Protected couplings and BPS black holes

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with Guillaume Bossard and Charles Cosnier-Horeau

Precision counting of BPS black holes I

- Since Strominger and Vafa's seminal 1995 work, a lot of work has gone into performing precision counting of BPS black hole micro-states in various string vacua with extended SUSY, and detailed comparison with macroscopic supergravity predictions.
- For string vacua with 16 or 32 supercharges, exact degeneracies are given by Fourier coefficients of (classical, or Jacobi, or Siegel) modular forms, giving access to their large charge behavior, and enabling detailed comparison with the Bekenstein-Hawking formula and its refinements.

Precision counting of BPS black holes II

- An important complication in $\mathcal{N} \leq 4$ string vacua in $D = 4$ is that multi-centered black hole solutions exist, and correspondingly, the spectrum of BPS states is subject to wall-crossing. Microstates of single-centered black holes are counted by mock modular forms, which affects the growth of Fourier coefficients.

Dabholkar Murthy Zagier 2012

- In string vacua with 8 supersymmetries, such as Calabi-Yau vacua, precision counting is much more difficult, as it involves detailed properties of the internal manifold (Gromov-Witten invariants, generalized Donaldson-Thomas invariants, etc), and complicated structure of walls of marginal stability.

Maldacena Strominger Witten 1998; Denef 2000; Denef Moore 2007

Counting black holes via protected couplings I

- For several years, I have advocated to approach the problem of precision counting of BPS states in $D+1$-dimensional string vacua by considering protected couplings in the low energy effective action in D dimensions after compactifying on a circle of radius R.

Gunaydin Neitzke BP Waldron 2005

- Indeed, a finite energy stationary solution in dimension $D+1$ produce a finite action solution in D Euclidean dimensions. States breaking $2k$ supercharges lead to instantons with $2k$ fermionic zero-modes, contributing to vertices with more than $2k$ fermions (or k derivatives) in the LEEA.
- The simplest example of this phenomenon are 't Hooft-Polyakov monopoles in $D=4$, which induce a scalar potential in 3D QED with compact $U(1)$, explaining confinement *[Polyakov 1977]*.

Counting black holes via protected couplings II

- Couplings in the LEEA in dimension D are functions $f^{(D)}(R, z^a, \phi^I)$ of the radius R, moduli z^a in dimension $D+1$, and holonomies ϕ^I of the $D+1$-dimensional gauge fields along the circle:

$$\mathcal{M}_D = \mathbb{R}^+ \times \mathcal{M}_{D+1} \times \mathcal{T}$$

- Any coupling has a Fourier expansion w.r.t the torus $\mathcal{T}$,

$$f^{(D)}(R, z^a, \varphi^I) = \sum_{Q\in\Lambda^+} \mathcal{F}_Q(R, z^a)\, e^{2\pi\mathrm{i}\langle Q,\phi\rangle} + \mathrm{cc}$$

where Λ_+ is a positive cone in the charge lattice Λ.

Counting black holes via protected couplings III

- For BPS saturated couplings, and for Q primitive, $\mathcal{F}_Q(R, z^a)$ is expected to receive contributions from BPS states of charge Q in dimension $D+1$, exponentially suppressed as $R \to \infty$ and weighted by a suitable BPS index $\Omega_k(Q)$, or helicity supertrace,

$$\mathcal{F}_Q(R, z^a) = \Omega_k(Q)\, \mathcal{K}_Q(R, z^a) \;, \quad \mathcal{K}_Q(R, z^a) \sim e^{-2\pi R \mathcal{M}(Q)}$$

- If Q is not primitive, i.e. $Q = \sum_{i=1}^{n} Q_i$ with $Q_i \in \Lambda^+, n > 1$, $\Omega_k(Q)$ may jump as a function of z^a, but contributions from multi-particle states of charge Q_i ensure that $\mathcal{F}_Q(R, z^a)$ is smooth across walls of marginal stability.

Alexandrov Moore Neitzke BP 2013

Counting black holes via protected couplings IV

- Thus, $f^{(D)}(R, z^a, \varphi^I)$ plays the rôle of a thermodynamical black hole partition function at temperature $T = 1/R$, chemical potentials φ^I, for fixed values $z^a \in \mathcal{M}_{D+1}$ of the moduli at spatial infinity.
- In contrast, the constant term $\mathcal{F}_0(R, z^a)$ typically grows as a power of R as $R \to \infty$, and matches terms in the LEEA in dimension $D+1$.

Counting black holes via protected couplings V

- For $D+1=4$, the moduli space $\mathcal{M}_3$ also includes the NUT potential σ, dual to the KK gauge field, and valued in a circle bundle over $\mathcal{T}$. The Fourier expansion includes non-Abelian Fourier coefficients

$$f^{(3)}(R, z^a, \varphi^I, \sigma) = \sum_{Q \in \Lambda} \mathcal{F}_Q(R, z^a)\, e^{2\pi \mathrm{i} \langle Q, \phi \rangle} + \sum_{k \neq 0} \mathcal{F}_k(R, z^a, \phi^I) e^{\mathrm{i}\pi k \sigma}$$

  where $\mathcal{F}_k(R, z^a, \phi^I)$ is a section of a circle bundle $\mathcal{L}^k$ over $\mathcal{T}$. It receives contributions from Taub-NUT instantons of charge k, suppressed as $e^{-\pi R^2/\ell_P^2}$ as $R \to \infty$.

- In that case, the black hole partition function is the constant term of $f^{(3)}(R, z^a, \varphi^I, \sigma)$ with respect to σ.

Counting black holes via protected couplings VI

- For vacua with $\mathcal{N} \geq 4$ supersymmetries, the moduli space is a symmetric space $\mathcal{M}_D = G_D/K_D$, exact at tree-level, and $f^{(D)}$ is an automorphic function under the U-duality group, an arithmetic subgroup $G_D(\mathbb{Z}) \subset G_D$.

Hull Townsend 1994; Witten 1995

- BPS indices in dimension $D+1$ thus arise as Fourier coefficients $\mathcal{F}_Q$ of an automorphic form under $G_D(\mathbb{Z})$. They are automatically invariant under the U-duality group $G_{D+1}(\mathbb{Z})$ in dimension $D+1$, while $G_D(\mathbb{Z})$ plays the role of a spectrum generating symmetry.

Breitenlohner Mason Gibbons 1988; Gunaydin Neitzke BP Waldron 2005

- In the remainder of this talk, I will discuss protected couplings in $D=3$ string vacua with 32 and 16 supercharges, and demonstrate their relationship to BPS indices in $D=4$.

Outline

1. From BPS indices to BPS-saturated couplings
2. Protected couplings in $\mathcal{N} = 8$ string vacua
3. Protected couplings in $\mathcal{N} = 4$ string vacua

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Protected couplings in $\mathcal{N} = 8$ string vacua I

- In type II string compactified on a torus T^d, the LEEA is expected to be invariant under the U-duality group $E_{d+1}(\mathbb{Z})$, generated by the combination of the T-duality group $SO(d, d, \mathbb{Z})$ and the group $SL(d + 1, \mathbb{Z})$ of global diffeomorphisms of the M-theory torus:

$$\begin{array}{ccccccccc} & & & & \circ_2 & & & & \\ & & & & | & & & & \\ \circ_1 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - \cdots - & \circ_{d+1} \end{array}$$

- Supersymmetric Ward identities and known perturbative contributions uniquely determine the $\mathcal{R}^4$ and $\nabla^4\mathcal{R}^4$ couplings to be given by Eisenstein series,

$$f^{(D)}_{\mathcal{R}^4} = 2\zeta(3)\, \mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{3}{2}\Lambda_1}\ , \quad f^{(D)}_{\nabla^4\mathcal{R}^4} = \zeta(5)\, \mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{5}{2}\Lambda_1}$$

Green Gutperle 97; BP Kiritsis 98; Obers BP 99; Green Vanhove Russo 2008-11

Protected couplings in $\mathcal{N}=8$ string vacua II

- In the limit where a circle decompactifies, one expects these couplings to receive contributions from 1/2-BPS and 1/4-BPS states in dimension $D+1$, respectively, weighted by the helicity supertraces $\Omega_8(Q)$ and $\Omega_{12}(Q)$, where

$$\Omega_n(Q) = \frac{(-1)^{n/2}}{n!} \mathrm{Tr}'_Q (-1)^{2J_3} (2J_3)^n$$

- In the perturbative spectrum of type II strings compactified on T^d,
  1. 1/2-BPS states arise from ground states $N=\bar{N}=m_i w^i = 0$ and have $\Omega_8 = 1$;
  2. 1/4-BPS states arise from $N = m_i w^i > 0, \bar{N} = 0$, or $\bar{N} = -m_i w^i > 0$, $N=0$. There is an exponentially large number of $1/4$-BPS states, but large cancellations in the helicity supertrace: $\Omega_{12} = \sigma_3(N)$.

Protected couplings in $\mathcal{N}=8$ string vacua III

- More generally, for general primitive charges Q,

$$\Omega_8(Q) = \begin{cases} 1 & (Q \times Q = 0) \\ 0 & (Q \times Q \neq 0) \end{cases}$$

$$\Omega_{12}(Q) = \begin{cases} \sigma_3[\mathrm{gcd}(Q \times Q)] & (I'_4(Q) = 0, Q \times Q \neq 0) \\ 0 & (I'_4(Q) \neq 0) \end{cases}$$

  where $Q \times Q$ is the Jordan quadratic product and $I'_4(Q)$ is the Freudenthal cubic product, which coincides with the gradient of the quartic invariant $I_4(Q)$.

- Does this match the large radius expansion of $f^{(D)}_{\mathcal{R}^4}$ and $f^{(D)}_{\nabla^4\mathcal{R}^4}$?

Protected couplings in $\mathcal{N}=8$ string vacua IV

- The Fourier expansion of general Eisenstein series with respect to non-minimal parabolic subgroups is not known in general. Here it can be obtained by analyzing the decompactification limit of the perturbative terms, and covariantizing the result under U-duality.
- For the $\mathcal{R}^4$ coupling, the result is known from previous work:

$$f^{(D)}_{\mathcal{R}^4} = R^{\frac{6}{8-d}} \left(f^{(D+1)}_{\mathcal{R}^4} + 4\pi\,\xi(d-2) R^{d-3} \right.$$
$$\left. + 4\pi R^{\frac{d-3}{2}} \sum_{Q\times Q=0}' \sigma_{d-3}(Q)\, \frac{K_{\frac{d-3}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d-3}{2}}}\, e^{2\pi i\langle Q,a\rangle} \right)$$

exhibiting the expected contributions of 1/2-BPS states with mass $\mathcal{M}=|Z(Q)|$, $\Omega_8=1$ (for Q primitive). NB: $\xi(s)\equiv\pi^{-\frac{s}{2}}\Gamma(s/2)\,\zeta(s)$.

Kazhdan BP Waldron 2001; Kazhdan Polishchuk 2002; BP 2010; Green Russo Vanhove 2010

- For the $\nabla^4\mathcal{R}^4$ coupling,

$$f^{(D)}_{\nabla^4\mathcal{R}^4} = R^{\frac{10}{8-d}} \left(f^{(D+1)}_{\nabla^4\mathcal{R}^4} + 2\xi(d-4)R^{d-5}\, f^{(D+1)}_{\mathcal{R}^4} + 8\pi\xi(4)\,\xi(d+2)R^{d+1} \right.$$

$$+16\pi\xi(4)R^{\frac{d+1}{2}} \sum_{Q\times Q=0}' \sigma_{d+1}(Q)\, \frac{K_{\frac{d+1}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d+1}{2}}} e^{2\pi\mathrm{i}\langle Q,a\rangle}$$

$$+16\pi\xi(3)R^{\frac{d-5}{2}} \sum_{Q\times Q=0}' \frac{\sigma_{d-5}(Q)\,\mathcal{E}^{E_{d-1}(\mathbb{Z})}_{\frac{3}{2}\Lambda_1}(g_Q)}{(\gcd Q)^{\frac{6}{d-10}}} \frac{K_{\frac{d-5}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d-5}{2}+\frac{6}{10-d}}} e^{2\pi\mathrm{i}\langle Q,a\rangle}$$

$$\left. +16\pi R^{d-2} \sum_{\substack{I_4'(Q)=0,\\ Q\times Q\neq 0}} \sum_{n|Q} n^{d+1}\sigma_3\!\left(\frac{Q\times Q}{n^2}\right) \frac{B_{\frac{d-2}{2},\frac{3}{2}}(R^2|Z(Q)|^2, R^2\sqrt{\Delta(Q)})}{\Delta(Q)^{\frac{3}{4}}} e^{2\pi\mathrm{i}\langle Q,a\rangle} \right)$$

$$+\ldots$$

Protected couplings in $\mathcal{N} = 8$ string vacua VI

- The result exhibits contributions from 1/2-BPS states with $Q \times Q = 0$, mass $\mathcal{M}_{1/2} = |Z(Q)|$, weighted by divisor sums and by Eisenstein series for the stabilizer $E_{d-1}(\mathbb{Z}) \subset E_d(\mathbb{Z})$ of the charge vector Q;
- In addition, there are contributions from 1/4-BPS states with $Q \times Q \neq 0$, $I'_4(Q) = 0$, with mass $\mathcal{M}_{1/4} = \sqrt{|Z(Q)|^2 + 2\sqrt{\Delta(Q)}}$, as follows from the asymptotics of the "double Bessel function"

$$B_{s,\nu}(x,y) = \int_0^\infty \frac{\mathrm{d}t}{t^{1+s}} e^{-\pi t - \frac{\pi x}{t}} K_\nu(2\pi y/t) \sim \frac{e^{-2\pi\sqrt{x+2y}}}{2y^{1/2}(x+2y)^{s/2}}$$

- For $D \geq 4$, the previous result is complete. For $D = 3$, it misses non-Abelian Fourier coefficients. For $s \neq \frac{5}{2}$, additional Fourier coefficients with $I'_4(Q) \neq 0$ will appear.

Protected couplings in $\mathcal{N}=8$ string vacua VII

- Similarly, one may expect that the $\nabla^6\mathcal{R}^4$ coupling in dimension D exhibits contributions from $1/8$-states, weighted by the helicity supertrace $\Omega_{14}(Q)$.
- For Q primitive, $\Omega_{14}(Q)$ is given by a Fourier coefficient of a weak Jacobi form,

$$\Omega_{14}(Q) = c(I_4(Q))\ , \quad -\frac{\theta_1^2(z,\tau)}{\eta^6} = \sum_{N,\ell} c(4N-\ell^2) q^N y^\ell$$

At large Q, $\Omega_{14}(Q) \sim e^{\pi\sqrt{I_4(Q)}}$, in agreement with the Bekenstein-Hawking entropy formula.

Maldacena Moore Strominger 1999; Shih Strominger Yin 2005; BP 2005

Protected couplings in $\mathcal{N} = 8$ string vacua VIII

- It is clear however that $f^{(D)}_{\nabla^6\mathcal{R}^4}$ is not simply an Eisenstein series, indeed SUSY Ward identities require

$$\left(\Delta_{E_{d+1}} - \frac{6(D-6)(14-D)}{D-2} \right) f^{(D)}_{\nabla^6\mathcal{R}^4} = -[f^{(D)}_{\mathcal{R}^4}]^2$$

  up to additional linear source terms in dimension $D = 4, 5, 6$ where the local and non-local parts of the 1PI effective action mix.

  Green Vanhove 2005, Green Russo Vanhove 2010; BP 2015; Bossard Verschinin 2015

- In order to satisfy this equation, $f^{(D)}_{\nabla^6\mathcal{R}^4}$ should also get contributions from pairs of 1/2-BPS and 1/2-$\overline{BPS}$ black holes !

Protected couplings in $\mathcal{N} = 8$ string vacua IX

- Based on these Ward identities and the known perturbative contributions up to 3 loops, one can show that the exact $\nabla^6\mathcal{R}^4$ coupling for $D = 6$ should be given by

$$f^{(6)}_{\nabla^6\mathcal{R}^4} = \pi\,\mathrm{R.N.}\int_{\mathcal{F}_2} \mathrm{d}\mu_2\,\Gamma_{5,5,2}\,\varphi_{KZ} + \frac{8}{189}\mathcal{E}^{SO(5,5,\mathbb{Z})}_{4\Lambda_5}$$

  where $\Gamma_{5,5,2}$ is the partition function of the 'non-perturbative' Narain lattice at genus-two, and φ_{KZ} is the Kawazumi-Zhang invariant, which appears naturally in the integrand of the two-loop $\nabla^6\mathcal{R}^4$ coupling.

  d'Hoker Green 2014; BP 2015

- Another non-perturbative completion has been proposed, in principle valid for any D, which involves a two-loop amplitude in exceptional field theory. It is not clear yet how to reconcile the two proposals.

  Bossard Kleinschmidt 2015

Protected couplings in $\mathcal{N} = 8$ string vacua X

- The large radius expansion is in principle computable from the Fourier expansion of φ_{KZ}, which follows from the theta lift representation

$$\varphi_{KZ}(\Omega) = -\frac{1}{2}\int_{\mathcal{F}_1} \mathrm{d}\mu_1 \left[\Gamma^{(0)}_{3,2,1}\, D_\tau h_0 + \Gamma^{(1)}_{3,2,1}\, D_\tau h_1\right]$$

where h_i are the coefficients of the theta series decomposition

$$\frac{\theta_1^2(z,\tau)}{\eta^6} = h_0(\tau)\,\theta_3(2z,2\tau) + h_1(\tau)\,\theta_2(2z,2\tau)\;.$$

BP 2015

- Remarkably, $\varphi_{KZ}(\Omega)$ knows about degeneracies of $1/8$-BPS black holes ! More work is needed to make this connection precise.

Outline

Protected couplings in $\mathcal{N}=4$ string vacua I

- in $D=4$ string vacua with 16 supercharges, the moduli space is

$$\mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(r-6,6)}{O(r-6)\times O(6)}$$

The highest rank $r=28$ is attained in Het$/T^6$ or its dual type II$/K3\times T^2$. A large set of CHL models with reduced rank can be obtained as freely acting $\mathbb{Z}_N$ orbifolds. The $SL(2)/U(1)$ factor corresponds to the heterotic axiodilaton $S=a+\mathrm{i}/g_4^2$.

Chaudhury Hockney Lykken 1995

- These 4D models are believed to be invariant under $G_4(\mathbb{Z})$, an arithmetic subgroup of $SL(2)\times O(r-6,6)$ preserving the charge lattice $\Lambda_e \oplus \Lambda_m$ (extended by "Fricke S-duality", which mixes the two factors)

Font Ibanez Lüst Quevedo 1990; Sen 1994; Persson Volpato 2015

Protected couplings in $\mathcal{N}=4$ string vacua II

- Degeneracies of 1/4-BPS dyons are given by Fourier coefficients of a meromorphic Siegel modular form of weight $-k=\frac{8-r}{2}$:

$$\Omega_6(Q,P,z^a)=(-1)^{Q\cdot P}\int_{\mathcal{C}}\mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}v\,\frac{e^{\mathrm{i}\pi(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi_k(\rho,\sigma,v)}$$

where $\mathcal{C}$ is a suitable contour, depending on $z^a\in\mathcal{M}_4$.

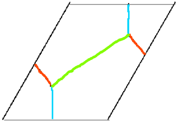
Dijkgraaf Verlinde Verlinde 1996; David Jatkar Sen 2005-06; Cheng Verlinde 2007

- Across walls of marginal stability, $\Omega_6(Q,P,z^a)$ jumps due to poles of $1/\Phi_k$ on the separating divisor $v=0$ (and its images), corresponding to bound states of two 1/2-BPS dyons.
- In particular, the BPS indices $\Omega_4(Q,0)$ and $\Omega_4(0,P)$ for purely electric or magnetic states are Fourier coefficients of $1/f_1(\rho)$ and $1/f_2(\sigma)$, such that $\Phi_k(\rho,\sigma,v)\sim v^2 f_1(\rho)f_2(\sigma)$ as $v\to 0$.

Protected couplings in $\mathcal{N}=4$ string vacua III

- For $r=28$, i.e. heterotic on T^6 or type II string on $K3\times T^2$, Φ_{10} is the weight 10 Igusa cusp form under $Sp(4,\mathbb{Z})$, and $f_1=f_2=1/\Delta$.
- Invariance under $G_4(\mathbb{Z})=SL(2,\mathbb{Z})\times O(\Lambda_e)$ is manifest, thanks to $SL(2,\mathbb{Z})\subset Sp(4,\mathbb{Z})$, but the physical origin of the $Sp(4,\mathbb{Z})$ symmetry is obscure.
- Gaiotto and Dabholkar proposed that 1/4-BPS dyons can be interpreted as heterotic strings wrapped on a genus-two Riemann surface Σ_2, or M5-branes wrapped on $K3\times\Sigma_2$, but left many questions unanswered (e.g. why higher genera are not allowed).

Gaiotto 2005; Dabholkar Gaiotto 2006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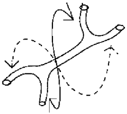

- After compactification on a circle, the moduli space extends to

$$\mathcal{M}_3 = \frac{O(r-4,8)}{O(r-4)\times O(8)} \supset \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \mathbb{R}^{2r+1} \\ \mathbb{R}^+_{1/g_3^2} \times \frac{O(r-5,7)}{O(r-5)\times O(7)} \times \mathbb{R}^{r+2} \end{cases}$$

  and the U-duality group enhances to an arithmetic subgroup $G_3(\mathbb{Z}) \subset O(r-4,8)$, containing both $G_4(\mathbb{Z})$ and the T-duality group in $D=3$.

  Markus Schwarz 1983, Sen 1994

- For $r=28$, $G_3(\mathbb{Z})$ is the automorphism group of the non-perturbative Narain lattice $\tilde{\Lambda} = \Lambda_e \oplus \Lambda_{2,2}$. For CHL orbifolds, noting that $\Lambda_e = \Lambda_m^* = \Lambda_m[N]$, it is natural to propose that $\tilde{\Lambda} = \Lambda_m \oplus \Lambda_{1,1} \oplus \Lambda_{1,1}[N]$.

  Cosnier-Horeau, Bossard, BP, 2017

Protected couplings in $\mathcal{N}=4$ string vacua V

- The 4-derivative and 6-derivative couplings in the LEEA

$$F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d + G_{ab,cd}(\Phi)\nabla(\nabla\Phi^a\nabla\Phi^b)\,\nabla(\nabla\Phi^c\nabla\Phi^d)$$

  are expected to satisfy non-renormalization theorems and get contributions from 1/2-BPS and 1/4-BPS instantons, respectively.
- Indeed, they satisfy supersymmetric Ward identities

$$\mathcal{D}^2_{ef}\,F_{abcd} = c_1\,\delta_{ef}\,F_{abcd} + c_2\,\delta_{e)(a}\,F_{bcd)(f} + c_3\,\delta_{(ab}\,F_{cd)ef}\ ,$$

$$\begin{aligned}\mathcal{D}^2_{ef}G_{ab,cd} =& c_4\delta_{ef}G_{ab,cd} + c_5\left[\delta_{e)(a}G_{b)(f,cd} + \delta_{e)(c}G_{d)(f,ab}\right]\\ &+c_6\left[\delta_{ab}\,G_{ef,cd} + \delta_{cd}\,G_{ef,ab} - 2\delta_{a)(c}\,G_{ef,d)(b}\right]\\ &+c_7\left[F_{abk(e}\,F_{f)cd}{}^{k} - F_{c)ka(e}\,F_{f)b(d}{}^{k}\right],\end{aligned}$$

$$\mathcal{D}_{[e}{}^{[\hat e}\mathcal{D}_{f]}{}^{\hat f]}F_{abcd} = 0\ ,\quad \mathcal{D}_{[e}{}^{[\hat e}\mathcal{D}_{f}{}^{\hat f}\mathcal{D}_{g]}{}^{\hat g]}G_{ab,cd} = 0\ .$$

Bossard, Cosnier-Horeau, BP, 2016

- The coupling $(\nabla\Phi)^4$ is a 3D version of the F^4 and $\mathcal{R}^2$ couplings which were analyzed in the past. The F^4 coupling is one-loop exact on the heterotic side in $D\geq 4$, while the $\mathcal{R}^2$ coupling is one-loop exact on the type II side in $D=4$.

Lerche Nilsson Schellekens Warner 1988; Harvey Moore 1996

- Requiring invariance under U-duality, it is natural to conjecture that the exact coefficient of the $(\nabla\Phi)^4$ in $D=3$ is *[Obers BP 2000]*

$$\boxed{F^{(r-4,8)}_{abcd} = \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\mathit{\Gamma}_{r-4,8,1}[P_{abcd}]}{\Delta_{k+2}}}$$

where Δ_{k+2} is a weight $k+2$ modular form, and $\Gamma_{r-4,8}$ is the Narain partition function of the lattice $\tilde{\Lambda}$ with polynomial insertion,

$$\mathit{\Gamma}_{r-4,8,1}[P_{abcd}] = \rho_2^4 \sum_{Q\in\tilde{\Lambda}} P_{abcd}(Q_L)\, e^{\mathrm{i}\pi Q_L^2\rho - \mathrm{i}\pi Q_R^2\bar{\rho}}$$

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua II

- This Ansatz satisfies the Ward identities and has the correct perturbative expansion on the heterotic side:

$$F^{(r-4,8)}_{\alpha\beta\gamma\delta} = \frac{c_0}{16\pi g_3^4}\,\delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{F^{(r-5,7)}_{\alpha\beta\gamma\delta}}{g_3^2} + 4\sum_{\ell=1}^{3}\sum'_{Q\in\Lambda_{r-5,7}} P^{(\ell)}_{\alpha\beta\gamma\delta}$$
$$\times\bar{c}(Q)\, g_3^{2\ell-9}\,|\sqrt{2}Q_R|^{\ell-\frac{7}{2}}\, K_{\ell-\frac{7}{2}}\left(\tfrac{2\pi}{g_3^2}|\sqrt{2}Q_R|\right)\, e^{-2\pi i a^I Q_I}$$

exhibiting the tree-level and one-loop contribution and an infinite sum of NS5-brane and KK5-brane instantons. Here $P^{(\ell)}_{\alpha\beta\gamma\delta}$ are degree $6-2k$ polynomials in Q_L, and

$$\bar{c}(Q) = \sum_{d|Q} d\, c\!\left(-\tfrac{|Q|^2}{2d}\right)\,, \quad \frac{1}{\Delta_k} = \sum_{N\geq 1} c(N) q^N\,.$$

- In the large radius limit, one finds instead

$$F^{(r-4,8)}_{\alpha\beta\gamma\delta} = R^2 \left(f_{\mathcal{R}^2}(S)\, \delta_{(\alpha\beta}\delta_{\gamma\delta)} + F^{(r-6,6)}_{\alpha\beta\gamma\delta} \right) + 4 \sum_{\ell=1}^{3} R^{5-\ell} \sum'_{\widetilde{Q}\in\Lambda_{r-6,6}} \sum'_{j,p}$$
$$c\left(-\tfrac{|\widetilde{Q}|^2}{2}\right) P^{(\ell)}_{\alpha\beta\gamma\delta} K_{\ell-\frac{7}{2}} \left(\frac{2\pi R|pS+j|}{\sqrt{S_2}} |\sqrt{2}\widetilde{Q}_R| \right) e^{-2\pi\mathrm{i}(ja^1+pa^2)\cdot\widetilde{Q}} + \mathcal{O}(e^{-R^2})$$

exhibiting the exact $\mathcal{R}^2$ and F^4 couplings in $D=4$, along with $\mathcal{O}(e^{-R})$ terms from 1/2-BPS dyons with charge $(Q,P)=(j,p)\widetilde{Q}$, with measure

$$\mu(Q,P) = \sum_{d|(Q,P)} c\left(-\tfrac{\gcd(Q^2,P^2,Q\cdot P)}{2d^2}\right) \overset{primitive}{=} \Omega_4(Q,P)\ .$$

The non-Abelian $\mathcal{O}(e^{-R^2})$ terms come from Taub-NUT instantons.

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua IV

- These expansions are easily obtained using the unfolding trick: for $\Gamma_{p,q} \to \Gamma_{p-1,q-1}$, the sum in $\Gamma_{1,1} = R\sum_{(\tilde m,n)} e^{-\pi R^2|\tilde m - n\rho|^2/\rho_2}$ can be restricted to $n=0$ provided it is integrated on the strip $\mathcal{S}=\mathcal{H}_1/\mathbb{Z}$.
- For $\Gamma_{p,q} \to \Gamma_{p-2,q-2}$, the sum over (dual momenta,windings) in $\Gamma_{2,2}$ has three orbits:

$$\begin{pmatrix} \tilde m_1 & n_1 \\ \tilde m_2 & n_2 \end{pmatrix}_{/SL(2,\mathbb{Z})} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ; \underset{(j,p)\neq(0,0)}{\begin{pmatrix} j & 0 \\ p & 0 \end{pmatrix}} ; \underset{0\leq j<k,\, p\neq 0}{\begin{pmatrix} j & k \\ p & 0 \end{pmatrix}} \right\}$$

integrated over $\mathcal{F}_1, \mathcal{H}_1/\mathbb{Z}, 2\mathcal{H}_1$, respectively. These produce the powerlike, Abelian and non-Abelian Fourier coefficients, respectively.

Dixon Kaplunovsky Louis 1990; Harvey Moore 1995

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua I

- Similarly, it is natural to conjecture that the exact coefficient of the $\nabla^2(\nabla\Phi)^4$ in $D=3$ is given by

$$\boxed{G^{(r-4,8)}_{ab,cd} = \int_{\mathcal{F}_2(N)} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\mathit{\Gamma}_{r-4,8,2}[R_{ab,cd}]}{\Phi_k}}$$

  where Φ_k is a cusp form of weight k under a suitable level N subgroup of the Siegel modular group, and $\mathit{\Gamma}_{24,8,2}$ is the genus-two partition function of the non pert. Narain lattice $\tilde{\Lambda}$,

$$\mathit{\Gamma}_{24,8,2}[R_{ab,cd}] = |\Omega_2|^4 \sum_{Q^i \in \tilde{\Lambda}^{\otimes 2}} R_{ab,cd}(Q_L)\, e^{\mathrm{i}\pi(Q^i_L \Omega_{ij} Q^j_L - Q^i_R \bar{\Omega}_{ij} Q^j_R)}$$

- Again, this ansatz satisfies the correct Ward identity, including the quadratic source term originating from the pole of $1/\Phi_k$ in the separating degeneration.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua II

- At weak heterotic coupling, it reproduces the known perturbative contributions,

$$G^{(r-4,8)}_{\alpha\beta,\gamma\delta} = \frac{G^{(r-5,7)}_{\alpha\beta,\gamma\delta}}{g_3^4} - \frac{\delta_{\alpha\beta}G^{(r-5,7)}_{\gamma\delta} + \delta_{\gamma\delta}G^{(r-5,7)}_{\alpha\beta} - 2\delta_{\gamma)(\alpha}G^{(r-5,7)}_{\beta)(\delta}}{12 g_3^6} - \frac{1}{2\pi g_3^8}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta}\right] + \mathcal{O}(e^{-1/g_3^2})$$

exhibiting the two-loop *[d'Hoker Phong 2005]*, one-loop *[Sakai Tanii 1987]*,

$$G^{(r-5,7)}_{ab} = \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\widehat{E}_2\,\Gamma_{r-5,7}[P_{ab}]}{\Delta_k}\,,$$

tree-level, and NS5/KK5-brane instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua III

- In the large radius limit, we find instead, schematically,

$$\begin{aligned} G^{(r-4,8)}_{\alpha\beta,\gamma\delta} =& R^4 \Big[G^{(r-6,6)}_{\alpha\beta,\gamma\delta} - f_{\mathcal{R}^2}(S) \left(\delta_{\alpha\beta} G^{(r-6,6)}_{\gamma\delta} + \delta_{\gamma\delta} G^{(r-6,6)}_{\alpha\beta} - 2\delta_{\gamma)(\alpha} G^{(r-6,6)}_{\beta)(\delta} \right) \\ & + [f_{\mathcal{R}^2}(S)]^2 (\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta} \Big] + G^{(1)}_{\alpha\beta,\gamma\delta} + \boxed{G^{(2)}_{\alpha\beta,\gamma\delta}} + G^{(\mathrm{KKM})}_{\alpha\beta,\gamma\delta} \end{aligned}$$

  exhibiting the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D=4$.
- The Abelian Fourier coefficients $G^{(1)}$ and $G^{(2)}$ are both $\mathcal{O}(e^{-R})$, and come from $1/2$-BPS and $1/4$-BPS states in $D=4$.
- The non-Abelian Fourier coefficient $G^{(\mathrm{KKM})}$ is $\mathcal{O}(e^{-R^2})$ and comes from Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua IV

- These expansions follow again from the unfolding trick: for $\Gamma_{p,q} \to \Gamma_{p-1,q-1}$, the sum over non-zero (dual momenta,windings) unfolds onto $\mathbb{R}^+ \times \mathcal{F}_1(N) \times T^{2+1}$.
- For $\Gamma_{p,q} \to \Gamma_{p-2,q-2}$, the sum has (for $N=1$) 4 orbits:

$$\left\{ 0, \underset{(m_1,m_2)\neq(0,0)}{\begin{pmatrix} 0 & m_1 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}}, \underset{0\le j<p,\, k\neq 0}{\begin{pmatrix} k & 0 & 0 & 0 \\ j & p & 0 & 0 \end{pmatrix}}, \underset{0\le j_1,j_2<p,\, k\neq 0}{\begin{pmatrix} j_1 & j_2 & 0 & p \\ 0 & k & 0 & 0 \end{pmatrix}} \right\}$$

  integrated over $\mathbb{R}^+ \times \mathcal{F}_1 \times T^{2+1}$, $\mathcal{P}_2 \times T^3$, $\mathcal{P}_2 \times \mathbb{R}^3$. These produce the powerlike, 1/2-BPS Abelian, 1/4-BPS Abelian and non-Abelian Fourier coefficients, respectively.
- Due to the pole in $1/\Phi_k$, the naive unfolding procedure misses a crucial contact term corresponding to $\nabla^2 F^4$ amplitude in $\mathcal{N}=4$ supergravity.

Bern Davies Dennen 2013

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua V

- We focus on the Abelian rank-two orbit $G^{(2)}$, integrated over $\mathcal{P}_2 \times T^3$. The integral over Ω_1 in T^3 extracts the Fourier coefficient

$$C\left[\begin{smallmatrix} -\frac{1}{2}|Q_1|^2 & -Q_1\cdot Q_2 \\ -Q_1\cdot Q_2 & -\frac{1}{2}|Q_2|^2 \end{smallmatrix} ;\Omega_2\right] = \int_{[0,1]^3} \mathrm{d}\rho_1 \mathrm{d}\sigma_1 \mathrm{d}v_1 \, \frac{e^{\mathrm{i}\pi(\rho Q_1^2+\sigma Q_2^2+2vQ_1\cdot Q_2)}}{\Phi_k(\rho,\sigma,v)}$$

  which is a locally constant function of Ω_2.

- For large R, the integral is dominated by a saddle point at

$$\Omega_2^\star = \frac{R}{\mathcal{M}(Q,P)} A^{\mathrm{T}} \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R\wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2 \end{pmatrix}\right] A\,.$$

  where $\left(\begin{smallmatrix} Q \\ P \end{smallmatrix}\right) = A\left(\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right)$, $A=\begin{pmatrix} k & 0 \\ j & p \end{pmatrix}$, $|P_R\wedge Q_R| = \sqrt{(P_R^2)(Q_R^2)-(P_R\cdot Q_R)^2}$.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua VI

- Approximating $C[\,;\Omega_2]$ by its saddle point value, we find

$$G^{(2)}_{\alpha\beta,\gamma\delta} = 2R^7 \sum_{Q,P\in\Lambda'_{r-6,6}} \sum_{\ell=1}^{3} P^{(\ell)}_{\alpha\beta,\gamma\delta} e^{-2\pi\mathrm{i}(a^1 Q + a^2 P)}$$
$$\times \frac{\mu(Q,P)}{|2P_R\wedge Q_R|^{\frac{4-\ell}{2}}} B_{\frac{1}{2},\frac{4-\ell}{2}}\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}|Q_R|^2 & P_R\cdot Q_R\\ P_R\cdot Q_R & |P_R|^2\end{pmatrix}\begin{pmatrix}1 & 0\\ S_1 & S_2\end{pmatrix}\right]$$

where

$$\mu(Q,P) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda^{\otimes 2}_{r-6,6}}} |A|\, C\left[A^{-1}\begin{pmatrix} -\frac{1}{2}|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac{1}{2}|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right]$$

and B is the same "double Bessel function" encountered in $\nabla^4\mathcal{R}^4$,

$$B_{\nu,\delta}(Z) = \int_0^\infty \frac{\mathrm{d}t}{t^{1+s}}\, e^{-\pi t - \frac{\pi \mathrm{Tr} Z}{t}} K_\delta\left(\frac{2\pi}{t}\sqrt{|Z|}\right)$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N}=4$ string vacua VII

- In the limit $R\to\infty$, using $B_{\nu,\delta}(Z)\sim e^{-2\pi\sqrt{\mathrm{Tr}Z+2\sqrt{|Z|}}}$, one finds that the contributions are suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$.
- In 'primitive' cases where only $A=1$ contributes, $\mu(Q,P)$ agrees with the helicity supertrace $\Omega_6(Q,P;z^a)$, evaluated with the correct contour prescription. It also refines earlier proposals for counting dyons with 'non-primitive' charges.

Cheng Verlinde 2007; Banerjee Sen Srivastava 2008; Dabholkar Gomes Murthy 2008

- There are exponentially suppressed corrections due to the discrepancy between $C[\,;\Omega_2]$ and its saddle point value. Presumably these terms give the instanton/anti-instanton effects sourced by the square of the $(\nabla\Phi)^4$ coupling in the Ward identity.

Conclusion - Outlook I

- $\nabla^2(\nabla\Phi)^4$ couplings in $D=3, \mathcal{N}=4$ string vacua nicely incorporate degeneracies of 1/4-BPS dyons in $D=4$, and explain their hidden modular invariance. They give a precise implementation of Gaiotto's idea that 1/4-BPS dyons are (U-duals of) heterotic strings wrapped on genus-two Riemann surfaces.
- A similar story presumably relates $\nabla^6\mathcal{R}^4$ couplings in $\mathcal{N}=8$ string vacua and degeneracies of 1/8-BPS dyons, but details remain to be worked out.

BP 2015; Bossard Kleinschmidt 2015

Conclusion - Outlook II

- In $D = 4, \mathcal{N} = 2$ string vacua, the appropriate coupling capturing degeneracies of $1/2$-BPS black holes is the metric on the vector-multiplet moduli space $\mathcal{M}_V$ after compactification on a circle. It is related by T-duality to the hypermultiplet moduli space $\mathcal{M}_H$. Hopefully, progress on understanding $\mathcal{M}_V$ and $\mathcal{M}_H$ will allow new precision tests of BPS black holes, and provide new ways of computing Donaldson-Thomas invariants...

  Alexandrov BP Vandoren 2008, Alexandrov Banerjee Manschot BP 2016

- From a mathematical viewpoint, higher-genus modular integrals are an interesting source of new automorphic objects beyond Eisenstein series, which satisfy Poisson-type equations with sources.