q^{(\kappa)}$ is obtained from $\Theta_q^{(1)}$ by acting with the Hecke-type operator *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

$$(\mathcal{T}_\kappa[\phi])_q(\tau) = \frac{1}{\kappa} \sum_{\substack{a,d>0\\ ad=\kappa}} \left(\tfrac{\kappa}{d}\right)^{w+\frac{1}{2}D} \delta_\kappa(q,d) \sum_{b=0}^{d-1} e^{-\pi i \frac{b}{a} q^2} \phi_{dq}\left(\tfrac{a\tau+b}{d}\right),$$

  with $q \in \Lambda^*/\Lambda(\kappa)$ and $\delta_\kappa(q,d) = 1$ if $q \in \Lambda^*/\Lambda(d)$ and 0 otherwise.

Mock modularity for one-modulus compact CY

- For $\kappa = 1$, a candidate for $g_q^{(1)}$ is well-known: the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}q + q^2 + \frac{4}{3}q^3 + \frac{3}{2}q^4 + \dots$$
$$H_1(\tau) = q^{\frac{3}{4}}\left(\frac{1}{3} + q + q^2 + 2q^3 + q^4 + \dots\right)$$

- For any κ, we can thus choose $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.
- The vv modular form $\widetilde{h}_{2,q}$ is uniquely specified by its polar terms (n_2 of them in the table below), but those must satisfy constraints for such a form to exist (C_2 of them), and integrality is not guaranteed !

Mock modularity for one-modulus compact CY

CICY	χ	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for one-modulus compact CY

- Mathematical results by Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus is prohibitive so far.
- Our naive $D6-\overline{D6}$ ansatz has a natural generalization for any D4-brane charge, allowing N units of flux on the $\overline{D6}$-brane:

$$\Omega(0,N,q,n) \stackrel{?}{=} (-1)^{\chi(\mathcal{O}_{N\mathcal{D}})-Nq-n+1}\left(\chi(\mathcal{O}_{N\mathcal{D}})-Nq-n\right) DT(q,n)$$

  but the resulting polar terms are not compatible with mock-modularity or integrality...☹

Conclusion

- The existence of an isometric action of S-duality on the vector-multiplet moduli space in $D = 3$, leads to strong modularity constraints on rank 0 DT invariants in the large volume limit.
- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p is a mock modular form of depth $n - 1$ with an explicit shadow, thus it is uniquely determined by its polar coefficients.
- While modularity is clear physically, its mathematical origin is mysterious. Perhaps Noether-Lefschetz theory or VOAs can help *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*
- Using modularity and GV/DT/PT relations, we can not only compute D4D2-D0 indices, but also push $\Psi_{\rm top}$ to higher genus !
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

Thanks for your attention !