Attractor indices, brane tilings and crystals

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based on arXiv:2004.14466 with Guillaume Beaujard, Jan Manschot

and arXiv:2012.14358 with Sergey Mozgovoy

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- The net number of BPS states with fixed electro-magnetic charge γ, called BPS index $\Omega(\gamma)$, is known exactly in most string backgrounds with $\mathcal{N} \geq 4$ supersymmetry. This is not yet so in $\mathcal{N} = 2$ string vacua such as type IIA on a generic CY3.

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- The net number of BPS states with fixed electro-magnetic charge γ, called BPS index $\Omega(\gamma)$, is known exactly in most string backgrounds with $\mathcal{N} \geq 4$ supersymmetry. This is not yet so in $\mathcal{N} = 2$ string vacua such as type IIA on a generic CY3.
- Part of the reason is that $\Omega(\gamma, z)$ depends on the moduli z in an intricate way, due to wall-crossing phenomena associated to BPS bound states with any number of constituents. The moduli space itself receives quantum corrections, unlike in $\mathcal{N} \geq 4$.

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- For D6-D2-D0 bound states for single unit of D6-brane charge at large volume, $\Omega(\gamma, z)$ are the standard Donaldson-Thomas invariants, related to higher-genus GV invariants.

  Thomas' 99; Maulik Nekrasov Okounkov Panharipande '04

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- D4-D2-D0 black holes can be realized by wrapping an M5 on a compact 4-cycle $P \subset X$, hence are described by a 2D superconformal field theory. The generating series of BPS indices (=VW invariants) is expected to be modular under $SL(2,\mathbb{Z})$. The central charge of the SCFT predicts the correct entropy at large charge, but exact indices are known only in a handful of cases.

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- Alternatively, by reducing along T^2, D4-D2-D0 branes on a rigid 4-cycle P are described by Vafa-Witten theory on P. Unless the divisor P is irreducible, the generating series of VW invariants is expected to be a (vector-valued) mock modular form, with a precise modular anomaly.

Minahan Nemeschansky Vafa Warner'98

Alexandrov Banerjee Manschot BP'16-19; Dabholkar Putrov Witten '20

Toric CY3, quivers and brane tilings

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- The nodes of Q corresponds to a basis of absolutely stable branes on X, whose bound states generate the BPS spectrum. For $X = \mathbb{C}^3/\Gamma$, these are the fractional branes; for $X = K_S$, these are elements of an exceptional collection on S.

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- The dimension vector d and stability parameters ζ can be deduced from the Chern vector γ and CY moduli z.

For toric CY3, attractor indices almost always vanish !

- Since the quiver has oriented loops, the indices $\Omega(\gamma, z) = \Omega(d, \zeta)$ are in general difficult to compute. We claim that quivers associated to toric CY3 are special: the attractor indices

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- More generally, for toric CY3 singularities, we claim that $\Omega_*(d) = 0$ unless $d_a = \delta_{a,\ell}$ or d lies in (a subspace of) the kernel of the Dirac pairing (i.e. $\langle d, d' \rangle = 0$ for all d').

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- The fact that $\Omega_\star(\gamma) = \mathcal{O}(1)$ (and apparently $\Omega_S^{L^2}(\gamma) = 0$!) is disappointing but consistent with gravity being decoupled.

Outline

1. The attractor flow tree formula for quivers
2. Toric CY3 and brane tilings
3. Unframed indices and VW invariants
4. Framed indices and molten crystals
5. Conclusion

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Quiver quantum mechanics

- Consider a SUSY quantum mechanics in $0+1$ dimensions, obtained by reducing $\mathcal{N}=1$ gauge theory in $3+1$ dimension, with matter content encoded in a quiver: each node $\ell = 1 \ldots K$ represents a $U(d_\ell)$ vector multiplet, each arrow from k to ℓ represents a chiral multiplet $\Phi^\alpha_{k,\ell}$ in $(d_\ell, \bar{d}_k)$ representation of $U(d_\ell) \times U(d_k)$. *[Denef '02]*

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- The ranks $\{d_\ell\}$ are encoded in a dimension vector $\gamma = \sum d_\ell \gamma_\ell$ in a lattice Γ, endowed with an antisymmetric Dirac-Schwinger pairing $\langle \gamma, \gamma' \rangle = \sum \gamma_{k\ell} d_k d'_\ell$ where $\gamma_{k\ell}$ is the skew-adjacency matrix (the number of arrows from node k to node ℓ counted with sign).

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- In addition, one must specify Fayet-Iliopoulos terms $\zeta_\ell \in \mathbb{R}$ and (in presence of closed oriented loops) a superpotential $W(\Phi)$.

Quiver quantum mechanics

- On the Higgs branch, the moduli space of classical SUSY vacua $\mathcal{M}_H(\gamma,\zeta)$ is the solutions of the F-term and D-term equations modulo set gauge equivalence,

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \Phi_{\ell k}^* T^a \Phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \Phi_{k\ell}^* T^a \Phi_{k\ell} = \zeta_\ell \operatorname{Tr}(T^a)$$

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- Equivalently, $\mathcal{M}_H$ is the moduli space of stable quiver representations with potential, an open subspace of solutions of F-term equations modulo the complexified gauge group.
- 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation with dimension vector $\gamma' < \gamma$, where $\mu(\gamma') = (\sum_\ell \zeta_\ell d'_\ell)/\sum d'_\ell$ is the slope. *[King'94]*

Quiver quantum mechanics

- BPS states correspond to Dolbeault cohomology classes of degree (p, q) on in $\mathcal{M}_H(\gamma, \zeta)$, counted by the Hodge polynomial

$$\Omega(\gamma, y, t, \zeta) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H(\gamma, \zeta))\,(-y)^{p+q-d}\,t^{p-q}$$

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- $\Omega(\gamma,y)$ also counts BPS states on the Coulomb branch, but that interpretation is subtle due to scaling solutions.

Primitive wall-crossing

- The DT invariants $\Omega(\gamma, y, \zeta)$ for $\gamma \in \mathrm{Span}(\gamma_1, \gamma_2)$ jump on walls where $\mu(\gamma_1) = \mu(\gamma_2)$. For primitive dimension vectors $\gamma_{1,2}$ with Dirac-Schwinger pairing $\gamma_{12} = \langle \gamma_1, \gamma_2 \rangle$,

$$\Delta\Omega(\gamma_1 + \gamma_2, y) = (-1)^{\gamma_{12}} \frac{y^{\gamma_{12}} - y^{-\gamma_{12}}}{y - 1/y} \Omega(\gamma_1, y)\Omega(\gamma_2, y)$$

  Physically, a two-centered bound state with spin degeneracy $2j + 1 = |\gamma_{12}|$ appears/disappears. *[Denef Moore '07]*

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- For more general charges, it is useful to introduce the rational invariants

$$\bar{\Omega}(\gamma, y) = \sum_{m|\gamma} \frac{1}{m} \frac{y - 1/y}{y^m - 1/y^m} \Omega(\gamma/m, y^m)$$

General wall-crossing

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Konsevitch Soibelman'08, Joyce Song'08

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- On physical grounds, we expect and get

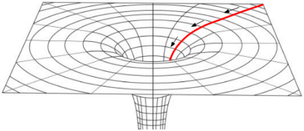
$$\bar{\Omega}(\gamma, y, \zeta_+) = \sum_{\gamma=\sum \alpha_i} \frac{g_{\rm WC}(\{\alpha_i\}, y)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}(\alpha_i, y, \zeta_-)$$

where $|{\rm Aut}(\{\alpha_i\})|$ is a Boltzmann symmetry factor, and $g_{\rm WC}(\{\alpha_i\}, y)$ is the index for Abelian quiver quantum mechanics with one node v_i for each α_i, and $\langle \alpha_i, \alpha_j \rangle$ arrows from v_i to v_j. This is computable using localisation on the Coulomb branch, or using Reineke's formula on the Higgs branch.

Reineke '02; Manschot BP Sen '10

Attractor flow and attractor indices

- For spherically symmetric black holes in $\mathcal{N}=2$ supergravity, the moduli flow from z_∞ to z_γ determined by the attractor mechanism:



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Ferrara Kallosh Strominger'95

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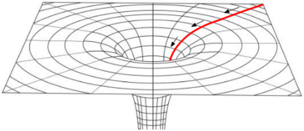

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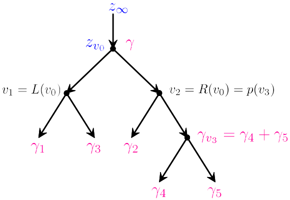
- Similarly, in quiver quantum mechanics there is a particular choice of stability parameters where 2-center bound states are ruled out,

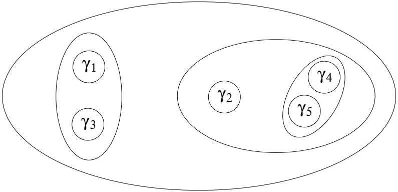
$$\zeta_k^\star(\gamma) = -\sum_\ell \gamma_{k\ell} d^\ell = -\langle \gamma_k, \gamma \rangle$$

known as attractor point or self-stability *[Manschot BP Sen '13, unpublished]*

Attractor flow and attractor indices

- The full spectrum can be constructed as bound states of these attractor BPS states, labelled by attractor flow trees:





Denef '00; Denef Greene Raugas '01; Denef Moore'07; Manschot '10

Wall-crossing and attractor indices

- The flow tree formula allows to express $\bar\Omega(\gamma, y, \zeta)$ in terms of the attractor indices $\bar\Omega_\star(\alpha_i, y) := \bar\Omega(\alpha_i, y, \zeta^*(\alpha_i))$

$$\bar\Omega(\gamma, y, \zeta) = \sum_{\gamma = \sum \alpha_i} \frac{g_{\rm tr}(\{\alpha_i\}, y, \zeta)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar\Omega_*(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18

where

$$g_{\rm tr}(\{\alpha_i\}, y, \zeta) = \sum_T \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all possible stable flow trees T ending on the leaves $\alpha_1, \ldots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

Wall-crossing and attractor indices

- The flow tree formula allows to express $\bar\Omega(\gamma, y, \zeta)$ in terms of the attractor indices $\bar\Omega_\star(\alpha_i, y) := \bar\Omega(\alpha_i, y, \zeta^*(\alpha_i))$

$$\bar\Omega(\gamma, y, \zeta) = \sum_{\gamma=\sum \alpha_i} \frac{g_{\rm tr}(\{\alpha_i\}, y, \zeta)}{|{\rm Aut}(\{\alpha_i\})|} \prod_i \bar\Omega_*(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18

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- The flow tree formula is combinatorial, and does not require integrating the attractor flow ! It is now a mathematical theorem.

Mozgovoy '20, Argüz Bousseau '21

Outline

1. The attractor flow tree formula for quivers
2. Toric CY3 and brane tilings
3. Unframed indices and VW invariants
4. Framed indices and molten crystals
5. Conclusion

Toric CY3 and brane tilings

- Toric CY3 are non-compact CY three-folds which admit an action of $(\mathbb{C}^\times)^3$ having a dense orbit. The category of coherent sheaves $\mathcal{D}(X)$ is isomorphic to the category of representations $\mathcal{D}(Q, W)$ of a quiver with superpotential.

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- The quiver (Q, W) are conveniently summarized by a brane tiling, i.e. a bipartite graph embedded in a two-torus. Tiles correspond to gauge groups, edges to chiral fields, and black/white vertices to monomials in the superpotential. The dual graph is a periodic quiver $\tilde{Q}$ covering Q.

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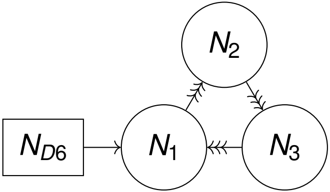
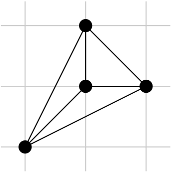
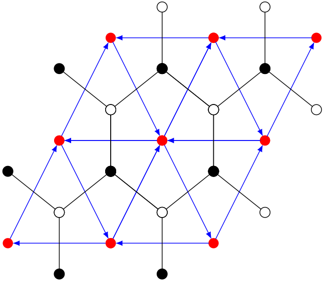
- Bound states with a D6-brane or a non-compact D4 are described by a framed quiver (Q_∞, W_∞) with an extra ungauged node and extra arrows $\infty \to \ell$ or $\ell \to \infty$.
- The same toric CY3 may be described by different tilings/quivers, related by Seiberg duality.

Example: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$



$$W = \epsilon_{ijk}\Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$





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Quivers from exceptional collections

- For local surfaces $X = K_S$, a basis of branes on $\mathcal{D}(X)$ (aka tilting sequence) can be constructed from an exceptional collection on S, i.e. an ordered sequence of (virtual) sheaves $(E_1, \dots, E_r)$ s.t.

$$\begin{aligned} \mathrm{Hom}(E_k, E_k) &= \mathbb{C}\,, \quad \mathrm{Ext}^m_S(E_k, E_k) \quad \forall m > 0 \\ \mathrm{Ext}^m_S(E_k, E_\ell) &= 0 \quad \forall (m \geq 0,\ 1 \leq \ell < k \leq r) \end{aligned}$$

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- There are two types of arrows $k \to \ell$: forward arrows from $\mathrm{Ext}^1(E_k, E_\ell)$ with $k < \ell$ and backward arrows from $\mathrm{Ext}^2(E_\ell, E_k)$ with $k > \ell$. The net number is computable from the Euler form

$$\chi(E, E') = \sum_{m \geq 0} (-1)^m \dim \mathrm{Ext}^m_S(E, E') = \int_S \mathrm{ch}(E^*)\,\mathrm{ch}(E')\ \mathrm{Td}(S)$$

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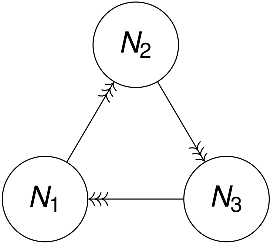
- The dimension vector d and FI parameters ζ can be related to the Chern vector γ and moduli z using $\gamma = \sum N_\ell \gamma_\ell$, $\zeta_\ell = \mathrm{Im}[Z_\gamma \overline{Z_{\gamma_\ell}}]$.

Sheaves on $\mathbb{P}^2$



$$E_1 = \mathcal{O} \qquad \gamma_1 = [1,0,0]$$
$$E_2 = \Omega(1)[1] \qquad \gamma_2 = [-2,1,\tfrac{1}{2}]$$
$$E_3 = \mathcal{O}(-1)[2] \qquad \gamma_3 = [1,-1,\tfrac{1}{2}]$$

$$\chi(E_k, E_\ell) = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}$$

[Le Potier'94]

Dimension vector: ($\propto (1,1,1)$ for D0-branes)

$$(N_1, N_2, N_3) = -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + \mathrm{rk}, \tfrac{1}{2}c_1 + \mathrm{ch}_2, -\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)$$

For canonical polarization $J = \rho c_1(S)$ with $\rho \gg 1$,

$$\zeta = 3\rho\,(N_2 - N_3, N_3 - N_1, N_1 - N_2) + \left(-\tfrac{N_2+N_3}{2}, \tfrac{N_1+3N_3}{2}, \tfrac{N_1-3N_2}{2}\right)$$

Canonical vs. attractor chamber

- For any $X = K_S$, the canonical chamber $J = \rho\, c_1(S)$ in the large volume limit translates into the anti-attractor chamber,

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- The dimension of the moduli space of stable sheaves coincides with the dimension of the moduli space of stable representations,

$$d_{\mathbb{C}} = \sum_{a \notin I} N_k N_\ell - \sum_{a \in I} N_k N_\ell - \sum N_\ell^2 + 1 = 1 - \chi(E, E)$$

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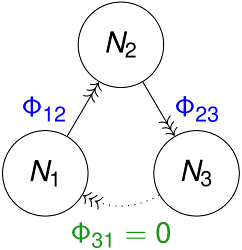
- In contrast, in the attractor chamber $\zeta_k = -\rho \sum_\ell \gamma_{k\ell} N_\ell$, the expected dimension is always negative, unless the dimension vector is one of the basis vectors γ_ℓ, or lies in the kernel of $\langle -, - \rangle$.



$$
\begin{aligned}
N_1 &= -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + \mathrm{rk}\right)\\
N_2 &= -\left(\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)\\
N_3 &= -\left(-\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)
\end{aligned}
$$

• In canonical (anti-attractor chamber), the expected dimension is positive at large instanton number $c_2 \sim -\mathrm{ch}_2$,

$$
\begin{aligned}
d_{\mathbb{C}} =& 3(N_1 N_2 + N_2 N_3 - N_3 N_1) - N_1^2 - N_2^2 - N_3^2 + 1\\
=& c_1^2 - 2\,\mathrm{rk}\,\mathrm{ch}_2 - \mathrm{rk}^2 + 1
\end{aligned}
$$

This requires $\zeta_1 \geq 0, \zeta_3 \leq 0$ hence $-\mathrm{rk} \leq c_1 \leq 0$.

Attractor invariants for $K_{\mathbb{P}^2}$

- In attractor chamber $\zeta^\star = 3\,(N_2 - N_3, N_3 - N_1, N_1 - N_2)$, the expected dimension is almost always negative:

$$d^\star_{\mathbb{C}} = 1 - \mathcal{Q}(\gamma) + \begin{cases} \frac{2}{3} N_3 \zeta^\star_3 - \frac{2}{3} N_1 \zeta^\star_1 & \zeta^\star_1 \geq 0, \zeta^\star_3 \leq 0 \quad (\Phi_{31} = 0) \\ \frac{2}{3} N_1 \zeta^\star_1 - \frac{2}{3} N_2 \zeta^\star_2 & \zeta^\star_2 \geq 0, \zeta^\star_1 \leq 0 \quad (\Phi_{12} = 0) \\ \frac{2}{3} N_2 \zeta^\star_2 - \frac{2}{3} N_3 \zeta^\star_3 & \zeta^\star_3 \geq 0, \zeta^\star_2 \leq 0 \quad (\Phi_{23} = 0) \end{cases}$$

$$\mathcal{Q}(\gamma) = \tfrac{1}{2}(N_1 - N_2)^2 + \tfrac{1}{2}(N_2 - N_3)^2 + \tfrac{1}{2}(N_3 - N_1)^2$$

hence $d^*_{\mathbb{C}} < 0$ unless $\gamma \in \{(1,0,0), (0,1,0), (0,0,1), (n,n,n)\}$.

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- We conjecture that $\Omega_\star(\gamma) = 0$ except in those cases. We set $\Omega_\star(1,0,0) = \Omega_\star(0,1,0) = \Omega_\star(0,0,1) = 1$, The index $\Omega_\star(n,n,n)$, corresponding to n D0-branes will be specified later.

VW invariants on $\mathbb{P}^2$

- Using the flow tree formula, and assuming the conjecture, we find that the index in canonical chamber agrees with VW invariants on $\mathbb{P}^2$ previously computed using blow-up/wall-crossing formulae !

Goettsche'90, Klyachko'91, Yoshioka'94, Manschot'11-14

$[N; c_1; c_2]$	(N_1, N_2, N_3)	$\Omega(\gamma, -\zeta^\star(\gamma))$
$[1; 0; 2]$	$(1, 2, 2)$	$y^4 + 2y^2 + 3 + \dots$
$[1; 0; 3]$	$(2, 3, 3)$	$y^6 + 2y^4 + 5y^2 + 6 + \dots$
$[2; 0; 3]$	$(1, 3, 3)$	$-y^9 - 2y^7 - 4y^5 - 6y^3 - 6y - \dots$
$[2; -1; 2]$	$(1, 2, 1)$	$y^4 + 2y^2 + 3 + \dots$
$[2; -1; 3]$	$(2, 3, 2)$	$y^8 + 2y^6 + 6y^4 + 9y^2 + 12 + \dots$
$[3; -1; 3]$	$(1, 3, 2)$	$y^8 + 2y^6 + 5y^4 + 8y^2 + 10 + \dots$
$[4; -2; 4]$	$(1, 3, 1)$	$y^5 + y^3 + y + \dots$

Attractor invariants for Fano surfaces

- We conjecture that the vanishing of attractor invariants holds for any CY threefold $X = K_S$ where S is a Fano surface. This includes the toric cases $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $S = dP_{k\leq 3}$, but also the non-toric del Pezzo surfaces $dP_{4\leq k\leq 8}$.

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- For those cases, we have computed VW invariants using the flow tree formula, under the assumption that $\Omega_\star(\gamma, y) = 0$ unless $\gamma = \gamma_k$ or $\langle\gamma, \cdot\rangle = 0$, and found agreement with independent results based on blow-up and wall-crossing formulae.

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- The vanishing of $\Omega_\star(\gamma, y)$ is supported by similar arguments about expected dimension, using ad hoc quadratic form $\mathcal{Q}(\gamma)$.
- The computation of D4-D2-D0 indices are insensitive to the value of $\Omega_*(n\delta)$, the BPS index for n D0-branes on X. This value can be fixed by considering D6-brane bound states.

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D6-D4-D2-D0 bound states

- In presence of a non-compact D6-brane, the quiver acquires an additional (ungauged) framing node with f_k arrows $\infty \to \ell$. For $X = K_S$, $f_k = \chi(\mathcal{O}_S, E_k)$.

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- For simplicity we assume a single framing arrow, $f_k = \delta_{k,\ell}$. The framed DT invariants $\Omega(1, d)$ in the non-commutative (NC) chamber $\zeta_\infty > 0, \zeta_k < 0$ can be computed by torus localization.

Mozgovoy Reineke'08

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- Let $J(Q, W)$ the Jacobian algebra (i.e. the path algebra modded out by relations $\partial_a W = 0$), and Δ_ℓ the set of equivalence classes of paths which start at the vertex ℓ. It admits a partial order with $u \leq v$ if there exists a path w such that $wu \sim v$. Δ_i can be represented as a pyramid or crystal.

D6-D4-D2-D0 bound states

- In the NC chamber, toric fixed points are in one-to-one correspondence with finite ideals $\mathcal{C} \subset \Delta_\ell$, i.e. subsets such that $u \in \mathcal{C}$ whenever $\exists v \in \mathcal{C}$ with $u \leq v$. They can be represented as molten pyramids or molten crystals.

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- Each ideal $\mathcal{C}$ contributes ± 1 to the (unrefined, framed) index $\Omega_{\rm NCDT}(1, d)$ with $d = \sum_{u \in \mathbb{C}} d_u$. The generating series is

$$Z_\ell(x) = \sum_{\mathcal{C} \subset \Delta_\ell} (-1)^{d_\ell + \chi_Q(d,d)}\, x^d$$

Mozgovoy Reineke'08

where $\chi_Q(d, d') = \sum_{a \in Q_0} d_a d'_a - \sum_{a: i \to j} d_a d'_b$ is the Euler form.

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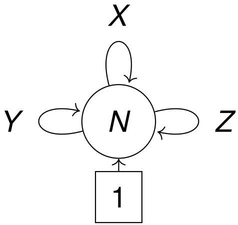
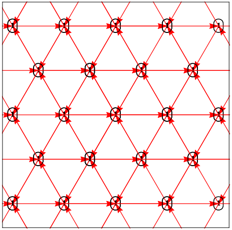
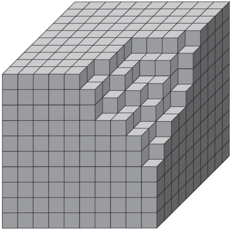
- Using the flow tree formula for quiver $\tilde{Q}$ with $\Omega_\star(1, d) = 0$ for $d \neq 0$, we can read off the (unrefined, unframed) attractor invariants $\Omega_\star(0, d)$.

Example: D6-D0/$\mathbb{C}^3$



$$W = x[y,z]$$





- The Jacobian algebra is $J(Q,W) = \mathbb{C}[x,y,z]$. Ideals correspond to plane partitions, or molten configurations of the crystal $\mathbb{N}^3$.

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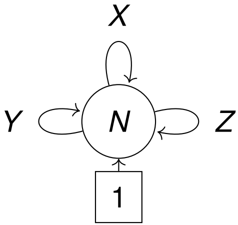

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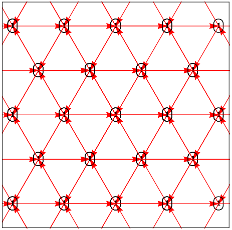

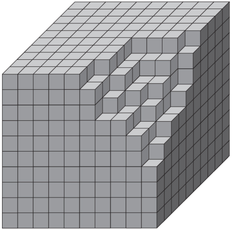

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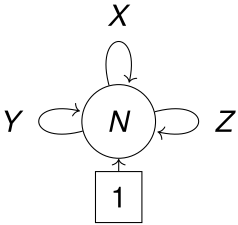
- The generating function of D6-D0 indices is *[MacMahon 1916]*

$$M(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + \ldots$$

Example: D6-D0/$\mathbb{C}^3$



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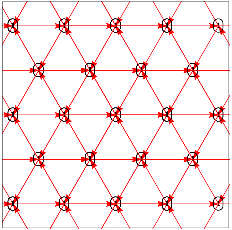

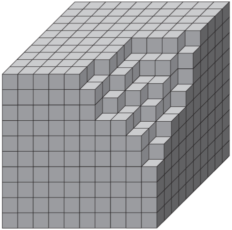

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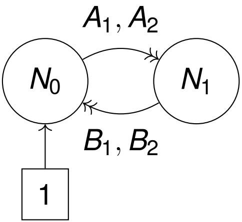
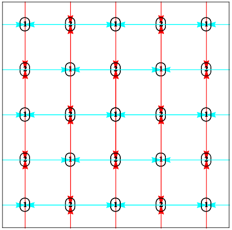
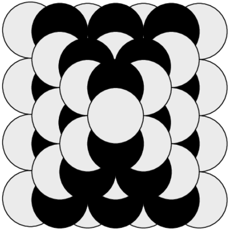
Example: D6-D2-D0 on the conifold



$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$





- The generating function of D6-D2-D0 indices is *[Szendroi'07]*

$$\begin{aligned} Z_0 &= M(-x_0 x_1)^2 \prod_{k\geq 1} (1 + x_0^k (-x_1)^{k-1})^k (1 + x_0^k (-x_1)^{k+1})^k \\ &= 1 + x_0 - 2x_0 x_1 + (x_0 x_1^2 - 4x_0^2 x_1) + (8x_0^2 x_1^2 - 2x_0^3 x_1) + \ldots \end{aligned}$$

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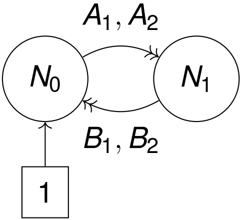

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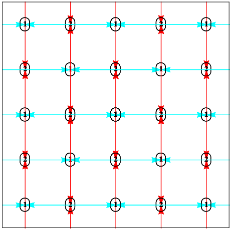

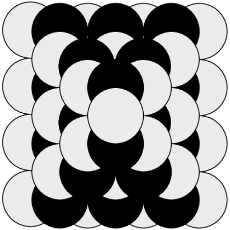

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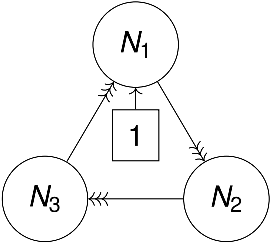
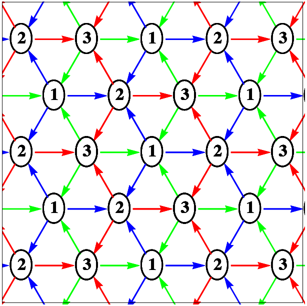
- The non-zero unframed indices are $\Omega(n, n) = -2$, $\Omega(n, n \pm 1) = 1$.

Example: D6-D4-D2-D0 on $\mathbb{C}^3/\mathbb{Z}_3$



$$W = \epsilon_{ijk}\Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$



- The generating function of D6-D4-D2-D0 indices is

$$Z_1 = 1 + x_1 + 3x_1x_2 + 3x_1x_2^2 - 3x_1x_2x_3 + 9x_1x_2^2x_3 + x_1x_2^3 - 3x_1^2x_2x_3 + \ldots$$

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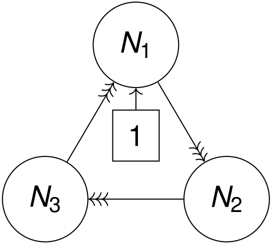

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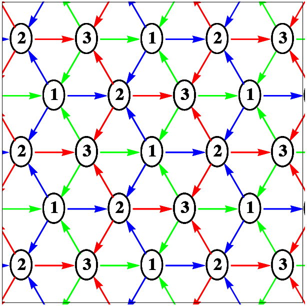

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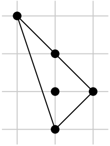
- This is consistent with the vanishing of all attractor indices except $\Omega_\star(n,n,n) = -3 = -\chi(K_{\mathbb{P}_2})$ for n D0-branes.

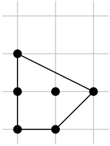
NCDT invariants from attractor indices

- This strategy applies to any brane tiling and allows to determine the (unframed, unrefined) attractor indices by counting molten crystals.
- This confirms our conjecture for Fano surfaces, and indicates that the vanishing of all attractor indices except $\Omega_\star(n\delta) = -\chi_X$ also holds for smooth toric threefolds with more than one compact divisor. Eg: $\mathbb{C}_3/\mathbb{Z}_5$, $Y^{3,2}$, ...
- For singular toric threefolds, such that the boundary of the toric diagram contains lattice points in addition to the corners, one finds $\Omega_\star(d) \neq 0$ for some d in the kernel of $\langle -, - \rangle$. Eg: $\mathbb{F}_2$, PdP_2, $\mathbb{C}^3/\mathbb{Z}_6$, ...

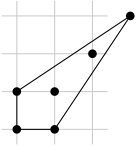
Toric CY threefolds



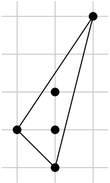
$\mathbb{F}_2$



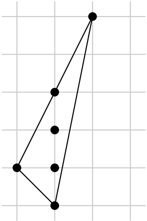
PdP_2



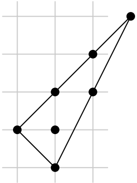
$Y^{3,2}$



$\mathbb{C}^3/\mathbb{Z}_5$



$\mathbb{C}^3/\mathbb{Z}_6(1,1,4)$



$\mathbb{C}^3/\mathbb{Z}_6(1,2,3)$

NCDT invariants from attractor indices

- Assuming the conjecture holds, refined NCDT invariants can be computed for all d once we know $\Omega_\star(n\delta, y)$. The latter can be extracted from the motivic D6-D0 invariants of X:

$$\Omega_\star(n\delta, y) = (-y)^{-3}\,[X] = -b_6/y^3 - b_4/y - yb_2 - y^3 b_0$$

  where b_i are Betti numbers for cohomology with compact support.

  Behrend Bryan Szendroï'09, Manschot BP Sen'10

- For toric CY threefold, $[X]$ can be read off from the toric diagram:

$$\Omega_\star(n\delta, y) = -y^{-3} - (i + b - 3)y^{-1} - i\,y$$

  where i and b are the number of internal and boundary lattice points. For $y = 1$, $\Omega_\star(n\delta) = -(2i + b - 2) = -\chi_X$ is the number of triangles in the toric diagram, by Pick's theorem.

Refined NCDT invariants for $\mathbb{C}^3/\mathbb{Z}_3$

- The generating function of refined framed indices is

$$
\begin{aligned}
Z_1 \;=\;& 1 + x_1 + \left(y^2 + 1 + 1/y^2\right)\left(x_1 x_2 + x_1 x_2^2\right) \\
-\;& \left(y^3 + y + 1/y\right)\left(x_1 x_2 x_3 + x_1^2 x_2 x_3\right) \\
+\;& \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3 + x_1 x_2^3 \\
-\;& \left(y^5 + y^3 + y + 1/y + 1/y^3 + 1/y^5\right) x_1 x_2^3 x_3 \\
+\;& \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3^2 \\
-\;& \left(y^5 + 2y^3 + 3y + 2y + 1/y^3\right)(x_1^2 x_2^2 x_3 + x_1^2 x_2^3 x_3) + \dots
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&+ \left(y^4 + 2y^2 + 3 + 2/y^2 + 1/y^4\right) x_1 x_2^2 x_3 + x_1 x_2^3 \\
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&- \left(y^5 + 2y^3 + 3y + 2y + 1/y^3\right) (x_1^2 x_2^2 x_3 + x_1^2 x_2^3 x_3) + \dots
\end{aligned}$$

- These invariants can be confirmed by computing (unframed, refined) DT invariants for trivial stability, and using wall-crossing.

Mozgovoy BP'20

Implications for L^2 and single-centered invariants

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- If so, $\Omega_\star^{L^2}(n\delta,y) = -i(y+1/y)$ for m D0-branes, where i is the number of internal points, or compact divisors.

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- Single-centered (or pure Higgs) invariants $\Omega_{\rm S}(\gamma, y)$ differ from $\Omega_\star(\gamma, y)$ due to scaling solutions. There is circumstancial evidence that $\Omega_{\rm S}^{L_2}(\gamma, y) = 0$ except for the basic D-branes !

Outline

1. The attractor flow tree formula for quivers
2. Toric CY3 and brane tilings
3. Unframed indices and VW invariants
4. Framed indices and molten crystals
5. Conclusion

Summary and Outlook

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- Does this shed light on the mock modular properties of generating series of VW invariants ? How about $\Omega_{\star/S}(\gamma)$ for compact CY3 ?

Thank you for your attention, and mind the wall !