Schwarzschild meets Ramanujan: From quantum black holes to mock modular forms

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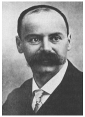
Introduction I

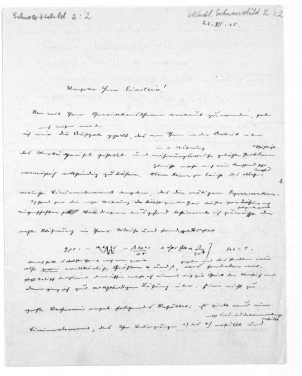
Karl Schwarzschild
(1879-1916)

Srinivasa Ramanujan
(1887-1920)

From Schwarzschild's metric to Hawking's paradox



Letter to A. Einstein (Dec 1915, Russian front): *"the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas."*



$$ds^2 = -\left(1 - \frac{r_S}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_S}{r}\right)^2} + r^2\, d\Omega^2 \,, \quad r_s = \frac{2GM}{c^2}$$

K. Schwarzschild, Sitz. Kön. Preuss. Ak. Wiss. 1916

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- In early 70s, *Bardeen, Bekenstein, Carter* discovered that black holes behave like thermodynamical objects, with energy $E = M$, entropy $S_{BH} = \frac{\mathcal{A}}{4G}$ and therefore temperature $T = dE/dS_{BH}$. *Hawking (1974)* realized that black holes emit thermal radiation, and evaporate on a time scale $t \propto G^2 M^3$.

- Q1: If this thermodynamical analogy is correct, what are the $\mathcal{O}(e^{S_{BH}})$ microscopic degrees of freedom responsible for the Bekenstein-Hawking entropy ?

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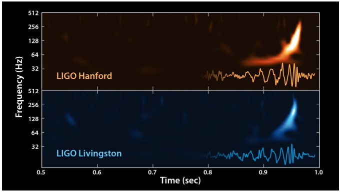
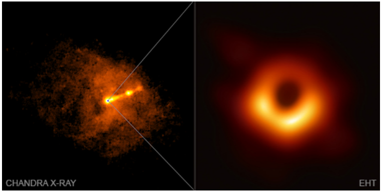
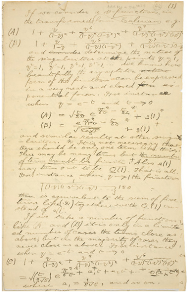
- Meanwhile, *LIGO (2016)*, *EHT (2019)*:





From Ramanujan to mock modular forms

Letter to G. H. Hardy (Jan 1920, Madras): *I am extremely sorry for not writing you a single letter up to now... I discovered very interesting functions recently which I call "Mock" theta-functions. Unlike the "False" theta-functions... they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.*

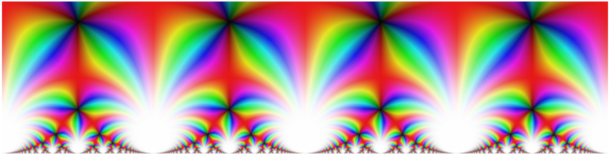


$$f(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\dots(1+q^n)^2} = 1 + q - 2q^2 + 3q^3 + \dots$$

From Ramanujan to mock modular forms

- Recall that $f(\tau) = \sum_{n\geq 0} a_n q^{n-\Delta}$ (with $q = e^{2\pi i\tau}$, $\mathrm{Im}\tau > 0$) is a *modular form* of weight k under $\Gamma \subset SL(2,\mathbb{Z})$ if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)\,(c\tau+d)^k\, f(\tau) \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$



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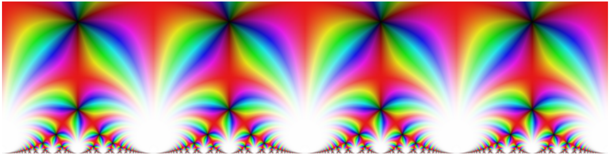

- Examples include *Dedekind*'s function $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ and *Jacobi*'s theta function $\theta(\tau) = \sum_{n=0}^{\infty} q^{n^2}$ which are modular forms of weight 1/2 under $SL(2,\mathbb{Z})$ and $\Gamma_0(4)$, respectively.

From Ramanujan to mock modular forms

- For the rest of the 20th century, Mock theta functions remained as troubling to number theorists as black holes to physicists...
- ... until *Sander Zwegers (2002)*, then PhD student under Don Zagier in Bonn, found several intrinsic characterizations of mock theta functions, in terms of
  1. Appell-Lerch sums,
  2. indefinite theta series,
  3. Fourier coefficients of meromorphic Jacobi forms
  4. harmonic Maass forms,

Bourbaki seminar, Zagier 2007

- Amazingly, each of these classes show up in physics !

Harmonic Mass forms and Eichler integrals

- With hindsight, one important clue can be traced to *Zagier and Hirzebruch (1975-76)*, who studied a certain limit of a non-holomorphic Eisenstein series $\hat{G}(\tau)$ of weight $3/2$, and found that it decomposes into an ordinary, holomorphic q-series plus a non-holomorphic remainder now called "Eichler integral":

$$\hat{G}(\tau,\bar\tau) := \lim_{s\to 0}\left[\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} (\mathrm{Im}\tau)^s|_{3/2}\gamma\right]$$

$$\stackrel{!}{=} G(\tau) - \frac{\mathrm{i}}{4\pi\sqrt{2}}\int_{-\bar\tau}^{\mathrm{i}\infty} \frac{\theta(u)\,\mathrm{d}u}{[-\mathrm{i}(\tau+u)]^{3/2}}$$

The sum $\hat{G}(\tau,\bar\tau)$ is an example of harmonic Maass form.

Harmonic Mass forms and Eichler integrals

- $G(\tau)$ is the generating function of Hurwitz class numbers, counting conjugacy classes of quadratic forms with fixed discriminant $-4n$,

$$G(\tau) = \sum_{n \geq 0} H(4n) q^n = -\frac{1}{12} + \frac{1}{2} q + q^2 + \frac{4}{3} q^3 + \frac{3}{2} q^4 + 2 q^5 + \dots$$

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- Since $\hat{G}$ is an ordinary (albeit non-holomorphic) modular form, its holomorphic part G must transform non-homogeneously,

$$G\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\frac{3}{2}} \left[G(\tau) - \frac{\mathrm{i}}{4\pi\sqrt{2}} \int_{-d/c}^{\mathrm{i}\infty} \frac{\theta(u)\,\mathrm{d}u}{[-\mathrm{i}(\tau+u)]^{3/2}} \right]$$

Indeed: $\int_{-\bar\tau}^{\mathrm{i}\infty} \mathrm{d}u \mapsto \int_{-\overline{\gamma(\tau)}}^{\mathrm{i}\infty} \mathrm{d}u \sim \int_{-\bar\tau}^{\gamma^{-1}(\mathrm{i}\infty)} \mathrm{d}u' = \int_{-\bar\tau}^{\mathrm{i}\infty} \mathrm{d}u' + \int_{\mathrm{i}\infty}^{\gamma^{-1}(\mathrm{i}\infty)} \mathrm{d}u'$

Mock modular forms in field theories

- On the physics side, the first connections of mock modular forms occurred as early as 1994, where evidence started to emerge that certain supersymmetric gauge theories in 3+1 dimensions are invariant under electromagnetic duality, acting by fractional linear transformations on the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$.

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- Using mathematical results by Klyachko and Yoshioka, *Vafa Witten (1994)* computed the partition function of topologically twisted $\mathcal{N} = 4$ gauge theory with gauge group $SU(2)$ on $\mathbb{P}^2$,

$$Z_{\mathbb{P}^2}^{SU(2)} = \sum_{n \geq 0} \chi(\mathcal{M}_n^{SU(2)})\, q^{n-1/4} = \frac{3G(\tau)}{\eta^6(\tau)}$$

where $\mathcal{M}_n^{SU(2)}$ is the moduli space of $SU(2)$ instantons of charge n on $\mathbb{P}^2$, and $G(\tau)$ is the generating function of Hurwitz class numbers.

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- Invariance under electric-magnetic duality appears to be broken, but might be recovered by including contributions from the boundary of $\mathcal{M}_n^{SU(2)}$ which (hopefully) turn G into its modular completion $\hat{G}$.

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- With hindsight, the first connection to physics can be traced to *Eguchi Taormina (1988)*'s study of characters of superconformal algebras in 2 dimensions, though a precise connection was only made by *Semikhatov Taormina Typunin (2003)*, just around the same time as Zwegers' thesis.

Modular forms and BPS black holes

- Connections between modular forms and black holes go back to *Strominger Vafa (1995)* seminal work, which provided for the first time a microscopic description of a class of supersymmetric black holes in $D = 5$ dimensions. Soon extended to supersymmetric Reissner-Nordström black holes in 4D by *Maldacena Strominger (96)*, *Johnson Khuri Myers (96)*, and *Dijkgraaf Verlinde ^{2} (96)*.

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- Supersymmetric black holes saturate the bound $M \geq |Q|$ (often called BPS bound, by analogy to a similar bound for monopoles in gauge theories due to *Bogolmolnyi Prasad Sommerfeld*). As a result, they have $T = 0$ and cannot decay, unlike Schwarzschild black holes. However, they do have finite entropy $S_{BH} = \mathcal{A}/(4G)$, suggestive to a large ground state degeneracy $\Omega(Q) \propto \mathcal{O}(e^{S_{BH}})$.

Modular forms and BPS black holes

- By viewing black holes as black strings wrapped on a circle, micro-states can be viewed as excitations in a 2D superconformal field theory. The partition function $Z_{T^2(\tau)}$ is then manifestly modular. BPS ground states are chiral excitations of the SCFT, counted (with sign) by a weakly holomorphic modular form.

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- E.g. in type II string theory compactified on T^6, $\Omega(Q) = c(I_4(Q))$ where I_4 is a quartic polynomial in Q and $c(n)$ are Fourier coefficients of the modular form

$$\frac{\theta(\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} c(n)\, q^n = \frac{1}{q} + 2 + 8q^3 + 12q^4 + 39q^7 + 56q^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

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- The Hardy-Ramanujan formula (aka. Cardy's formula in physics) gives $c(n) \sim e^{\pi\sqrt{n}}$, in perfect agreement with gravity's prediction $\Omega(Q) \sim e^{\pi\sqrt{I_4(Q)}}$.

Modular forms and BPS black holes

- Given beautiful matching in the large charge limit $Q \to \infty$, one may ask if subleading corrections to the Bekenstein-Hawking entropy, coming from stringy corrections to Einstein-Hilbert action, also match subleading terms in the asymptotics of $c(n)$.

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- Alternatively, can one extend this counting to BPS black holes in string vacua with $\mathcal{N} < 8$ supersymmetries (ideally, $\mathcal{N} = 0$) ? This is where connections to mock theta functions come about.

Manschot 2009, Dabholkar Murthy Zagier 2009-12

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- I will first discuss the $\mathcal{N} = 4$ case, before turning to $\mathcal{N} = 2$, more closely connected to my own work.

Mock modular forms and $\mathcal{N} = 4$ dyons

- In a visionary paper from 1996, *Dijkgraaf Verlinde*[2] proposed a magic formula claiming to give the exact number of micro-states of a generic 1/4-BPS black hole with charge $\gamma = (Q, P)$ in type II strings compactified on $K3 \times T^2$:

$$\Omega(\gamma) = \int_{\mathcal{C}} \mathrm{d}\rho\, \mathrm{d}v\, \mathrm{d}\sigma\, \frac{e^{\mathrm{i}\pi(\rho Q^2 + \sigma P^2 + v Q\cdot P)}}{\Phi_{10}(\rho, v, \sigma)}\,, \quad \tau = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \in \mathcal{H}_2$$

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  where Φ_{10} is Igusa's cusp form (the product of genus-two ThetaNullWerte with even characteristics).

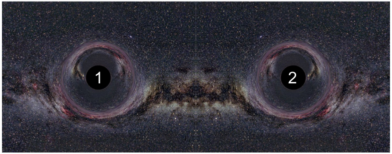
- Due to the pole at $v = 0$ and its images under $Sp(4, \mathbb{Z})$,

$$\Phi_{10}(\rho, v, \sigma) \sim v^2\, \eta^{24}(\rho)\, \eta^{24}(\sigma) + \mathcal{O}(v^4)$$

  the RHS depends on the choice of integration contour. But the LHS is also ambiguous, due to wall-crossing phenomena.

Mock modular forms and $\mathcal{N} = 4$ dyons

- For suitable values of the moduli z, there can exist stationary BPS solutions made of two black holes with charges $\gamma_i = (Q_i, P_i)$, $i = 1, 2$:



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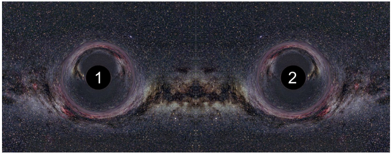

- The distance between the two centers is fixed to be $r_{12} = \frac{\kappa}{\vartheta}$ where $2\kappa = Q_1 P_2 - Q_2 P_1$ is the Dirac-Schwinger-Zwanziger product, and ϑ^2 is proportional to the binding energy,

$$\frac{\vartheta^2}{2M} = M_1 + M_2 - M_{1+2} \ , \qquad M = \frac{M_1 M_2}{M_1 + M_2}$$

The solution exists only in the chamber where $\mathrm{sgn}(\vartheta) = \mathrm{sgn}(\kappa)$.

Denef 2000

Mock modular forms and $\mathcal{N}=4$ dyons

- The wall-crossing phenomenon at $\vartheta=0$ is captured by a simple supersymmetric Hamiltonian describing the relative motion of two mutually non-local particles,

$$H=\frac{1}{2M}(\vec{p}-\kappa\vec{A})^2+\frac{1}{2M}\left(\vartheta-\frac{\kappa}{r}\right)^2-\frac{\kappa}{2M}\vec{B}\cdot\vec{\sigma}\otimes(1_2-\sigma_3)$$

where $\vec{B}=\nabla\times\vec{A}=\frac{\vec{r}}{r^3}$ is a unit magnetic monopole at the origin.

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- Incidentally, *Harish Chandra (1948)* studied the similar problem of an electron in a magnetic monopole background, without the potential required for SUSY.

- For $\kappa\vartheta > 0$, there exists 2κ BPS bound states filling out a multiplet of spin $j = \kappa - \frac{1}{2}$ under rotations (as well as non-BPS bound states which cancel in the index). For $\kappa\vartheta < 0$, no bound states whatsoever. In either case, there is a continuum of scattering states.

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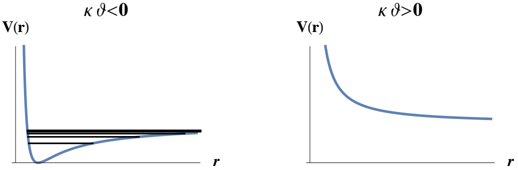

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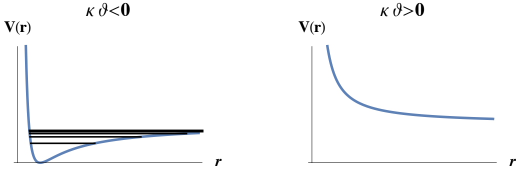

- In $\mathcal{N} = 4$ string vacua, composite 1/4-BPS states can only arise from pairs of 1/2-BPS states, whose electric and magnetic charges are collinear: $(Q_i, P_i) = h \cdot (\tilde{Q}_i, 0)$ with $h \in SL(2, \mathbb{Z})$.

Mock modular forms and $\mathcal{N} = 4$ dyons

- The BPS index carried by the i-th component is given by the number of colored partitions of $N_i = \frac{1}{2}\tilde{Q}_i^2$:

$$\Omega(\gamma_i) = p_{24}\left(\tfrac{1}{2}\tilde{Q}_i^2\right) \;, \quad \sum_{N\geq 0} p_{24}(N)\, q^{N-1} = \frac{1}{\eta^{24}(\tau)}$$

Dabhokar Harvey 1989

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- *Cheng and Verlinde (2007)* devised a prescription for the contour $\mathcal{C}(z)$, in such a way that whenever z crosses a wall of marginal stability for a bound state of charge $\gamma_1+\gamma_2$, $\Omega(\gamma, z)$ jumps by the required amount

$$\Delta\Omega(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\,\langle\gamma_1,\gamma_2\rangle\,\Omega(\gamma_1)\,\Omega(\gamma_2)$$

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- The DVV formula was proven by D-brane techniques, or extracted from non-perturbative effective interaction in $D = 3$.

Shih Strominger Yin 2005, David Sen 2006; Bossard Cosnier Horeau BP 2016

Appell-Lerch sums and indefinite theta series I

- For the purpose of matching the Bekenstein-Hawking entropy with the number of micro-states, we need to subtract the contribution of all two-centered bound states. The latter can be expressed as

$$Z(\rho, v, \sigma) = \sum_{m \geq 0} \left[\frac{p_{24}(m+1)}{\eta^{24}(\rho)} \mathcal{A}_m(\rho, v) \right] e^{2\pi \mathrm{i} m \sigma}$$

Dabholkar Murthy Zagier (2012)

where $\mathcal{A}_m(\rho, v)$ is a Appell-Lerch sum, or indefinite theta series

$$\mathcal{A}_m(\rho, v) = \sum_{s \in \mathbb{Z}} \frac{q^{ms^2+s} \, e^{2\pi \mathrm{i}(2ms+1)v}}{(1 - q^s e^{2\pi \mathrm{i} v})^2}$$

$$= \frac{1}{2} \sum_{s, \ell \in \mathbb{Z}} \ell \left[\mathrm{sgn} \left(s + \frac{\mathrm{Im} v}{\mathrm{Im} \rho} \right) + \mathrm{sgn} \, \ell \right] q^{ms^2 + \ell s} e^{2\pi \mathrm{i}(2ms + \ell) v}$$

Note that only terms with $\ell \left(s + \frac{\mathrm{Im} v}{\mathrm{Im} \rho} \right) > 0$ contribute, analogous to the condition $\kappa \vartheta > 0$ for stable bound states.

Appell-Lerch sums and indefinite theta series II

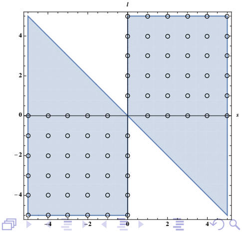
- $\mathcal{A}_m(\rho, v)$ is a basic example of a mock Jacobi forms. It can be written as a theta series of the form

$$\vartheta[\Phi](\rho, c - b\rho) = \rho_2^{-\lambda/2} \sum_{k \in \Lambda + b} \Phi(k\sqrt{2\rho_2})\, e^{\mathrm{i}\pi\rho Q(k) + 2\pi\mathrm{i}(c,k)}$$

with signature $(1,1)$ quadratic form $Q(k) = (k,k) = 2(ms^2 + \ell s)$ and kernel

$$\Phi(x) = \frac{1}{2}(C', x)\left(\mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)\right)$$

where $C = (0, -1)$, $C' = (1, -2m)$ are two negative vectors with $(C, C') < 0$. The sum converges since it is supported inside the positive cone, but $\vartheta[\Phi](\rho, v)$ is not modular.



Appell-Lerch sums and indefinite theta series III

- In his PhD thesis, Zwegers explained that $\mathcal{A}_m(\rho, v)$ is the holomorphic part of a real-analytic modular form, basically obtained by replacing

$$\mathrm{sgn}(C,k) \to \mathrm{Erf}\left(\sqrt{\frac{2\pi\rho_2}{-Q(C)}}(C,k)\right)$$

- In fact, it follows from *Vignéras 1978* that the modular completion is found by replacing the kernel $\Phi(x)$ by a smooth kernel $\widehat{\Phi}(x)$ with the same asymptotics as $|x| \to \infty$, and satisfying

$$\left[Q^{-1}(\partial_x) + 2\pi x \partial_x\right] \widehat{\Phi}(x) = 2\pi\lambda\,\widehat{\Phi}(x)$$

which ensures invariance under Poisson resummation.

Appell-Lerch sums and indefinite theta series IV

- In the case at hand one finds the modular completion

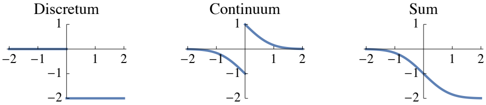
$$\widehat{\mathcal{A}}_m(\rho,v)=\sum_{s,\ell\in\mathbb{Z}} q^{ms^2+\ell s}e^{2\pi\mathrm{i}(2ms+\ell)v}$$
$$\times\left\{\frac{\ell}{2}\left[\mathrm{sgn}\left(s+\frac{\mathrm{Im}v}{\mathrm{Im}\rho}\right)+\mathrm{Erf}\left(\ell\sqrt{\frac{\pi\rho_2}{m}}\right)+\frac{1}{2\pi}\sqrt{\frac{m}{\rho_2}}e^{-\frac{\pi\ell^2\rho_2}{m}}\right]\right\}$$

- The difference $\widehat{\mathcal{A}}_m(\rho,v)-\mathcal{A}_m(\rho,v)$ is proportional to a non-holomorphic Eichler integral $\int_{-\bar\rho}^{\mathrm{i}\infty}\frac{\vartheta_{m,\ell}(u)\,\mathrm{d}u}{(\rho+u)^{3/2}}$. Hence $\mathcal{A}_m(\rho,v)$ transforms under $SL(2,\mathbb{Z})$ with a non-homogeneous term of the form $\int_{-d/c}^{\mathrm{i}\infty}\frac{\vartheta_{m,\ell}(u)\,\mathrm{d}u}{(\rho+u)^{3/2}}$.

Physics of the modular completion I

- Physically, the difference $\mathrm{Erf}x - \mathrm{sgn}(x) = \mathrm{Erfc}(|x|)$ can be understood as the contribution from the continuum of scattering states in the spectrum of the Hamiltonian H

$$\mathrm{Tr}_{\mathrm{cont}}(-1)^F e^{-\beta H} = -\frac{2q\vartheta}{\pi} \int_{k=|\vartheta|}^{\infty} \frac{e^{-\frac{\beta k^2}{2M}} \mathrm{d}k}{k\sqrt{k^2-\vartheta^2}} = -q\,\mathrm{Erfc}\left(|\vartheta|\sqrt{\frac{\beta}{2M}}\right)$$



BP 2015, Murthy BP 2018

Physics of the modular completion II

- Since the sum of single-centered and two-centered contributions is an ordinary meromorphic Jacobi form, the single-centered piece itself must be a mock Jacobi form.

Dabholkar Murthy Zagier 2012

- In particular the detailed asymptotics of the number of micro-states associated to a single horizon deviates from the usual Hardy-Ramanujan-Cardy prediction.

Bringmann Manschot 2010; Bringmann Nazaroglu 2018

- It would be very interesting to understand the origin of these corrections from the macroscopic side, i.e. from quantum gravity in $AdS_2 \times T_2 \times K_3 \times T^2$.

Beyond mock modular forms: $\mathcal{N} = 2$ dyons

- In the remainder, I will briefly discuss that happens for BPS black holes in string vacua with $\mathcal{N} = 2$ supersymmetry, such as type II strings compactified on a genuine Calabi-Yau 3-fold.

Manschot 2009, . . . , Alexandrov Banerjee Manschot BP 2016-18

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Manschot 2009, ..., Alexandrov Banerjee Manschot BP 2016-18

- Unlike the case with $\mathcal{N} \geq 4$ SUSY, there can now be black hole bound states involving an arbitrary number of constituents. The positions of the centers are constrained to satisfy

$$\forall i, \quad \sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i(z) \,, \qquad \sum_{i=1}^{n} c_i = 0$$

Denef 2000

where $\gamma_{ij} \equiv \langle \gamma_i, \gamma_j \rangle$ is the Dirac-Schwinger-Zwanziger product, and c_i are some real constants depending on the moduli z.

Beyond mock modular forms: $\mathcal{N} = 2$ dyons

- The space of solutions modulo translations is a $2n - 2$ dimensional symplectic space $(\mathcal{M}_n, \omega_n)$, with ω_n the restriction of the full symplectic form to the BPS sector. The number of configurational states, or Coulomb index $g(\{\gamma_i, c_i\})$ is the Dirac index of $(\mathcal{M}_n, \omega_n)$.

de Boer El Showk Messamah Van den Bleeken 2008

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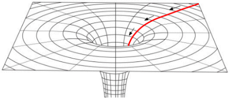
- In the vicinity of each center, the solution is indistinguishable from a spherically symmetric BPS black hole, in particular the moduli z asymptote to the attractor point $z_*(\gamma_i)$. *Ferrara Kallosh Strominger 1995*



Beyond mock modular forms: $\mathcal{N} = 2$ dyons

- Taking into account statistics, the contribution of all bound states is

$$\bar{\Omega}(\gamma, z) = \sum_{n\geq 1} \sum_{\gamma=\gamma_1+\dots+\gamma_n} \frac{g(\{\gamma_i, c_i\})}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\gamma_i)$$

where $|\mathrm{Aut}(\{\gamma_i\})|$ is a Boltzmann symmetry factor and $\bar{\Omega}(\gamma)$ is the attractor index. More precisely: $\bar{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2} \Omega(\frac{\gamma}{d}, z_*(\gamma))$

Manschot BP Sen 2011

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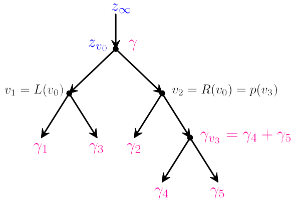
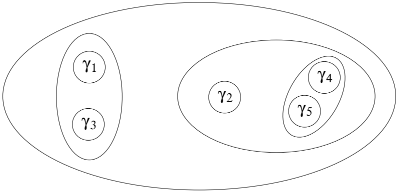
  Manschot BP Sen 2011

- One way to compute the Coulomb index $g(\{\gamma_i, c_i\})$ is to localize with respect to rotations around a fixed axis: fixed points are collinear multi-black hole configurations, which can be enumerated. *The supersymmetric n-body problem is solvable !*

  Manschot BP Sen 2011-13

- Another way is to use the attractor flow conjecture, which postulates that every bound state can be decomposed as a hierarchical sequence of two-center bound states:

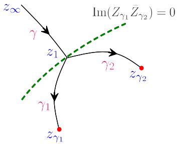




Denef '00; Denef Green Raugas '01; Denef Moore'07

Beyond mock modular forms: $\mathcal{N} = 2$ dyons

- Along each edge, the moduli flow as in a spherically symmetric black hole, until they hit the wall of marginal stability for the decay $\gamma_v \to \gamma_{L(v)} + \gamma_{R(v)}$:



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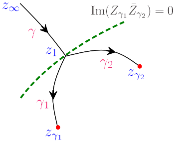

- Effectively, the stability parameters $c_{v,i}$ at each vertex follow the discrete attractor flow *(Alexandrov BP '18)*

$$c_{v,i} = c_{p(v),i} - \frac{\langle\gamma_v,\gamma_i\rangle}{\langle\gamma_v,\gamma_{L(v)}\rangle} \sum_{j=1}^{n} m^j_{L(v)} c_{p(v),j}$$

where m^i_v are the components of $\gamma_v = \sum_{i=1}^n m^i_v \gamma_i$. This ensures $\sum m^i_{L(v)} c_{v,i} = \sum m^i_{R(v)} c_{v,i} = 0$ for each of the two partitions.

Beyond mock modular forms: $\mathcal{N} = 2$ dyons

- The contribution of the flow tree T to the index $g(\{\gamma_i, c_i\})$ is then

$$\frac{(-1)^{n-1}}{2^{n-1}} \prod_{v \in V_T} \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \left[\mathrm{sgn}\left(\sum_i m^i_{L(v)} c_{v,i} \right) + \mathrm{sgn} \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \right]$$

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- Skipping over details, it is not hard to imagine that the modularity of generating functions of the attractor indices $\bar{\Omega}(\alpha)$ boils down to studying indefinite theta series $\vartheta[\Phi_r]$ with kernel

$$\Phi_r(x) = \frac{1}{2^r} \prod_{i=1}^{r} (C''_i, x) \, \left[\mathrm{sgn}(C_i, x) - \mathrm{sgn}(C'_i, x) \right]$$

where x is summed over a signature $(n - r - r)$ lattice Λ.

Alexandrov Banerjee Manschot BP 2016-17, Alexandrov BP 2018

Indefinite theta series and generalized error functions

- The convergence of such indefinite theta series requires certain incidence conditions on the negative vectors C_i, C_i' which are not so easy to state, but are guaranteed by physics.

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- The convergence of such indefinite theta series requires certain incidence conditions on the negative vectors C_i, C_i' which are not so easy to state, but are guaranteed by physics.
- To characterize the modular anomaly of $\vartheta[\Phi_r]$, it suffices to construct its modular completion $\vartheta[\widehat{\Phi}_r]$, which amounts to finding a smooth solution $\widehat{\Phi}_r(x)$ of Vignéras' equation which asymptotes to $\Phi(x)$ as $|x| \to \infty$:

$$\left[Q^{-1}(\partial_x) + 2\pi x \partial_x \right] \widehat{\Phi}_r(x) = 2\pi\lambda \widehat{\Phi}_r(x)$$

with $\lambda = \deg(\Phi_r)$.

Indefinite theta series and generalized error functions

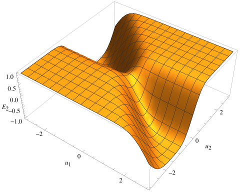
- Expanding out the product, and ignoring the prefactor $\prod_{i=1}^{r}(C_i'', x)$, it suffices to find solutions which asymptote to $\prod_{i=1}^{r} \mathrm{sgn}(C_i, x)$. Fortunately, convolution with a Gaussian does the job:

$$E_r(C_1, \dots, C_r; x) = \int_{\langle C_1,\dots,C_r \rangle} \mathrm{d}^r y \, e^{\pi Q(y-x_+)} \prod_{i=1}^{r} \mathrm{sgn}(C_i, y)$$

where x_+ is the orthogonal projection on the plane $\langle C_1, \dots, C_r \rangle$.

For $r = 1$, this reduces to the standard error function $E_1(C, x) = \mathrm{Erf}(\sqrt{\pi}(C, x)/\sqrt{-Q(C)})$.

For $r = 2$:



Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016; Zagier Zwegers ca. 2005

Indefinite theta series and generalized error functions

- Acting with the shadow operator $\rho_2^2 \partial_{\bar\rho}$ produces another theta series with kernel

$$x\partial_x \, E_r(C_1,\dots,C_r;x) = e^{\frac{\pi(C_1,x)^2}{Q(C_1)}} \, E_{r-1}(C_{2\perp 1},\dots,C_{r\perp 1};x) + \text{cycl.}$$

where $C_{i\perp j} = C_i - \frac{(C_i,C_j)}{Q(C_j)} C_j$

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- The holomorphic theta series $\vartheta[\Phi_r]$ may be viewed as mock modular form of depth r, with Ramanujan's mock theta functions corresponding to $r=1$.

Indefinite theta series and generalized error functions

- *Kudla Funke (2016-17)* have given a nice conceptual understanding of our completed indefinite theta series:

$$\vartheta[\widehat{\Phi}_r] = \int_{\mathcal{C}(\{C_i, C_i'\})} \vartheta_{KM}$$

where $\vartheta_{KM} \in H^r(G_{n-r,r})$ is the Kudla-Milsson theta form on $G_{n-r,r}$, the Grassmannian of negative r-planes inside $\mathbb{R}^n$, and $\mathcal{C}(\{C_i, C_i'\})$ is the image of the map

$$\begin{array}{rcl} [0,1]^r & \to & G_{n-r,r} \\ (s_1, \dots, s_r) & \mapsto & Span[s_1 C_1 + (1-s_1) C_1', \dots, s_r C_r + (1-s_r) C_r'] \end{array}$$

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- The shadow of $\vartheta[\widehat{\Phi}_r]$ is a sum of theta series associated to the faces of the r-cube.

Mock modularity of Donaldson-Thomas invariants

- By considering a certain protected coupling in the low energy effective action of type II strings compactified on a CY 3-fold $\mathcal{X}$, and enforcing invariance under S-duality, we predict that generating functions $h_p(\tau)$ of generalized Donaldson-Thomas invariants $\Omega_*(p,q)$ supported on a fixed divisor class $p \in H_4(\mathcal{X})$ admit a non-holomorphic modular completion of the form

$$\widehat{h}_p(\tau) = h_p(\tau) + \sum_{n=2}^{\infty} \sum_{p=\sum_{i=1}^n p_i} \vartheta_{p_1,\dots,p_n}(\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

  where $\vartheta_{p_1,\dots,p_n}(\tau,\bar\tau)$ is an indefinite theta series of signature $(n-1)(b_2-1,1)$. Can this be used to compute $h_p(\tau)$?

  Alexandrov BP 2018

- A very similar structure arises for higher rank Vafa-Witten invariants of 4-manifolds with $b_2 = 1$.

  Manschot 2017; Alexandrov BP Manschot, in progress

Conclusion

Thank you for your attention !