Indefinite Theta Series and Generalized Erf

Workshop "Indefinite Theta functions and Applications in Physics & Geometry",

Trinity College, June 6–9, 2017

Part I

Holomorphic Θ series show up in many contexts, ranging from number theory, representation theory, differential geometry, algebraic geo combinatorics, ... and string theory:

$$\Theta_{Q,\mathcal{C},\mu}(\tau,z) = \sum_{k\in\mathbb{Z}^n\cap\mathcal{C}+\mu} q^{-\frac{1}{2}Q(k)}\, e^{2\pi i B(z,k)}$$

notice perverse changed sign!

$q = e^{2\pi i\tau} \in \mathcal{H}$, $z\in\mathbb{C}^n$, $\mu\in\mathbb{R}^n$, $\mathcal{C}\subset\mathbb{R}^n$

$Q(k) = B(k,k)$ int. quadratic form of signature $(r, n-r)$

$r=0$, $\mathcal{C}=\mathbb{R}^n$ $\Rightarrow$ Jacobi theta series, modular of weight $\frac{n}{2}$ under some congruence subgroup of $SL(2,\mathbb{Z})$

$r>0$: indefinite case

Choose $\mathcal{C}$ a cone such that $k\in\mathcal{C} \Rightarrow Q(k)\leq 0$

Then the sum converges, but is in general not modular.

But it may be modular for some choices of $\mathcal{C}$ (Hecke 1925)

Q: Can one characterize the behavior of $\Theta_{Q,\mathcal{C},\mu}$ under $SL(2,\mathbb{Z})$, at least for some class of cones?

Strategy: construct a "non-holomorphic correction" Θ^* such that $\widehat{\Theta} = \Theta + \Theta^*$ is modular (but not holomorphic), and modular transf of Θ^* are known (eg, an Eichler integral of ordinary theta series)

In 1606.05495 with Alexandrov, Banerjee, Manschot, we answered this question for $r=2$ a class of cubical cones and some of their degenerations, and outlined the extension to arbitrary r.

Subsequent fundational work

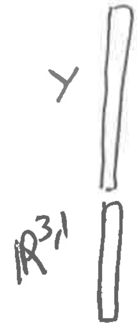
1608.03534 Kudla
1608.08874 ~~Nazaroglu~~ Westerholt-Raum
1609.01224 Nazaroglu

Applications: Bringmann Kaszian Rolen 1608.08588
" " Milas 1704.06891

Also secret work by Zwegers & Zagier since 2005, which is slowly starting to be declassified...

I will spend the 1st half of my lecture to explain the physics problem which got us (Sergei, Sibasish, Jan and myself) thinking about this problem, and gave us the clues which made this work possible.

Type IIA string theory compactified on CY threefold Y



- gravity
- gauge interactions
- massless scalar fields describing the shape of Y at any pt in $\mathbb{R}^{3,1}$

valued in a product $\mathcal{M}_Y^{CX} \times \tilde{\mathcal{M}}_Y^{K}$

$\mathcal{M}_Y^{CX} \to$ special Kähler manifold controlled by periods of hol 3-form on Y, well understood

$\tilde{\mathcal{M}}_Y^{K} \to$ Quaternion-Kähler manifold, still mysterious; governs Kähler structure on Y, string coupling g_s and more

Topologically, as $g_s \to 0$, $\tilde{\mathcal{M}}_Y^K$ is a double bundle

$$\begin{array}{ccc}
\mathbb{R}^+ \times S_1 = \mathbb{C}^\times & \longrightarrow & \tilde{\mathcal{M}}_Y^K \\
 & & \downarrow \\
T^{2b_2} = H_{ev}(Y,\mathbb{R})/H_{ev}(Y,\mathbb{Z}) & \longrightarrow & \\
 & & \downarrow \\
 & & \mathcal{M}_Y^K
\end{array}$$

$\mathcal{M}_Y^K$: special Kähler manifold, $= \mathcal{M}^{CX}_{\check{Y}}$ if $\check{Y}$ is the CY mirror to Y

determined by intersection numbers of Y
Euler numbers,
Gromov Witten invariants

Metrically, as $g_s \to 0$, the QK metric on $\tilde{\mathcal{M}}^K_Y$ goes exponentially fast to the "semiflat" or "c-map" metric,

$$ds^2\left(\hat{\mathcal{M}}^K_Y\right) \xrightarrow{g_s \to 0} ds^2_{\text{semi-flat}} + \mathcal{O}(e^{-1/g_s}) + \mathcal{O}(e^{-1/g_s^2})$$

- The $\mathcal{O}(e^{-1/g_s})$ physically come from D-branes wrapping Y

Mathematically, stable coherent sheaves $\mathcal{E}$

- Chern character $\gamma = ch(\mathcal{E}) \in H_{ev}(Y, \mathbb{Z})$:
  "electric magnetic charge"
- Stability condition depends on $t \in \mathcal{M}^K_Y$
  $\arg Z_\gamma(t)$
- Contributions determined by generalized DT invariant
  $$\Omega(\gamma; t) = \text{Euler}\,(\mathcal{M}_{\gamma, t})$$
  ↳ not to be confused with Y!

  locally cst function of t, away from "walls of marginal stability"

- The $\mathcal{O}(e^{-1/g_s^2})$ corrections are still mysterious, but in principle determined by existence of isometric action of $SL(2,\mathbb{Z})$:

$\gamma \in H_2 + H_0 \longrightarrow$ related to GW invariants

$\gamma \in H_4 + H_2 + H_0$ ↺ : Should be modular by itself!

$\gamma \in H_6 + H_4 + H_2 + H_0 \longrightarrow$ related to NS5-branes

Let us focus on coherent sheaves supported on an effective divisor $\mathcal{D}$

$$\gamma = (p^a, q_a, q_0) \in H_4 + H_2 + H_0 \quad , \; p^a \geqslant 0$$

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When $\mathcal{P}$ is irreducible ($p \neq p_1 + p_2$ with p_1, p_2 effective),
modularity leads to following constraint:

$$\Omega(p^a, q_a, q_0) = \Omega^{MSW}(p^a, \underbrace{q_a \bmod \kappa_{ab} p^b}_{\mu_a}, \underbrace{q_0 - \tfrac{1}{2} \kappa^{ab} p_a p_b}_{N})$$

(no wall crossing)

$$\kappa_{ab} = \kappa_{abc}\, p^c, \quad \kappa^{ab} = (\kappa_{ab})^{-1}$$

MSW = "Maldacena Strominger Witten"

Define $$F_{p,\mu} = \sum_{N \gg -\infty} \Omega^{MSW}(p^a, \mu_a, N)\, q^N$$

Thm MSW: $F_{p,\mu}$ must be vv. modular form of weight $-\frac{b_2}{2} - 1$
(specific multiplier system)

To show that this ensures the existence of an isometric $SL(2,\mathbb{Z})$ action,
introduce the twistor space

$$\begin{array}{ccc} \mathbb{P}_1 & \longrightarrow & \mathcal{Z} \\ & & \downarrow \\ & & \tilde{\mathcal{M}}^K_Y \end{array}$$

: complex contact manifold

The sum over p^a, q_a, q_0 produces $\sum_\mu F_{p,\mu}\, \Theta_{p,\mu}$

non hol theta series of sig $(1, b_2 - 1)$

which can be lifted to a hol theta series of sig $(1, b_2 - 1)$
using Zwegers' thesis trick. The modular completion comes from
a contour integral over $\mathbb{P}_1$, which produces an Eichler integral.
or error function

Higher signature theta series come by looking at reducible divisors:

$\Omega(\gamma, t)$ is no longer indep. of t, but can be decomposed as

$$\Omega(\gamma,t) = \Omega_{MSW}(\gamma) +$$

$$+ \sum_{\gamma=\gamma_1+\gamma_2} \frac{1}{2}\left(\operatorname{sgn}(\langle\gamma_1,\gamma_2\rangle) + \operatorname{sign} \operatorname{Im} Z_{\gamma_1}\bar{Z}_{\gamma_2}\right)\Omega_{MSW}(\gamma_1)\,\Omega_{MSW}(\gamma_2)$$

$$+ \sum_{\gamma=\gamma_1+\gamma_2+\gamma_3} \cdots$$

where $\Omega_{MSW}(\gamma)$ are independent of t ($=$ DT invariants in specific chamber)

$$\Omega_{MSW}(\gamma) = \Omega_{MSW}(p^a, \mu_a, N)$$

If p is reducible, $F_{p,\mu}$ is no longer v.v. modular form, but existence of $\Omega(\gamma, z_a)$ fixes its modular anomaly.

Corrections to QK metric on $\hat{\mathcal{M}}_Y^K$ with fixed total charge γ look like

$$\sum_\mu F_{p,\mu}\,\Theta_{p,\mu} + \sum_{\substack{\mu_1,\mu_2\\ p=p_1+p_2}} F_{p_1,\mu_1}F_{p_2,\mu_2}\,\Theta_{p_1,p_2,\mu_1,\mu_2}$$

non hol theta series of signature $2(1, b_2-1)$

which can be lifted to <u>holomorphic</u> indefinite theta series on Z

The twistorial construction naturally produces the modular completion in the form of a double contour integral which produces a double error function, and eventually a double Eichler integral.

Our main tool: Vignéras' theorem 1977

Λ n-dim lattice with even quadratic form, signature $(r, n-r)$

$$Q(x) = B(x,x) \in 2\mathbb{Z} \text{ if } x \in \Lambda \qquad (\text{could be relaxed to } \in \mathbb{Z})$$

$\mu \in \Lambda^*/\Lambda$

$\lambda \in \mathbb{Z}$

$\tau \in H$, $q = e^{2\pi i \tau}$, $b, c \in \Lambda \otimes \mathbb{R}$, $\tau = \tau_1 + i\tau_2$

$\Phi : \Lambda \otimes \mathbb{R} \to \mathbb{C}$ such that $\Phi(x) e^{\frac{\pi}{2} Q(x)} \in L_1(\Lambda \otimes \mathbb{R})$

Define
$$\Theta_\mu[\Phi, \lambda](\tau, b, c) = \tau_2^{-\lambda/2} \sum_{k \in \Lambda + \mu} \Phi(\sqrt{2\tau_2}(k+b))\, q^{-\frac{1}{2}Q(k+b)} \times e^{2\pi i B(c, k + \frac{1}{2} b)}$$

This satisfies the elliptic properties
$$\Theta_\mu(\Phi, \lambda)(\tau, b+k, c) = e^{-i\pi B(k,c)}\, \Theta_\mu(\Phi, \lambda)(\tau, b, c)$$
$$\text{''} \quad (\tau, b, c+k) = e^{i\pi B(k,b)} \quad \text{''}$$
$$(\tau+1, b, c) = e^{-2\pi i Q(\mu)} \quad \text{''}$$

i.e. $\Theta_\mu(\Phi, \lambda)$ is a section of the Θ line bundle over $\Lambda \otimes \mathbb{C}/\Lambda$.

<u>Rk</u> $\tilde{\Theta}_\mu(\tau, z = b\tau - c) = e^{-i\pi B(b, b\tau - c)}\, \Theta_\mu(\tau, b, c)$

satisfies the more familiar elliptic properties of Jacobi forms (EZ)

<u>Thm</u> Assume (i) $f(x) \equiv \Phi(x) e^{\frac{\pi}{2} Q(x)}$ is such that
$$R(x) f,\ R(\partial_x) f \in L^2(\Lambda \otimes \mathbb{R})$$
for any quad. polynomial

$$\text{(ii)} \quad \left[B^{-1}(\partial_x, \partial_x) + 2\pi x \partial_x\right] \Phi = 2\pi \lambda \Phi$$

then Θ_μ transform like a v-v modular form of weight $\lambda + \frac{n}{2}$:

$$\Theta_\mu(\Phi, \lambda)[-1/\tau, c, -b] = \frac{(-i\tau)^{\lambda + \frac{n}{2}}}{\sqrt{|\Lambda^*/\Lambda|}} \cdot e^{2\pi i \frac{\lambda + r}{4}} \cdot \sum_{\nu \in \Lambda^*/\Lambda} e^{2\pi i B(\mu, \nu)}\, \Theta_\nu(\tau, b, c)$$

Ex Choose r-dim positive plane $P \in \Lambda \otimes \mathbb{R}$

$$x = \underset{\substack{\cap \\ P}}{x_+} + \underset{\substack{\cap \\ P^\perp}}{x_-}$$

let $\Phi_P(x) = e^{-\pi Q(x_+)}$: satisfies assumptions of Vigneras' thm with $\lambda = r$

$\Theta_\mu(\Phi_P, -r)$ is then the usual (genus 1) Siegel theta series of signature $(r, n-r)$

Rk The Maass lowering operator $\tau_2^2 \partial_{\bar\tau}$ ("shadow operator") sends

$$\Theta_\mu[\phi, \lambda] \longrightarrow \Theta_\mu\left[\tfrac{i}{4}(x\partial_x - \lambda)\phi, \lambda - 2\right]$$

so ϕ is holomorphic in τ if $(x\partial_x - \lambda)\phi = 0$

One could chose ϕ to be a harmonic / homogeneous polynomial of degree λ,

but then $\phi e^{\frac{\pi}{2}Q} \notin L_1$: tension between holomorphy & modularity.

Instead, we shall chose ϕ to be locally a harmonic homogeneous polynomial of degree λ such that $\phi e^{\frac{\pi}{2}Q} \in L_1$ but not in L_2.
$\Theta_\mu[\phi, \lambda]$ will then be holomorphic but not modular.
To find its modular completion, we look for a C^∞ solution $\hat\phi$ of Vigneras equation which asymptotes to ϕ in all radial directions.

If $\Psi = \frac{i}{4}(x\partial_x - \lambda)\hat\Phi$ is its shadow, then

$$\hat\Phi(x) = \Phi(x) + 4i \int_1^\infty \frac{dt}{t^{\lambda+1}} \Psi(tx)$$

So

$$\Theta_\mu[\hat\phi, \lambda](\tau, b, c) = \Theta_\mu[\phi, \lambda](\tau, b, c)\ 4 \int_{-i\infty}^{\bar\tau} \frac{d\bar w}{(\tau - \bar w)^2}\, \Theta_\mu[\Psi, \lambda-2](\tau, \bar w, b, c)$$

ie we succeeded to write $\underset{\text{modular}}{\hat\Theta} = \underset{\text{hol}}{\Theta} + \underset{\text{Eichler integral}}{\Theta^*}$

Part II Generalized error functions & convergent Θ series

Signature $(1, n-1)$:

Choose C, C' pair of timelike, linearly indep. vectors in same component of the + light-cone

$Q(C)$, $Q(C')$, $B(C,C') > 0$

let
$$\Phi_1(x) = \frac{1}{2}\Big(\mathrm{Sgn}[B(C,x)] - \mathrm{Sgn}[B(C',x)]\Big)$$
$$\widehat{\Phi}_1(x) = \frac{1}{2}\big(E_1(C;x) - E_1(C',x)\big)$$

where $E_1(C,x) = E_1\left(\frac{B(C,x)}{\sqrt{Q(C)}}\right)$, $E_1(u) = \mathrm{Erf}(\sqrt{\pi}\, u)$

such that $\widehat{\Phi}_1(x) \longrightarrow \Phi_1(x)$ in all radial directions

Thm (Gøttsche, Zagier ; Zwegers)

1) $\Theta_\mu[\Phi_1, 0]$ is a convergent theta series, holomorphic in τ and z away from codim 1 loci in b-space

2) $\Theta_\mu[\widehat{\Phi}_1, 0]$ is a real-analytic convergent theta series, transforms as a vv modular (Jacobi) form of weight $\left(\frac{n}{2}, 0\right)$

3) The difference $\Theta_\mu[\widehat{\Phi}_1 - \Phi_1]$ is the Eichler integral of the Gaussian theta series $\Theta_\mu[\Psi_1]$ with
$$\Psi_1(x) = \frac{i}{4}\frac{B(C,x)}{\sqrt{Q(C)}}\, e^{-\pi B(C,x)^2/Q(C)}$$

Proof :

- $Q(k) < 0$ whenever $\mathrm{sgn}\, B(C,k) \neq \mathrm{sgn}\, B(C',k)$
  (In fact $Q(k) < Q_0(k) < 0$, Q_0 negative definite)
- $E_1(u)$ is a C^∞ solution of $(\partial_u^2 + 2\pi u \partial_u) E_1(u) = 0$
  So $E_1(C,x)$ is a C^∞ solution of Vignéras eq with $\lambda = 0$
- Decompose $E_1(u) = \mathrm{sgn}(u) + M_1(u)$: 
  where $M_1(u) = -\mathrm{sgn}(u)\, \mathrm{Erfc}(\sqrt{\pi}|u|)$ satisfies $|M_1(u)| \leq e^{-\pi u^2}$
  and $(\partial_u^2 + 2\pi u \partial_u) M_1 = 0$ away from $u = 0$

$\underline{Rk_1}$ - In the limit $Q(C) \to 0$, $E_1(C,x) \to \mathrm{sign}(C,x)$
$M_1(C,x) \to 0$

If both $\begin{cases} Q(C) \to 0 \\ Q(C') \to 0 \end{cases}$, $\hat{\Phi}_1$ coincides with Φ_1

so $\theta_\mu[\Phi_1, 0]$ is modular and hol in τ, but meromorphic in z.

$\underline{Rk_2}$ - To prepare the ground for higher signature, note the integral representations

$$(*) \quad M_1(u) = \frac{i}{\pi} \int_{\mathbb{R}-iu} \frac{dz}{z}\, e^{-\pi z^2 - 2\pi i z u}$$

$$(**) \quad E_1(u) = \int_{\mathbb{R}} du'\, \mathrm{sgn}(u')\, e^{-\pi(u-u')^2}$$

These reps make it manifest that

- $|M_1(u)| \sim e^{-\pi u^2}$, $M_1(0^+) - M_1(0^-) = -2$,

$$\left[\partial_u^2 + 2\pi u \partial_u\right] M_1 = \frac{i}{\pi} \int_{\mathbb{R}-iu} \frac{dz}{z} \cdot 2\pi z \partial_z \left[e^{-\pi z^2 - 2\pi i z u}\right] = 0$$

$$\frac{i}{4} u \partial_u M_1 = \frac{i}{\pi} \cdot \frac{i}{4} \cdot (-2\pi i)\, u \int_{\mathbb{R}-iu} dz\, e^{-\pi z^2 - 2\pi i z u} = \frac{i}{2}\, u\, e^{-\pi u^2}$$

- $E_1(u)$ C^∞, $E_1(u) \to \mathrm{sgn}(u)$ as $|u| \to \infty$

$$\left[\partial_u^2 + 2\pi u \partial_u\right] E_1 = -\int \underbrace{du'}_{\mathrm{sgn}(u')} \left[\partial_u \partial_{u'} + 2\pi u \partial_{u'}\right] e^{-\pi(u-u')^2}$$

$$= + \int \underbrace{du'}_{\partial_u + 2\pi u} e^{-\pi(u-u')^2} \cdot \partial_{u'} \mathrm{sgn}(u') = 0$$

- The identity $E_1 \overset{(u)}{=} M_1(u) + \mathrm{sgn}(u)$ can be shown by using $\mathrm{Pv}\left(\frac{1}{z}\right) = \mathrm{Fourier}(-i\pi\, \mathrm{sgn}\, u)$.

$\underline{Rk_3}$ (*) was naturally suggested by twistorial construction.
We understood (**) only in the final stage of the project! (cf Westerholt Raum independently)

Signature $(2, n-2)$

A natural extension of Zwegers theta series is to consider

$$\Phi_2(x) = \frac{1}{4}\left(\mathrm{sgn}[B(C_1,x)] - \mathrm{sgn}\,B(C_1',x)\right)\left(\mathrm{sgn}\,B(C_2,x) - \mathrm{sgn}\,B(C_2',x)\right)$$

where (C_1,C_2), (C_1,C_2'), (C_1',C_2), (C_1',C_2') span 4 positive 2-planes: $\Delta_{12}>0$, $\Delta_{12'}>0$, $\Delta_{1'2}>0$, $\Delta_{1'2'}>0$, $Q(C_1)>0$, $Q(C_2)>0$, $Q(C_1')>0$, $Q(C_2')>0$

with $\Delta_{ij} = \Delta(C_i,C_j) = Q(C_i)\,Q(C_j) - B(C_i,C_j)^2$

With suitable further conditions to be discussed below, the theta series $\Theta_\mu[\Phi_2,0]$ is convergent and holomorphic but not modular.
To find its modular completion, we need a C^∞ solution $E_2(C_1,C_2,x)$ of Vignéras eq. which asymptotes to $\mathrm{sgn}(C_1,x)\,\mathrm{sgn}(C_2,x)$ in all radial directions.

Using $GL(n,\mathbb{R})$ transformations, we can assume

$$Q(x) = x_1^2 + x_2^2 - \sum_{i=3}^{n} x_i^2$$

$$C_1 \propto (1,\alpha,0 \text{---}) \qquad \text{where} \quad \alpha = \frac{B(C_1,C_2)}{\sqrt{\Delta_{12}}}$$

$$C_2 \propto (0,1,0 \text{---})$$

$$\frac{B(C_1,x)}{\sqrt{Q(C_1)}} = \frac{u_1+\alpha u_2}{\sqrt{1+\alpha^2}}, \qquad \frac{B(C_2,x)}{\sqrt{Q(C_2)}} = u_2$$

So we can look for a C^∞ solution $E_2(\alpha;u_1,u_2)$ of Vignéras' equation

$$\left[\partial_{u_1}^2 + \partial_{u_2}^2 + 2\pi\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right)\right] E_2(\alpha;u_1,u_2)$$

such that $E_2(\alpha;u_1,u_2) \longrightarrow \mathrm{sgn}(u_1+\alpha u_2)\,\mathrm{sgn}(u_2)$

Def

$$\begin{cases} E_2(\alpha,u_1,u_2) = \displaystyle\int_{\mathbb{R}^2} du_1'\,du_2'\; e^{-\pi(u_1-u_1')^2-\pi(u_2-u_2')^2}\,\mathrm{sgn}(u_1'+\alpha u_2')\,\mathrm{sgn}(u_2') \\[2ex] M_2(\alpha,u_1,u_2) = -\dfrac{1}{\pi^2}\displaystyle\int_{\mathbb{R}-iu_1} dz_1 \int_{\mathbb{R}-iu_2} dz_2\; \frac{e^{-\pi z_1^2-\pi z_2^2-2\pi i(u_1z_1+u_2z_2)}}{z_1(z_2-\alpha z_1)} \end{cases}$$

Noting that $u_1 = \dfrac{B(C_{1\perp 2}, x)}{\sqrt{Q(C_{1\perp 2})}}$

$$\frac{u_2 - \alpha u_1}{\sqrt{1+\alpha^2}} = \frac{B(C_{2\perp 1}, x)}{\sqrt{Q(C_{2\perp 1})}}$$

where
$$C_{1\perp 2} = C_1 - \frac{(C_1, C_2)}{Q(C_2)} C_2$$
$$C_{2\perp 1} = C_2 - \frac{(C_1, C_2)}{Q(C_1)} C_1$$
is the dual basis to (C_1, C_2) in the plane spanned by (C_1, C_2)

we can define the "boosted error functions"

$$E_2(C_1, C_2, x) = E_2\left(\frac{B(C_1,C_2)}{\sqrt{\Delta_{12}}} ; \frac{B(C_{1\perp 2}, x)}{\sqrt{Q(C_{1\perp 2})}}, \frac{B(C_2, x)}{\sqrt{Q(C_2)}} \right)$$

$$M_2 \quad " \quad = M_2 \quad "$$

<u>Thm</u> ABMP 2016

- $E_2(C_1, C_2, x)$ is a C^∞ solution of Vignéras' eq with $\lambda = 0$ which asymptotes to $\operatorname{sgn}(C_1, x)\operatorname{sgn}(C_2, x)$ radially
- $M_2(C_1, C_2, x)$ is a solution of Vignéras V with $\lambda=0$, C_∞ away from the loci $(C_{2\perp 1}, x) = 0$ and $(C_{1\perp 2}, x) = 0$, exp. suppressed in all directions, $|M_2(C_1, C_2, x)| \leq 2\, e^{-\pi Q(x_+)}$ where x_+ is the projection of x on $\operatorname{Span}(C_1, C_2)$
- $$\begin{aligned} E_2(C_1, C_2, x) = {} & M_2(C_1, C_2, x) + \operatorname{sgn} B(C_1, x) \operatorname{sgn} B(C_2, x) \\ & + \operatorname{sgn} B(C_{1\perp 2}, x)\, M_1(C_2, x) \\ & + \operatorname{sgn} B(C_{2\perp 1}, x)\, M_1(C_1, x) \end{aligned}$$
- $$\operatorname{Shadow}\begin{pmatrix} E_2 \\ M_2 \end{pmatrix} = \frac{i}{2}\left[\frac{B(C_1, x)}{\sqrt{Q(C_1)}} e^{-\pi \frac{B(C_1,x)^2}{Q(C_1)}} \begin{pmatrix} E_1 \\ M_1 \end{pmatrix}(C_{2\perp 1}, x) + (1 \leftrightarrow 2) \right]$$
- E_2, M_2 are odd under $C_i \mapsto -C_i$, even under $C_1 \leftrightarrow C_2$
- If $(C_1, C_2) = 0$, $\begin{pmatrix} E_2 \\ M_2 \end{pmatrix} \longrightarrow \begin{pmatrix} E_1 \\ M_1 \end{pmatrix}(C_1, x) \begin{pmatrix} E_1 \\ M_1 \end{pmatrix}(C_2, x)$

Convergence conditions

In ABMP16, we stated complicated conditions for the convergence of $\Theta_\mu[\Phi_2]$, which provided an explicit bound

$$Q(k) \leq Q_0(k) < 0 \qquad \text{whenever} \quad \Phi_2(k) \neq 0$$

explicit negative definite metric

We provided a different set of conditions for the convergence of $\Theta_\mu[\hat{\Phi}_2]$ (which conjecturally follow from previous one).

The latter are easily obtained by requiring that the shadow converges:

$$\Psi_2 = \frac{i}{8}\Big\{ (C_1,x)\, e^{-\pi(C_1,x)^2}\big(E_1(C_{2\perp 1},x) - E_1(C_{2'\perp 1},x)\big)$$
$$+ (C_2,x)\, e^{-\pi(C_2,x)^2}\big(E_1(C_{1\perp 2},x) - E_1(C_{1'\perp 2},x)\big)$$
$$+ (C_1 \leftrightarrow C_1',\ C_2 \leftrightarrow C_2')\Big\}$$

So we see that we must require

$$\begin{cases} (C_{2\perp 1}, C_{2'\perp 1}) \geq 0 \\ (C_{1\perp 2}, C_{1'\perp 2}) \geq 0 \\ (C_{2\perp 1'}, C_{2'\perp 1'}) \geq 0 \\ (C_{1\perp 2'}, C_{1'\perp 2'}) \geq 0 \end{cases}$$

The geometric meaning of these conditions was clarified by Kudla, who showed that they are also sufficient to ensure convergence of $\Theta_\mu[\Phi_2]$ and $\Theta_\mu[\hat{\Phi}_2]$.

(Subject to additional condition $2(C_1,C_1')(C_2,C_2') - (C_1,C_2)(C_1',C_2') - (C_1,C_2')(C_1',C_2) \geq 0$ which conjecturally follows from the others)

In a nutshell, these conditions ensure that for all $s,t \in [0,1]$, the vectors $(sC_1 + (1-s)C_1',\ tC_2 + (1-t)C_2')$ span a positive 2-plane in $\mathbb{R}^{2,n-2}$. Thus, they define a compact 2 dim surface S inside $Gr^+_{2,n-2}$. The theta series $\hat{\Theta}_\mu[\hat{\Phi}_2]$ is then $-\int_S \Theta_{KM}$, where Θ_{KM} is the (co) homological theta series defined by Kudla & Millson.

<u>Rk</u> In order for the 2-planes $(C_1,C_2^{\prime})$, (C_1',C_2), (C_1,C_2'), (C_1',C_2'), be distinct, it is necessary to assume that $\Delta_{11'22'}>0$. This rules out the case $n<4$. However, nothing prevents us from taking $C_1=C_2$. In this case, we showed in ABMP16 that the conditions

$$\Delta_{1'2'},\ \Delta_{12'},\ \Delta_{11'} > 0,$$
$$(C_{1\perp 2'}, C_{1'\perp 2'}) \geq 0,$$
$$(C_{1\perp 1'}, C_{2'\perp 1'}) \geq 0$$
$$(C_{2'\perp 1}, C_{1'\perp 1}) \leq 0$$

are sufficient for the convergence of both $\hat{\Theta}_\mu[\Phi_2]$ and $\hat{\Theta}_\mu[\hat{\Phi}_2]$. This allows to cover various interesting cases (Ramanujan's fifth order mock theta, quantum invariants of torus knots, open GW invariants ...)

Rk the terms in the modular completion can be written as Eichler integrals:

$$M_1(C, \sqrt{2\tau_2}(k+b))\, q^{-\frac{1}{2}Q(k+b)} = -2\int_{-i\infty}^{\bar\tau} \frac{d\bar w}{\sqrt{i(\tau-\bar w)}}\, B(\underline{C}, k+b)\, e^{i\pi(\tau-\bar w)B(\underline{C},k+b)^2}\, q^{-\frac{1}{2}Q(k+b)}$$

$$M_2(C_1, C_2, \sqrt{2\tau_2}(k+b))\, q^{-\frac{1}{2}Q(k+b)}$$

$$= 4\int_{-i\infty}^{\bar\tau} \frac{d\bar w_1}{\sqrt{i(\tau-\bar w_1)}} \int_{-i\infty}^{\bar w_1} \frac{d\bar w_2}{\sqrt{i(\tau-\bar w_2)}} \cdot q^{-1/2\, Q(k+b)}$$

$$\cdot\, B(\underline{C_2}, k+b)\, . B(\underline{C_{1\perp 2}}, k+b)\, e^{i\pi(\tau-\bar w_1)B(\underline{C_2},k+b)^2 + i\pi(\tau-\bar w_2)B(\underline{C_{1\perp 2}},k+b)^2}$$

which transform under $SL(2,\mathbb{Z})$ according to

$$\int_{-i\infty}^{\bar\tau} f \,\Big|_{1-\gamma} = \int_{-i\infty}^{d/c} f \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\int_{-i\infty}^{\bar\tau} f_1 \int_{-i\infty}^{w_1} f_2 \,\Big|_{1-\gamma} = \int_{-i\infty}^{d/c} f_1 \int_{d/c}^{w_1} f_2 + \int_{-i\infty}^{\bar\tau} f_1 \times \int_{-i\infty}^{d/c} f_2$$

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Higher signature

A similar construction works for

$$\phi_r(x) = \frac{1}{2^r}\prod_{i=1}^{r}\left(\operatorname{sgn}[B(C_i,x)] - \operatorname{sgn} B[(C_i',x)]\right)$$

where $\{C_i, C_i'\}$ are chosen such that $\forall s_i \in [0,1]$,

$$\operatorname{Span}\left(\{s_i C_i + (1-s)C_i'\}\right)_{i=1\dots r} \text{ is a positive } r\text{-plane.}$$

Analogues of E_2 and M_2 are

$$E_r(C_1,\dots,C_r,x) = \int_{\langle C_1,\dots,C_r\rangle} dy^r \prod \operatorname{sgn} B(C_i,y)\, e^{-\pi Q(y-x_+)}$$

$$M_r(C_i^*,\dots,C_r^*,x) = \left(\frac{i}{\pi}\right)^r \int_{\langle C_1^*,\dots,C_r^*\rangle - ix_+} d^r z\, \frac{\sqrt{\Delta(C_i^*)}}{\prod_{i=1}^{r} B(C_i^*,z)}\, e^{-\pi Q(z)-2\pi i B(x,z)} \qquad C_i^* = \mathrm{dual}(C_i)$$

Thm

- $E_r(C_1,\dots,C_r,x)$ is a C^∞ solution of Vignéras' with $\lambda=0$
- $M_r(C_1^*,\dots,C_r^*,x)$ is a solution of Vignéras w/ $\lambda=0$, C^∞ away from the locus $(C_i^*,x)=0$
- $|M_r(C_1,\dots,C_r,x)| \leq 2^r e^{-\pi Q(x_+)}$ (Nazaroglu)
- $E_r(C_1,\dots,C_r,x) = \prod_{i=1}^{r} \operatorname{sgn} B(C_i,x) + \dots + M_r(C_1\dots,C_r,x)$, where the middle terms are $\prod_{i=1}^{r-s} \operatorname{sgn} B(\ ,x)\,.\, M_s(\ ,x)$
- $\operatorname{Shadow}\begin{pmatrix}E_r\\ M_r\end{pmatrix} \sim x e^{-\pi x^2}\begin{pmatrix}E_{r-1}\\ M_{r-1}\end{pmatrix}$

The modular completion of $\Theta_\mu[\Phi_r, \lambda=0]$ is provided by expanding out the product and replacing $\prod_{i=1}^{r} \operatorname{sgn} B(C_i,x) \longrightarrow E_r(C_1,\dots,C_r;x)$

Rk – The "tetrahedral" case is obtained by equating $C_1=C_2=\dots=C_r$.

– Presumably the same $\Theta_\mu[\hat{\Phi}_r]$ is equal to the integral of KM theta form.