Protected couplings and BPS black holes

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in memoriam Ioannis Bakas

based on arXiv:1608.01660 and work in progress
with Guillaume Bossard and Charles Cosnier-Horeau

Precision counting of BPS black holes I

- Since Strominger and Vafa's seminal 1995 work, a lot of work has gone into performing precision counting of BPS black hole micro-states in various string vacua with extended SUSY, and detailed comparison with macroscopic supergravity predictions.
- For string vacua with 16 or 32 supercharges, exact degeneracies are given by Fourier coefficients of (classical, or Jacobi, or Siegel) modular forms, giving access to their large charge behavior, and enabling comparison with the Bekenstein-Hawking formula and its refinements.

Precision counting of BPS black holes II

- An important complication in $\mathcal{N} \leq 4$ string vacua in $D = 4$ is that multi-centered black hole solutions exist, and correspondingly, the spectrum of BPS states is subject to wall-crossing. Microstates of single centered black holes are counted by mock modular forms, which affects the growth of Fourier coefficients.

  Dabholkar Murthy Zagier 2012

- In string vacua with 8 supersymmetries, such as Calabi-Yau vacua, precision counting is much more difficult, as it involves detailed properties of the internal manifold (Gromov-Witten invariants, generalized Donaldson-Thomas invariants, etc), and complicated structure of walls of marginal stability.

  Maldacena Strominger Witten 1998; Denef 2000; Denef Moore 2007

Counting black holes via protected couplings I

- For several years, I have advocated to approach the problem of precision counting of BPS states in $D+1$-dimensional string vacua by considering protected couplings in the low energy effective action in D dimensions after compactifying on a circle of radius R.

Gunaydin Neitzke BP Waldron 2005

- Indeed, finite energy stationary solutions in dimension $D+1$ produce finite action solutions in D Euclidean dimensions. States breaking k supercharges lead to instantons with $2k$ fermionic zero-modes, contributing to BPS saturated couplings, i.e. vertices with more than $2k$ fermions (or k derivatives) in the LEEA.
- The simplest example of this phenomenon are 't Hooft-Polyakov monopoles in $D=4$, which induce a scalar potential in 3D QED with compact $U(1)$, explaining confinement *[Polyakov 1977]*.

Counting black holes via protected couplings II

- Couplings in the LEEA in dimension D are functions $f^{(D)}(R, z^a, \phi^I)$ of the radius R, of the moduli z^a in dimension $D+1$, and of the holonomies ϕ^I of the $D+1$-dimensional gauge fields along the circle:

$$\mathcal{M}_D = \mathbb{R}^+ \times \mathcal{M}_{D+1} \times \mathcal{T}$$

- Any coupling has a Fourier expansion w.r.t $\mathcal{T}$,

$$f^{(D)}(R, z^a, \varphi^I) = \sum_{Q \in \Lambda^+} \mathcal{F}_Q(R, z^a)\, e^{2\pi \mathrm{i} \langle Q, \phi \rangle} + \mathrm{cc}$$

- For BPS saturated couplings, and for Q primitive, $\mathcal{F}_Q(R, z^a)$ is expected to receive contributions from BPS states of charge Q in dimension $D+1$, exponentially suppressed as $R \to \infty$ and weighted by a suitable BPS index $\Omega_k(Q)$, or helicity supertrace,

$$\mathcal{F}_Q(R, z^a) = \Omega_k(Q)\, \mathcal{K}_Q(R, z^a)\,, \quad \mathcal{K}_Q(R, z^a) \sim e^{-2\pi R \mathcal{M}(Q)}$$

Counting black holes via protected couplings III

- If Q is not primitive, i.e. $Q = \sum_{i=1}^{n} Q_i$ with $Q_i \in \Lambda^+$, $n > 1$, $\Omega_k(Q)$ may depend on z^a, and there are also contributions from multi-particle states of charge Q_i which ensure that $\mathcal{F}_Q(R, z^a)$ is smooth across walls of marginal stability.
- In contrast, the constant term $\mathcal{F}_0(R, z^a)$ typically grow as a power of R as $R \to \infty$, and matches terms in the LEEA in dimension $D + 1$.
- Thus, $f^{(D)}(R, z^a, \varphi^I)$ plays the rôle of a thermodynamical black hole partition function at temperature $T = 1/R$, chemical potentials φ^I, for fixed values $z^a \in \mathcal{M}_{D+1}$ of the moduli at spatial infinity.

Counting black holes via protected couplings IV

- For $D+1=4$, the moduli space $\mathcal{M}_3$ also includes the NUT potential σ, dual to the KK gauge field, and valued in a circle bundle over $\mathcal{T}$. The Fourier expansion includes non-Abelian Fourier coefficients

$$f^{(3)}(R, z^a, \varphi^I, \sigma) = \sum_{Q\in\Lambda} \mathcal{F}_Q(R, z^a)\, e^{2\pi \mathrm{i}\langle Q,\phi\rangle} + \sum_{k\neq 0} \mathcal{F}_k(R, z^a, \phi^I) e^{\mathrm{i}\pi k\sigma}$$

  where $\mathcal{F}_k(R, z^a, \phi^I)$ is a section of a line bundle $\mathcal{L}^k$ over $\mathcal{T}$. It receives contributions from Taub-NUT instantons of charge k, suppressed as $e^{-\pi R^2/\ell_P^2}$ as $R\to\infty$.
- In that case, the black hole partition function is the constant term of $f^{(3)}(R, z^a, \varphi^I, \sigma)$ with respect to σ.

Counting black holes via protected couplings V

- For vacua with $\mathcal{N} \geq 4$ supersymmetries, the moduli space is a symmetric space $\mathcal{M}_D = G_D/K_D$, exact at tree-level, and $f^{(D)}$ is an automorphic function under the U-duality group, an arithmetic subgroup $G_D(\mathbb{Z}) \subset G_D$.

Hull Townsend 1994; Witten 1995

- BPS indices in dimension $D + 1$ thus arise as Fourier coefficients $\mathcal{F}_Q$ of an automorphic form under $G_D(\mathbb{Z})$. They are automatically invariant under the U-duality group $G_{D+1}(\mathbb{Z})$ in dimension $D + 1$, while $G_D(\mathbb{Z})$ plays the role of a spectrum generating symmetry.

Breitenlohner Mason Gibbons 1988; Gunaydin Neitzke BP Waldron 2005

Counting black holes via protected couplings VI

- In very recent work with G. Bossard we have shown that $\mathcal{R}^4$ and $\nabla^4\mathcal{R}^4$ couplings in $\mathcal{N}=8$ string vacua correctly reproduce the helicity supertraces Ω_8 and Ω_{12}, which count 1/2-BPS and 1/4-BPS small black holes.

Bossard BP, arXiv:1610.06693

- In the remainder of this talk, I will discuss protected couplings in $D=3$ string vacua with 16 supercharges, and demonstrate their relation to BPS indices in dimension $D=4$.

Protected couplings in $\mathcal{N} = 4$ string vacua I

- in $D = 4$ string vacua with 16 supercharges, the moduli space is

$$\mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(r-6,6)}{O(r-6) \times O(6)}$$

where $r \leq 28$. The highest rank is attained in Het$/T^6$ or its dual type II$/K3 \times T^2$. A large set of CHL models with reduced rank can be obtained by freely acting orbifolds. The $SL(2)/U(1)$ factor corresponds to the heterotic axiodilaton $S = a + \mathrm{i}/g_4^2$.

Chaudhury Hockney Lykken 1995

- These 4D models are believed to be invariant under $G_4(\mathbb{Z})$, an arithmetic subgroup of $SL(2) \times O(r-6,6)$ preserving the charge lattice $\Lambda_e \oplus \Lambda_m$.

Font Ibanez Lüst Quevedo 1990; Sen 1994

Protected couplings in $\mathcal{N}=4$ string vacua II

- Degeneracies of 1/4-BPS dyons are given by Fourier coefficients of a meromorphic Siegel modular form of weight $-k=\frac{8-r}{2}$:

$$\Omega_6(Q,P,z^a)=(-1)^{Q\cdot P}\int_{\mathcal{C}} \mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}v\,\frac{e^{\mathrm{i}\pi(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi_k(\rho,\sigma,v)}$$

where $\mathcal{C}$ is a suitable contour, depending on $z^a\in\mathcal{M}_4$.

Dijkgraaf Verlinde Verlinde 1996; David Jatkar Sen 2005-06; Cheng Verlinde 2007

- Across walls of marginal stability, $\Omega_6(Q,P,z^a)$ jumps due to poles of $1/\Phi_k$ on the separating divisor $v=0$ (and its images), corresponding to bound states of two 1/2-BPS dyons.

Protected couplings in $\mathcal{N} = 4$ string vacua III

- For $r = 28$, i.e. heterotic on T^6 or type II string on $K3 \times T^2$, Φ_{10} is the weight 10 Igusa cusp form under $Sp(4, \mathbb{Z})$, and $f_1 = f_2 = 1/\Delta$.
- Invariance under $G_4(\mathbb{Z}) = SL(2, \mathbb{Z}) \times O(\Lambda_e)$ is manifest, thanks to $SL(2, \mathbb{Z}) \subset Sp(4, \mathbb{Z})$, but the physical origin of the $Sp(4, \mathbb{Z})$ symmetry is obscure.
- Gaiotto and Dabholkar proposed that 1/4-BPS dyons can be interpreted as heterotic strings wrapped on a genus-two Riemann surface Σ_2, or M5-branes wrapped on $K3 \times \Sigma_2$, but it was not clear why higher genera are not allowed.

Gaiotto 2005; Dabholkar Gaiotto 2006

Protected couplings in $\mathcal{N}=4$ string vacua IV

- After compactification on a circle, the moduli space extends to

$$\mathcal{M}_3 = \frac{O(r-4,8)}{O(r-4)\times O(8)} \supset \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \mathbb{R}^{2r+1} \\ \mathbb{R}^+_{1/g_3^2} \times \frac{O(r-5,7)}{O(r-5)\times O(7)} \times \mathbb{R}^{r+2} \end{cases}$$

  and the U-duality group enhances to an arithmetic subgroup $G_3(\mathbb{Z}) \subset O(r-4,8)$, containing both $G_4(\mathbb{Z})$ and the T-duality group in $D=3$.

  Markus Schwarz 1983, Sen 1994

- For $r=28$, $G_3(\mathbb{Z}) = O(\tilde{\Lambda})$ with $\tilde{\Lambda} = \Lambda_e \oplus \Lambda_{2,2}$. For CHL orbifolds, noting that $\Lambda_m = \Lambda_e^* = \Lambda_e[N]$, it is natural to propose that $G_3(\mathbb{Z}) = O(\tilde{\Lambda})$ with $\tilde{\Lambda} = \Lambda_e \oplus \Lambda_{1,1} \oplus \Lambda_{1,1}[N]$.

  Cosnier-Horeau, Bossard, Pioline, to appear

- The 4-derivative and 6-derivative couplings in the LEEA

$$F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d + G_{ab,cd}(\Phi)\nabla(\nabla\Phi^a\nabla\Phi^b)\,\nabla(\nabla\Phi^c\nabla\Phi^d)$$

  are expected to satisfy non-renormalization theorems and get contributions from 1/2-BPS and 1/4-BPS instantons, respectively.

- Indeed, they satisfy supersymmetric Ward identities

$$\mathcal{D}^2_{ef}\,F_{abcd} = c_1\,\delta_{ef}\,F_{abcd} + c_2\,\delta_{e)(a}\,F_{bcd)(f} + c_3\,\delta_{(ab}\,F_{cd)ef}\;,$$

$$\begin{aligned}\mathcal{D}^2_{ef}G_{ab,cd} =& c_4\delta_{ef}G_{ab,cd} + c_5\left[\delta_{e)(a}G_{b)(f,cd} + \delta_{e)(c}G_{d)(f,ab}\right]\\ &+c_6\left[\delta_{ab}\,G_{ef,cd} + \delta_{cd}\,G_{ef,ab} - 2\delta_{a)(c}\,G_{ef,d)(b}\right]\\ &+c_7\left[F_{abk(e}\,F_{f)cd}{}^{k} - F_{c)ka(e}\,F_{f)b(d}{}^{k}\right],\end{aligned}$$

$$\mathcal{D}_{[e}{}^{[\hat e}\mathcal{D}_{f]}{}^{\hat f]}F_{abcd} = 0\;,\quad \mathcal{D}_{[e}{}^{[\hat e}\mathcal{D}_{f}{}^{\hat f}\mathcal{D}_{g]}{}^{\hat g]}G_{ab,cd} = 0\;.$$

Bossard, Cosnier-Horeau, BP, 2016

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua I

- The coupling $(\nabla\Phi)^4$ is a 3D version of the F^4 and $\mathcal{R}^2$ couplings which were analyzed in the past. The F^4 coupling is one-loop exact on the heterotic side in $D \geq 4$, while the $\mathcal{R}^2$ coupling is one-loop exact on the type II side in $D=4$.

Lerche Nilsson Schellekens Warner 1988; Harvey Moore 1996

- Requiring invariance under U-duality, it is natural to conjecture that the exact coefficient of the $(\nabla\Phi)^4$ in $D=3$ is *[Obers BP 2000]*

$$F^{(r-4,8)}_{abcd} = \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\partial^4}{(2\pi\mathrm{i})^4 \partial y^a \partial y^b \partial y^c \partial y^d}\Big|_{y=0} \frac{\Gamma_{r-4,8}}{\Delta_{k+2}}$$

where Δ_{k+2} is a weight $k+2$ modular form, and $\Gamma_{r-4,8}$ is the Narain partition function of the lattice $\tilde{\Lambda}$,

$$\Gamma_{r-4,8} = \rho_2^4 \sum_{Q\in\tilde{\Lambda}} e^{\mathrm{i}\pi Q_L^2 \rho - \mathrm{i}\pi Q_R^2 \bar{\rho} + 2\pi\mathrm{i} Q_L\cdot y + \frac{\pi (y\cdot y)}{2\rho_2}}$$

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua II

Obers BP, 2000

- This Ansatz satisfies the Ward identities and has the correct perturbative expansion on the heterotic side:

$$F^{(r-4,8)}_{\alpha\beta\gamma\delta} = \frac{c_0}{16\pi g_3^4}\,\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{F^{(r-5,7)}_{\alpha\beta\gamma\delta}}{g_3^2} + 4\sum_{\ell=1}^{3}\sum_{Q\in\Lambda_{r-5,7}}{}' P^{(\ell)}_{\alpha\beta\gamma\delta}$$
$$\times\bar c(Q)\, g_3^{2\ell-9}\,|\sqrt{2}Q_R|^{\ell-\frac{7}{2}}\,K_{\ell-\frac{7}{2}}\left(\frac{2\pi}{g_3^2}|\sqrt{2}Q_R|\right)\,e^{-2\pi i a^I Q_I}$$

exhibiting the tree-level and one-loop contribution and an infinite sum of NS5-brane and KK5-brane instantons. Here $P^{(\ell)}_{\alpha\beta\gamma\delta}$ are degree $6-2k$ polynomials in Q_L, and

$$\bar c(Q) = \sum_{d|Q} d\, c\!\left(-\tfrac{|Q|^2}{2d}\right)\,, \qquad \frac{1}{\Delta_k} = \sum_{N\geq 1} c(N) q^N\,.$$

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua III

- In the large radius limit, one finds instead

$$F^{(r-4,8)}_{\alpha\beta\gamma\delta} = R^2 \left(f_{\mathcal{R}^2}(S)\, \delta_{(\alpha\beta}\delta_{\gamma\delta)} + F^{(r-6,6)}_{\alpha\beta\gamma\delta} \right) + 4 \sum_{\ell=1}^{3} R^{5-\ell} \sum'_{Q'\in\Lambda_{r-6,6}} \sum'_{j,p}$$
$$c\left(-\tfrac{|Q'|^2}{2}\right) P^{(\ell)}_{\alpha\beta\gamma\delta} K_{\ell-\frac{7}{2}} \left(\frac{2\pi R|pS+j|}{\sqrt{S_2}} |\sqrt{2} Q'_R| \right) e^{-2\pi i (ja^1+pa^2)\cdot Q'} + \mathcal{O}(e^{-R^2})$$

exhibiting the exact $\mathcal{R}^2$ and F^4 couplings in $D=4$, along with $\mathcal{O}(e^{-R})$ terms from 1/2-BPS dyons with charge $(Q,P)=(j,p)Q'$, with measure

$$\mu(Q,P) = \sum_{d|(Q,P)} c\left(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2d^2}\right) \overset{primitive}{=} \Omega_4(Q,P)\,.$$

The non-Abelian $\mathcal{O}(e^{-R^2})$ terms come from Taub-NUT instantons.

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua IV

- These expansions are easily obtained using the unfolding trick: for $\Gamma_{p,q} \to \Gamma_{p-1,q-1}$, the sum in $\Gamma_{1,1} = R\sum_{(\tilde m,n)} e^{-\pi R^2|\tilde m - n\rho|^2/\rho_2}$ can be restricted to $n=0$ provided it is integrated on the strip $\mathcal{S} = \mathcal{H}_1/\mathbb{Z}$.
- For $\Gamma_{p,q} \to \Gamma_{p-2,q-2}$, the sum over (dual momenta,windings) in $\Gamma_{2,2}$ has three orbits:

$$\begin{pmatrix} \tilde m_1 & n_1 \\ \tilde m_2 & n_2 \end{pmatrix}_{/SL(2,\mathbb{Z})} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ; \underset{(j,p)\neq(0,0)}{\begin{pmatrix} j & 0 \\ p & 0 \end{pmatrix}} ; \underset{0\leq j<k,\, p\neq 0}{\begin{pmatrix} j & k \\ p & 0 \end{pmatrix}} \right\}$$

integrated over $\mathcal{F}_1, \mathcal{H}_1/\mathbb{Z}, 2\mathcal{H}_1$, respectively. These produce the powerlike, Abelian and non-Abelian Fourier coefficients, respectively.

Dixon Kaplunovsky Louis 1990; Harvey Moore 1995

- Similarly, it is natural to conjecture that the exact coefficient of the $\nabla^2(\nabla\Phi)^4$ in $D=3$ is given by

$$G^{(r-4,8)}_{ab,cd} = \int_{\mathcal{F}_2(N)} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\frac{1}{2}(\varepsilon_{il}\varepsilon_{jm}+\varepsilon_{im}\varepsilon_{jl})\partial^4}{(2\pi\mathrm{i})^4 \partial y^a_i \partial y^b_j \partial y^c_l \partial y^d_m}\Big|_{y=0} \frac{\Gamma_{r-4,8,2}}{\Phi_k}$$

  where Φ_k is a cusp form of weight k under a suitable level N subgroup of the Siegel modular group, and $\Gamma_{24,8,2}$ is the genus-two Narain partition function of the lattice $\tilde\Lambda$,

$$\Gamma_{24,8,2} = |\Omega_2|^4 \sum_{Q^i\in\tilde\Lambda^{\otimes 2}} e^{\mathrm{i}\pi(Q^i_L\Omega_{ij}Q^j_L - Q^i_R\bar\Omega_{ij}Q^j_R + 2Q^i_L y_i) + \frac{\pi}{2} y^a_i \Omega_2^{-1\,ij} y_{ja}}$$

- Again, this ansatz satisfies the correct Ward identity, including the quadratic source term originating from the pole of $1/\Phi_k$ in the separating degeneration.

- At weak heterotic coupling, it reproduces the known perturbative contributions,

$$G^{(r-4,8)}_{\alpha\beta,\gamma\delta} = \frac{G^{(r-5,7)}_{\alpha\beta,\gamma\delta}}{g_3^4} - \frac{\delta_{\alpha\beta} G^{(r-5,7)}_{\gamma\delta} + \delta_{\gamma\delta} G^{(r-5,7)}_{\alpha\beta} - 2\delta_{\gamma)(\alpha} G^{(r-5,7)}_{\beta)(\delta}}{12 g_3^6} - \frac{1}{2\pi g_3^8}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta}\right] + \mathcal{O}(e^{-1/g_3^2})$$

exhibiting the two-loop *[d'Hoker Phong 2005]*, one-loop *[Sakai Tanii 1987]*,

$$G^{(r-5,7)}_{\alpha\beta} = \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\partial^2}{(2\pi\mathrm{i})^2 \partial y^\alpha \partial y^\beta}\Big|_{y=0} \frac{\widehat{E}_2\, \Gamma_{r-5,7}}{\Delta_k},$$

tree-level, and NS5/KK5-brane instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua III

- In the large radius limit, we find instead

$$
\begin{aligned}
G^{(r-4,8)}_{\alpha\beta,\gamma\delta} =& R^4 \Big[G^{(r-6,6)}_{\alpha\beta,\gamma\delta} - f_{\mathcal{R}^2}(S) \left(\delta_{\alpha\beta} G^{(r-6,6)}_{\gamma\delta} + \delta_{\gamma\delta} G^{(r-6,6)}_{\alpha\beta} - 2\delta_{\gamma)(\alpha} G^{(r-6,6)}_{\beta)(\delta} \right) \\
& + g(S)(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta} \Big] + G^{(1)}_{\alpha\beta,\gamma\delta} + \boxed{G^{(2)}_{\alpha\beta,\gamma\delta}} + G^{(\mathrm{KKM})}_{\alpha\beta,\gamma\delta}
\end{aligned}
$$

  exhibiting the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D=4$. The term proportional to $g(S)$ is required by the Ward identity, but hard to compute.
- The Abelian Fourier coefficients $G^{(1)}$ and $G^{(2)}$ are both $\mathcal{O}(e^{-R})$, and come from 1/2-BPS and 1/4-BPS states in $D=4$.
- The non-Abelian Fourier coefficient $G^{(\mathrm{KKM})}$ is $\mathcal{O}(e^{-R^2})$ and comes from Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua IV

- These expansions follow again from the unfolding trick: for $\Gamma_{p,q} \to \Gamma_{p-1,q-1}$, the sum over non-zero (dual momenta,windings) unfolds onto $\mathbb{R}^+ \times \mathcal{F}_1 \times T^{2+1}$.
- For $\Gamma_{p,q} \to \Gamma_{p-2,q-2}$, the sum has 4 orbits:

$$\left\{ 0, \underset{(m_1,m_2)\neq(0,0)}{\begin{pmatrix} 0 & m_1 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}}, \underset{0\leq j<p,\, k\neq 0}{\begin{pmatrix} k & 0 & 0 & 0 \\ j & p & 0 & 0 \end{pmatrix}}, \underset{0\leq j_1,j_2,j_3<p,\, k\neq 0}{\begin{pmatrix} j_1 & j_2 & j_3 & p \\ 0 & k & 0 & 0 \end{pmatrix}} \right\}$$

integrated over $\mathbb{R}^+ \times \mathcal{F}_1 \times T^{2+1}$, $\mathcal{P}_2 \times T^3$, $\mathbb{R}^+ \times \mathcal{F}_1 \times \mathbb{R}^3$. These produce the powerlike, 1/2-BPS Abelian, 1/4-BPS Abelian and non-Abelian Fourier coefficients, respectively.

- We focus on the Abelian rank-two orbit $G^{(2)}$, integrated over $\mathcal{P}_2 \times T^3$. The integral over Ω_1 in T^3 extracts the Fourier coefficient

$$C\left[\begin{array}{cc} -\frac{1}{2}|Q_1|^2 & -Q_1\cdot Q_2 \\ -Q_1\cdot Q_2 & -\frac{1}{2}|Q_2|^2 \end{array};\Omega_2\right] = \int_{[0,1]^3} \mathrm{d}\rho_1 \mathrm{d}\sigma_1 \mathrm{d}v_1 \, \frac{e^{\mathrm{i}\pi(\rho Q_1^2+\sigma Q_2^2+2vQ_1\cdot Q_2)}}{\Phi_k(\rho,\sigma,v)}$$

which is a locally constant function of Ω_2.

- For large R, the integral is dominated by a saddle point at

$$\Omega_2^\star = \frac{R}{\mathcal{M}(Q,P)} A^\intercal \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R\wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2 \end{pmatrix}\right] A\,.$$

where $\binom{Q}{P} = A\binom{Q_1}{Q_2}$, $A = \begin{pmatrix} k & 0 \\ j & p \end{pmatrix}$, $|P_R\wedge Q_R| = \sqrt{(P_R^2)(Q_R^2)-(P_R\cdot Q_R)^2}$.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua VI

- Approximating $C[\,;\Omega_2]$ by its saddle point value, we find

$$G^{(2)}_{\alpha\beta,\gamma\delta} = 2R^7 \sum_{Q,P\in\Lambda'_{r-6,6}} \sum_{\ell=1}^{3} P^{(\ell)}_{\alpha\beta,\gamma\delta}\, e^{-2\pi\mathrm{i}(a^1 Q+a^2 P)}$$

$$\times \frac{\mu(Q,P)}{|2P_R\wedge Q_R|^{\frac{4-\ell}{2}}}\, B_{\frac12,\frac{4-\ell}{2}}\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}|Q_R|^2 & P_R\cdot Q_R\\ P_R\cdot Q_R & |P_R|^2\end{pmatrix}\begin{pmatrix}1 & 0\\ S_1 & S_2\end{pmatrix}\right]$$

where

$$\mu(Q,P) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda^{\otimes 2}_{r-6,6}}} |A|\, C\left[A^{-1}\begin{pmatrix}-\frac12|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac12|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right]$$

and B is a kind of matrix-variate modified Bessel function,

$$B_{\nu,\delta}(Z) = \int_0^\infty \frac{\mathrm{d}t}{t^{1+s}}\, e^{-\pi t-\frac{\pi\mathrm{Tr}Z}{t}}\, K_\delta\left(\frac{2\pi}{t}\sqrt{|Z|}\right)$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua VII

- In the limit $R\to\infty$, using $B_{\nu,\delta}(Z)\sim e^{-2\pi\sqrt{\mathrm{Tr}Z+2\sqrt{|Z|}}}$, one finds that the contributions are suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$.
- In 'primitive' cases where only $A=1$ contributes, $\mu(Q,P)$ agrees with the helicity supertrace $\Omega_6(Q,P;z^a)$, evaluated with the correct contour prescription. It also refines earlier proposals for counting dyons with 'non-primitive' charges.

Cheng Verlinde 2007; Banerjee Sen Srivastava 2008; Dabholkar Gomes Murthy 2008

- There are exponentially suppressed corrections due to the discrepancy between $C[\ ;\Omega_2]$ and its saddle point value, which are necessary to match the F^2_{abcd} term in the Ward identity.
- The detailed analysis of $(\nabla\Phi)^4$ and $\nabla^2(\nabla\Phi)^4$ couplings in CHL models is subtle and in progress...

Conclusion - Outlook I

- $\nabla^2(\nabla\Phi)^4$ couplings in $D=3, \mathcal{N}=4$ string vacua nicely incorporate degeneracies of 1/4-BPS dyons in $D=4$, and explain their hidden modular invariance. They give a precise implementation of the idea that 1/4-BPS dyons are (U-duals of) heterotic strings wrapped on genus-two Riemann surfaces.

Gaiotto 2005; Dabholkar Gaiotto 2006

- A similar story presumably relates $\nabla^6\mathcal{R}^4$ couplings in $\mathcal{N}=8$ string vacua and degeneracies of 1/8-BPS dyons. In $D=6$, the exact $f_{\nabla^6\mathcal{R}^4}$ is given by a genus-two theta lifting of the Kawazumi-Zhang invariant, which is itself a genus-one theta lifting of the partition function of 1/8-BPS dyons... however generalisation to $D<6$ is unclear.

BP 2015; Bossard Kleinschmidt 2015

Conclusion - Outlook II

- In $D = 4, \mathcal{N} = 2$ string vacua, the appropriate coupling capturing degeneracies of $1/2$-BPS black holes is the metric on the vector-multiplet moduli space $\mathcal{M}_V$ after compactification on a circle, which is dual to the hypermultiplet moduli space $\mathcal{M}_H$. Hopefully, progress on understanding $\mathcal{M}_V$ and $\mathcal{M}_H$ will lead to new ways of computing Donaldson-Thomas invariants...

  Alexandrov BP Vandoren 2008, Alexandrov Banerjee Manschot BP 2016

- From a mathematical viewpoint, higher-genus theta liftings are an interesting source of new automorphic objects beyond Eisenstein series.