Exact BPS couplings and black hole counting

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Precision counting of BPS black holes I

- Since Strominger and Vafa's seminal 1995 work, a lot of work has gone into performing precision counting of BPS black hole micro-states in various string vacua with extended SUSY, and detailed comparison with macroscopic supergravity predictions.
- For string vacua with 16 or 32 supercharges, the exact BPS degeneracies are given by Fourier coefficients of (classical, or Jacobi, or Siegel) modular forms. This gives access to their large charge behavior, and enables detailed comparison with the Bekenstein-Hawking formula and and quantum corrections to it.

Precision counting of BPS black holes II

- In string vacua with 8 supercharges, such as type II strings on Calabi-Yau threefolds, precision counting is much more difficult, since generalized Donaldson-Thomas invariants depends on the details properties of the internal manifold. Constrains from modularity are not fully understood.

  Maldacena Strominger Witten 1998; Gaiotto Strominger Yin 2005; Cheng et al 2006; Denef Moore 2007; . . . ; Alexandrov Banerjee Manschot BP 2016-17

- An important complication in $\mathcal{N} \leq 4$ string vacua in $D = 4$ is that the spectrum of BPS states is subject to wall-crossing, due to the (dis)appearance of multi-centered black hole bound states.

  Denef 2000; Denef Moore 2007; Manschot BP Sen 2011

- Interesting challenges are a) to compute the exact BPS index $\Omega(\gamma, z)$ at an arbitrary point in moduli space, and b) determine what part comes from single centered black holes.

Black hole counting from BPS couplings I

- For some years, I have advocated to approach the problem of precision counting of BPS states in $D+1$-dimensional string vacua by considering protected couplings in the low energy effective action in D dimensions after compactifying on a circle of radius R.

Gunaydin Neitzke BP Waldron 2005

- Indeed, a finite energy stationary solution in dimension $D+1$ descends to a finite action solution in D Euclidean dimensions. A famous example are t Hooft-Polyakov monopoles in $D=4$, are responsible for confinement in 3D QED.

Polyakov 1977

Black hole counting from BPS couplings II

- In supersymmetric theories, states which break k out of $\mathcal{N}$ supercharges in dimension $D+1$ descend to instantons which carry k fermionic zero-modes. Hence they contribute to only to fermionic vertices in the low energy effective action in dimension D with at least k fermions (or bosonic vertices with $k/2$ derivatives)
- BPS couplings are vertices with $k < \mathcal{N}$ fermions, which only get corrections from instantons preserving some fraction of SUSY:

$\mathcal{N}$	$(\mathcal{N}-k)/\mathcal{N}$	$k/2$	BPS couplings
32	$1/2$	8	$\mathcal{R}^4$
32	$1/4$	12	$\nabla^4\mathcal{R}^4$
32	$1/8$	14	$\nabla^6\mathcal{R}^4$
16	$1/2$	4	$F^4, \mathcal{R}^2$
16	$1/4$	6	$\nabla^2 F^4, F^2\mathcal{R}^2$
8	$1/2$	2	$(\nabla\phi)^2$

Black hole counting from BPS couplings III

- The coefficients of these couplings are functions $f^{(D)}(R, z, \phi^I)$ of the radius R, moduli z in dimension $D+1$, and holonomies ϕ^I of the $D+1$-dimensional gauge fields along the circle:

$$\mathcal{M}_D = \mathbb{R}^+ \times \mathcal{M}_{D+1} \times \mathcal{T}$$

  When $D=3$, the torus is doubled and there also the NUT potential σ dual to the KK gauge field, parametrizing a circle bundle over $\mathcal{T}$.

- In the large radius limit $R \to \infty$, $f^{(D)}(R, z, \phi^I)$ is expected to behave schematically as

$$f^{(D)}(R, z, \varphi^I) = f_0(R, z) + \sum_{Q \in \Lambda} \Omega_k(Q, z)\, e^{-2\pi R \mathcal{M}(Q,z) + 2\pi \mathrm{i} \langle Q, \phi \rangle} + \dots$$

  where f_0 is a power-growing term, independent of φ, $\mathcal{M}(Q, z)$ is the BPS mass, and $\Omega_k(Q, z)$ is a suitable helicity supertrace counting BPS states with charge Q and k fermionic zero-modes.

Black hole counting from BPS couplings IV

- The power-growing term reproduces the same BPS coupling in dimension $D+1$, along with with lower order couplings in the derivative expansion due to threshold effects.
- The dots include subleading corrections to the exponential behavior, and possibly multi-instanton contributions which smoothen the jumps of $\Omega_k(Q,z)$ across walls of marginal stability.
- For $D=4$ contributions there are also from Taub-NUT instantons of order $\mathcal{O}(e^{-\pi k R^2+2\pi i k\sigma})$, which resolve the ambiguity of the divergent sum $\sum_Q e^{S_{BH}(Q)-R\mathcal{M}(Q)}$ [BP Vandoren (2009)]
- The main message is that $f^{(D)}(R,z,\varphi)$ provides a well-defined BPS black hole partition function at temperature $T=1/R$, chemical potentials φ^I, and fixed values $z\in\mathcal{M}_{D+1}$ of the moduli at spatial infinity. Fourier coefficients of $f^{(D)}$ encode BPS indices in dimension $D+1$.

Black hole counting from BPS couplings V

- For vacua with $\mathcal{N} \geq 4$ supersymmetries, the moduli space is a symmetric space $\mathcal{M}_D = G_D/K_D$, exact at tree-level. The low-energy effective action is expected to be invariant under the U-duality group, an arithmetic subgroup $G_D(\mathbb{Z}) \subset G_D$.

Hull Townsend 1994; Witten 1995

- BPS indices in dimension $D+1$ thus arise as Fourier coefficients of an automorphic form $f^{(D)}$ under $G_D(\mathbb{Z})$. They are invariant under the U-duality group $G_{D+1}(\mathbb{Z})$ in dimension $D+1$, acting linearly on the charge Q, but further constrained by invariance under the larger group $G_D(\mathbb{Z})$.
- In the remainder of this talk, I will focus on 1/4-BPS couplings in $D=3$ string vacua with 16 supercharges, and their relationship to 1/4-BPS black holes in $D=4$.

Four-dimensional string vacua with 16 supercharges I

- In heterotic string compactified on T^6, the moduli space is

$$\mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(22,6)}{O(22)\times O(6)}$$

  where the first factor is the heterotic axiodilaton $S = a + \mathrm{i}/g_4^2$, and the second are the Narain moduli. The same theory arises by compactifying type II on $K3 \times T^2$.
- A wider class of $\mathcal{N} = 4$ models with $r < 22$ multiplets can be obtained by freely acting orbifolds *[Chaudhury Hockney Lykken 1995]*, but for brevity we shall focus on the maximal rank case.
- These 4D models are believed to be invariant under $G_4(\mathbb{Z})$, an arithmetic subgroup of $SL(2) \times O(22,6)$ preserving the charge lattice $\Lambda_e \oplus \Lambda_m$. *[Font Ibanez Lüst Quevedo 1990; Sen 1994]*

Black hole counting and Siegel modular forms I

- Degeneracies of 1/4-BPS dyons are given by Fourier coefficients of a meromorphic Siegel modular form:

$$\Omega_6(Q,P;z) = (-1)^{Q\cdot P} \int_{\mathcal{C}} \mathrm{d}^3\Omega \frac{e^{\mathrm{i}\pi(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi_{10}(\Omega)}$$

  where $\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \in \mathcal{H}_2$, and Φ_{10} is the Igusa cusp form of weight 10 under $Sp(4,\mathbb{Z})$. *[Dijkgraaf Verlinde Verlinde 1996; David Jatkar Sen 2005-06]*

- The integration contour is chosen as $\mathcal{C} = [0,1]^3 + \mathrm{i}\Omega_2^*$ with

$$\Omega_2^\star = \Lambda \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R \wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2 \end{pmatrix} \right]$$

  with $\Lambda \gg 1$. This ensures that $\mathcal{C}$ crosses a zero of Φ_k whenever z crosses a wall of marginal stability. *[Cheng Verlinde 2007]*

Black hole counting and Siegel modular forms II

- By virtue of

$$\frac{1}{\Phi_{10}(\Omega)} \overset{v\to 0}{\sim} \frac{1}{v^2} \times \frac{1}{\Delta(\rho)} \times \frac{1}{\Delta(\sigma)}$$

where $1/\Delta = \sum_{N\geq -1} c(N)\, q^N$ is the generating function of the BPS indices $\Omega_4(Q,P)$ counting 1/2-BPS states, the jump in $\Omega_6(Q,P;z)$ matches the contribution of bound states of two 1/2-BPS dyons:

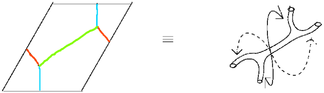
$$\Delta\Omega_6(Q,P) = \pm(P_1 Q_2 - P_2 Q_1)\,\Omega_4(Q_1,P_1)\,\Omega_4(Q_2,P_2)$$

where $P_1 \parallel Q_1, P_2 \parallel Q_2, (Q,P) = (Q_1,P_1) + (Q_2,P_2)$.

Denef Moore 2007

Black hole counting and Siegel modular forms III

- Invariance under $G_4(\mathbb{Z}) = SL(2,\mathbb{Z}) \times O(22,6)$ is manifest, thanks to $SL(2,\mathbb{Z}) \subset Sp(4,\mathbb{Z})$, but the physical origin of the $Sp(4,\mathbb{Z})$ symmetry and contour prescriptions are obscure.
- Gaiotto (2005) proposed that 1/4-BPS dyons can be interpreted as M5-branes wrapped on $K3 \times \Sigma_2$ where Σ_2 is a genus-two Riemann surface embedded in T^4; equivalently, as heterotic strings wrapped on Σ_2.



Gaiotto 2005; Dabholkar Gaiotto 2006

- Our aim will be to flesh out this picture, by computing exact six-derivative couplings in $D = 3$, and extracting the Fourier coefficients in the limit $R \to \infty$.

Exact BPS couplings in $D = 3$ I

- After compactification on a circle, the moduli space extends to

$$\mathcal{M}_3 = \frac{O(24,8)}{O(24)\times O(8)} \supset \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \mathbb{R}^{56+1} \\ \mathbb{R}^+_{1/g_3^2} \times \frac{O(23,7)}{O(23)\times O(7)} \times \mathbb{R}^{23+7} \end{cases}$$

Markus Schwarz 1983

- Accordingly, the U-duality group enhances to an arithmetic subgroup $G_3(\mathbb{Z}) \subset O(24,8)$, which is the automorphism group of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{23,7} \oplus \Lambda_{1,1}$.

Sen 1994

- The loci of enhanced gauge symmetry occur in codimension 6 in $D = 4$, but are expected to occur only in codimension 8 in $D = 3$.

Exact BPS couplings in $D = 3$ II

- The 4-derivative and 6-derivative couplings in the LEEA

$$F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d + G_{ab,cd}(\Phi)\nabla(\nabla\Phi^a\nabla\Phi^b)\,\nabla(\nabla\Phi^c\nabla\Phi^d)$$

  are expected to get contributions from 1/2-BPS and 1/4-BPS instantons, respectively.

- SUSY requires that the coefficients satisfy various differential constraints. Among them, and very schematically,

$$\mathcal{D}^2_{ef}\,F_{abcd} = 0\ , \quad \mathcal{D}^2_{ef}\,G_{ab,cd} = F_{abk(e}\,F_{f)cd}{}^k$$

  where $\mathcal{D}^2_{ef}$ is a second order differential operator on $\mathcal{M}_3$. These constraints imply that F_{abcd} is perturbatively exact at one-loop, while $G_{ab,cd}$ is perturbatively exact at two-loop on the heterotic side.

Bossard, Cosnier-Horeau, BP, 2016

Exact $(\nabla\Phi)^4$ coupling in $D=3$ I

- The coupling $(\nabla\Phi)^4$ is a 3D version of the F^4 coupling analyzed long ago. Up to non-perturbative effects,

$$g_3^2\, F_{abcd} = \frac{c_0}{g_3^2}\,\delta_{(ab}\delta_{cd)} + \mathrm{RN}\int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\mathit{\Gamma}_{\Lambda_{23,7}}[P_{abcd}]}{\Delta(\rho)} + \mathcal{O}(e^{-1/g_3^2})$$

  where $\mathit{\Gamma}_{\Lambda_{23,7}}$ is the partition function of the perturbative Narain lattice with polynomial insertion,

$$\mathit{\Gamma}_{\Lambda_{p,q}}[P_{abcd}] = \rho_2^{q/2}\sum_{Q\Lambda} P_{abcd}(Q_L)\,e^{\mathrm{i}\pi Q_L^2\rho-\mathrm{i}\pi Q_R^2\bar\rho}$$

  Lerche Nilsson Schellekens Warner 1988

- $\mathcal{F}_1$ is the standard fundamental domain of $SL(2,\mathbb{Z})$ on $\mathcal{H}_1$, and RN indicates a specific regularization of infrared divergences.

Exact $(\nabla\Phi)^4$ coupling in $D = 3$ II

- Requiring invariance under U-duality, it is natural to conjecture that the exact coefficient of the $(\nabla\Phi)^4$ in $D = 3$ is *[Obers BP 2000]*

$$F_{abcd} = \mathrm{RN} \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{24,8}}[P_{abcd}]}{\Delta}$$

- This satisfies $\mathcal{D}^2_{ef} F_{abcd} = 0$. The limit $g_3 \to 0$ can be extracted using the orbit method, and reproduces the tree-level and one-loop terms, plus instantons from NS5 and KK5-branes.
- In the large radius limit, one finds (schematically)

$$F^{(r-4,8)}_{\alpha\beta\gamma\delta} = R^2 \left(f_{\mathcal{R}^2}(S)\, \delta_{(\alpha\beta}\delta_{\gamma\delta)} + F^{(22,6)}_{\alpha\beta\gamma\delta} \right) + \sum'_{\widetilde{Q}\in\Lambda_{22,6}} \sum'_{m,n}$$

$$c_k\left(-\frac{|\widetilde{Q}|^2}{2}\right) P_{\alpha\beta\gamma\delta} K_\nu \left(\frac{2\pi R|mS+n|}{\sqrt{S_2}} |\widetilde{Q}_R| \right) e^{-2\pi \mathrm{i}(ma^1+na^2)\cdot\widetilde{Q}} + \mathcal{O}(e^{-R^2})$$

Exact $(\nabla\Phi)^4$ coupling in $D = 3$ III

- The power-like terms (from the trivial orbit and zero-mode of the rank one orbit) reproduce the exact $\mathcal{R}^2$ and F^4 couplings in $D = 4$.

Harvey Moore, Kiritsis Obers BP, 2000

- The $\mathcal{O}(e^{-R})$ terms (from the rank one orbit) correspond to 1/2-BPS dyons with charge $(Q, P) = (j, p)\widetilde{Q}$, with weight $c(-\frac{\tilde{Q}^2)}{2}) = \Omega_4(Q, P)$ (assuming that (Q, P) is a primitive).
- The $\mathcal{O}(e^{-R^2})$ terms (from the rank two orbit) have the expected form of Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ I

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ II

- The $\nabla^2(\nabla\Phi)^4$ coupling is a 3D version of the D^2F^4 coupling. Perturbatively, it receives up to two-loop corrections,

$$g_3^6\, G_{\alpha\beta,\gamma\delta} = \frac{c_0'}{g_3^2}\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\beta}G^{(23,7)}_{\gamma\delta} + g_3^2\, G^{(23,7)}_{\alpha\beta,\gamma\delta} + \mathcal{O}(e^{-1/g_3^2})$$

where the one-loop correction is given by *[Sakai Tanii 1987]*

$$G^{(23,7)}_{ab} = \mathrm{RN}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\widehat{E}_2\,\Gamma_{\Lambda_{23,7}}[P_{ab}]}{\Delta_k}\,,$$

while the two-loop correction is *[d'Hoker Phong 2005]*,

$$G^{(23,7)}_{ab,cd} = \mathrm{RN}\int_{\mathcal{F}_2}\frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\,\frac{\Gamma^{(2)}_{\Lambda_{23,7}}[R_{ab,cd}]}{\Phi_k}$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ III

- Here, $\mathit{\Gamma}^{(2)}_{\Lambda_{p,q}}[R_{ab,cd}]$ is the genus-two Siegel-Narain theta series

$$\mathit{\Gamma}^{(2)}_{\Lambda_{p,q}}[R_{ab,cd}] = |\Omega_2|^{q/2} \sum_{Q^i \in \Lambda_{p,q}^{\otimes 2}} R_{ab,cd}(Q_L)\, e^{\mathrm{i}\pi (Q_L^i \Omega_{ij} Q_L^j - Q_R^i \bar{\Omega}_{ij} Q_R^j)}$$

  and $R_{ab,cd}$ is a polynomial in Q_L^i.
- $\mathcal{F}_2$ is a fundamental domain for the action of $\Gamma_0^{(2)}(N)$ on the Siegel upper-half plane $\mathcal{H}_2$.
- RN denotes a regularization procedure which removes infrared divergences, both primitive and one-loop subdivergences.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ IV

- It is natural to conjecture that the exact coefficient of the $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ is given by

$$G_{ab,cd} = \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{24,8}}[R_{ab,cd}]}{\Phi_k}$$

- This ansatz satisfies the differential constraint $\mathcal{D}^2 G = F^2$, where the source term originates from the pole of $1/\Phi_k$ in the separating degeneration.
- The limit $g_3 \to 0$ can be extracted using the orbit method (extended to genus two), and reproduces the known perturbative terms, plus an infinite series of NS5/KK5-brane instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ V

- In the large radius limit, we find, schematically,

$$G_{\alpha\beta,\gamma\delta} = R^4 \left[G^{(22,6)}_{\alpha\beta,\gamma\delta} - f_{\mathcal{R}^2}(S)\delta_{\alpha\beta} G^{(22,6)}_{\gamma\delta} + [f_{\mathcal{R}^2}(S)]^2 \delta_{\alpha\beta}\delta_{\gamma\delta} \right] + G^{(1/2)}_{\alpha\beta,\gamma\delta} + G^{(1/4)}_{\alpha\beta,\gamma\delta} + G^{(TN)}_{\alpha\beta,\gamma\delta}$$

- The $\mathcal{O}(R^4)$ term (from trivial orbit and zero-mode of rank one orbits0 predicts the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D = 4$.
- The terms $G^{(1/2)}$ and $G^{(1/4)}$ (from the Abelian rank 1 and 2 orbits) come from 1/2-BPS and 1/4-BPS black holes in $D = 4$. and are both $\mathcal{O}(e^{-R\mathcal{M}(Q,P)})$.
- The term $G^{(TN)}$ (from the non-Abelian rank 2 orbit) is $\mathcal{O}(e^{-R^2})$ and can be ascribed to Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ VI

- In $G^{(1/4)}$, the fundamental domain $\mathcal{F}_2$ can be unfolded to $\mathcal{P}_2 \times T^3$, where $\mathcal{P}_2$ is the space of positive definite matrices Ω_2. The integral over Ω_1 extracts the Fourier coefficient

$$C\left[\begin{pmatrix} -\frac{1}{2}|Q_1|^2 & -Q_1\cdot Q_2 \\ -Q_1\cdot Q_2 & -\frac{1}{2}|Q_2|^2\end{pmatrix};\Omega_2\right] = \int_{[0,1]^3} \mathrm{d}\rho_1 \mathrm{d}\sigma_1 \mathrm{d}v_1 \, \frac{e^{\mathrm{i}\pi(\rho Q_1^2+\sigma Q_2^2+2vQ_1\cdot Q_2)}}{\Phi_k(\rho,\sigma,v)}$$

  which is a locally constant function of Ω_2.

- For large R, the integral over Ω_2 is dominated by a saddle point at

$$\Omega_2^\star = \frac{R}{\mathcal{M}(Q,P)} A^\intercal \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2\end{pmatrix} + \frac{1}{|P_R\wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2\end{pmatrix}\right] A\,.$$

  where $\binom{Q}{P} = A\binom{Q_1}{Q_2}$, $A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D=3$ VII

- Approximating $C[M;\Omega_2]$ by its saddle point value, we find (schematically)

$$G^{(2)}_{\alpha\beta,\gamma\delta} = R^7 \sum_{Q,P\in\Lambda'_{22,6}} P_{\alpha\beta,\gamma\delta}\, e^{-2\pi i(a^1 Q + a^2 P)}$$
$$\times \frac{\mu(Q,P)}{|2P_R\wedge Q_R|^{\frac{4-\ell}{2}}}\, B_\nu\left[\frac{2R^2}{S_2}\begin{pmatrix}1 & S_1\\ 0 & S_2\end{pmatrix}\begin{pmatrix}|Q_R|^2 & P_R\cdot Q_R\\ P_R\cdot Q_R & |P_R|^2\end{pmatrix}\begin{pmatrix}1 & 0\\ S_1 & S_2\end{pmatrix}\right]$$

where B_ν is the "matrix variate Bessel function"

$$B_\nu(Z) = \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}}\, e^{-\pi t - \frac{\pi \mathrm{Tr} Z}{t}}\, K_\nu\left(\frac{2\pi}{t}\sqrt{\det Z}\right)$$

and

$$\mu(Q,P) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda^{\otimes 2}_{22,6}}} |A|\, C\left[A^{-1}\begin{pmatrix}-\frac12|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac12|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right]$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ VIII

- In the limit $R \to \infty$, using $B_{\nu,\delta}(Z) \sim e^{-2\pi\sqrt{\mathrm{Tr}Z+2\sqrt{|Z|}}}$, one finds that the contributions are suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$.
- In 'primitive' cases where only $A = 1$ contributes, $\mu(Q, P)$ agrees with the helicity supertrace $\Omega_6(Q, P; z^a)$, evaluated with the correct contour prescription. It also refines earlier proposals for counting dyons with 'non-primitive' charges.

Cheng Verlinde 2007; Banerjee Sen Srivastava 2008; Dabholkar Gomes Murthy 2008

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ IX

- Corrections come from the difference between $C\,[M;\Omega_2]$ and $C\left[M;\Omega_2^*\right]$ at large Ω_2, due to wall-crossing. These corrections are of order $e^{-2\pi R(\mathcal{M}(Q_1,P_1)+\mathcal{M}(Q_2,P_2))}$ and are exponentially suppressed away from the wall. They are required by the source term in the differential constraint and ensure the smoothness across the wall.
- In addition, there are also contributions from the region where $\det(\Omega_2) < 1$ due to deep poles at

$$m_2 - m_1\rho + n_1\sigma + n_2(\rho\sigma - v^2) + jv = 0 \quad \text{with} \quad n_2 \neq 0$$

While the integral over Ω_1 is no longer well-defined, one can estimate that these corrections are of order $e^{-2\pi k R^2}$ and resolve the ambiguities of the sum over 1/4-BPS charges.

Conclusion I

- $\nabla^2(\nabla\Phi)^4$ couplings in $D=3$ string vacua with 16 supercharges nicely incorporate degeneracies of 1/4-BPS dyons in $D=4$, and explain their hidden modular invariance. They give a precise implementation of Gaiotto's idea that 1/4-BPS dyons are (U-duals of) heterotic strings wrapped on genus-two Riemann surfaces.
- A similar story presumably relates $\nabla^6\mathcal{R}^4$ couplings in $\mathcal{N}=8$ string vacua and degeneracies of 1/8-BPS dyons, but details remain to be worked out.
- From a mathematical viewpoint, higher-genus modular integrals are an interesting source of automorphic objects, which unlike Eisenstein series satisfy Poisson-type equations with sources.