Black Hole Degeneracies, Topological Strings and Quantum Attractors

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Lecture 3

- Higher derivative interactions and topological string amplitude
- The Ooguri-Strominger-Vafa conjecture
- Putting OSV to the test: precision counting of small black holes

Black hole entropy beyond leading order

The success of the string theory derivation of the thermodynamic entropy of BPS black holes has so far relied on the "thermodynamical" limit $A \gg G_N$, or $Q \gg 1$, where classical gravity can be trusted.

In order to push this beyond leading order, we require

- On the macroscopic side: to take into account quantum gravity effects, in the form of higher derivative corrections to Einstein's action $\Rightarrow$ Topological String Amplitude
- On the microscopic side: a refined understanding of the spectrum of the "black string" CFT $\Rightarrow$ Rademacher formula
- An identification of the appropriate thermodynamical ensemble implicit in the macroscopic entropy computation $\Rightarrow$ The OSV conjecture ?

Higher derivative F-terms

- Higher derivative interactions are usually hard to compute in string theory. Furthermore, for consistency all interactions related by supersymmetry at a given order must be included.
- Typically, the computation simplifies for "F-term" interactions, i.e. those described by chiral integrals in superspace. Holomorphicity considerably restricts the kind of contributions they can receive.
- In (the conformal approach to) $N = 2$ supergravity, an infinite family of such interactions can be described using the Weyl superfield and matter superfields

De Wit Lauwers Van Proyen; Cremmer, Kounnas ,Van Proyen, Derendinger, Ferrara de Wit Girardello

$$W_{\mu\nu}(x,\theta) = T_{\mu\nu} - \frac{1}{2} R_{\mu\nu\rho\sigma} \epsilon_{ij} \theta^i \sigma_{\lambda\rho} \theta^j + \dots$$

$$\Phi^I(x,\theta) = X^I + \frac{1}{2} \mathcal{F}^I_{\mu\nu} \epsilon_{ij} \theta^i \sigma^{\mu\nu} \theta^j + \dots$$

Here $T_{\mu\nu}$ is an auxiliary "graviphoton" Maxwell field; at tree-level, it can be eliminated in terms of $\mathcal{F}^I_{\mu\nu}$ appearing in the vector multiplets by $T = -2i\, e^{K/2}\, X^I \Im \mathcal{N}_{IJ} \mathcal{F}^J$.

Generalized prepotential

- From W one may construct the scalar chiral superfield

$$W^2(x,\theta) = T_{\mu\nu}T^{\mu\nu} - 2\epsilon_{ij}\theta^i\sigma^{\mu\nu}\theta^j R_{\mu\nu\lambda\rho}T^{\lambda\rho} - (\theta^i)^2(\theta^j)^2 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \ldots$$

- For any holomorphic function, homogeneous of degree two, known as the generalized prepotential,

$$F(\Phi, W^2) := \sum_{g=0}^{\infty} F_g(\Phi) W^{2g}$$

the chiral integral is well-defined:

$$\int d^4\theta d^4x \; F(\Phi, W^2) = S_{tree} +$$

$$+ \sum_{g=1}^{\infty} F_g(X) \left(g \;\; R^2 T^{2g-2} + 2g(g-1)(RT)^2 T^{2g-4} \right) + \text{ferm.}$$

(anti-self dual parts of R and T are understood)

Higher derivative F-terms and topological strings

Bershadsky Cecotti Ooguri Vafa;Antoniadis Gava Narain Taylor

- Consider type IIA compactified on a CY 3-fold Y. Due to the fact that the dilaton is a hypermultiplet, F_g can only occur at genus g.
- It can be extracted from the scattering amplitude of two gravitons and $2g-2$ graviphotons, at leading order in momenta. Their vertex operators are

$$V_g^{(0)} = h_{\mu\nu}(\partial X^\mu + ip\cdot\psi\,\psi^\mu)(\bar\partial X^\mu + ip\cdot\tilde\psi\,\tilde\psi^\mu)e^{ipX}$$

$$V_T^{(-1/2)} = \epsilon_\mu p_\nu e^{-(\phi+\tilde\phi)/2}\left(S\sigma_{\mu\nu}\tilde S\Sigma + cc\right)e^{ipX}p_\nu$$

where S, $\tilde S$ are spin fields in 4D, and Σ is the element of the chiral ring of the $N=(2,2)$ SCFT associated to the covariantly constant spinor on the CY: after bosonizing the $U(1)$ current $J = i\sqrt{3}\partial H$,

$$\Sigma = \exp\left[i\frac{\sqrt{3}}{2}\left(H(z) - \tilde H(\bar z)\right)\right]$$

ϕ is the free scalar in the bosonization of the β,γ superghost system.

Higher derivative F-terms and topological strings

- In order to have a non-vanishing result, it is necessary to cancel the ghost charge. The integration over supermoduli already provides $2g-2$ insertions of the picture changing operator $e^{\phi}T_F$. By changing $g-1$ of the graviphotons into the +1/2 ghost picture, we get the correct ghost charge $3g-3$. In total, we have $3g-3$ picture changing operators:

$$A_g = \int_{\mathcal{M}_g} \langle \psi\psi\ \psi\psi\ \prod_{i=1} 2g-2 e^{-\phi/2} S\Sigma \prod_{a=1}^{3g-3} e^{\phi} T_F \times \text{“cc”} \rangle$$

  Furthermore $T_F = G_- + G_+$.

- Recall that the topological twist, $L_0 \to L_0 - \frac{1}{2}J$, is related to the spectral flow $NS \to R$, and is equivalent to adding a background charge for J, i.e. adding $\int \frac{\sqrt{3}}{2} R^{(2)} H$ to the sigma model action. This is exactly the effect of inserting $2g-2$ operators Σ.
- Similarly, the insertion of $2g-2$ spin fields S has exactly the same effect in the space-time SCFT: all bosonic and fermionic fluctuations cancel (after summing over all spin structures)

Higher derivative F-terms and topological strings

- Altogether,

$$A_g = (g!)^2 \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} (\mu_a G_-)(\tilde{\mu}_a \tilde{G}_-) \langle = (g!)^2 F_g$$

This shows that in type IIA/CY, the F-term F_g in the $N = 2$ SUGRA is equal to the genus-g A-model topological string vacuum amplitude. The precise identification is

$$F^{hol}_{top} = \frac{i\pi}{2} F_{SUGRA} \ , \quad t^A = \frac{X^A}{X^0} \ , \quad \lambda = \frac{\pi}{4} \frac{W}{X^0}$$

where F^{hol}_{top} is the holomorphic topological amplitude and F_{SUGRA} is the generalized prepotential of the Wilsonian action. The actual scattering amplitude does have non-holomorphic dependence, as a result of massless singularities. Accordingly, the 1PI effective action is usually non-local due to infrared singularities from massless particles, and cannot be easily described with chiral superfields.

- Similarly, the same computation in type IIB shows that F_g is now given by the genus-g B-model topological string vacuum amplitude, in accord with mirror symmetry.
- When Y is K3-fibered, type IIA/Y is equivalent to Heterotic/$K_3 \times T^2$type II duality. In the

regime where the base of the K3 fibration is large, the F-terms can also be obtained from a one-loop computation in the heterotic string.

Quaternionic topological string amplitude

- Incidentally, note that a similar computation using the vertex operators of the universal hypermultiplet rather than the gravity multiplet, shows that in type IIA, the topological B string computes an amplitude

$$\tilde{S} = \int d^4x \sum_{g=1}^{\infty} \tilde{F}_g(X) \left[g(\partial\partial S)^2 (\partial Z)^{2g-2} + 2g(g-1)(\partial\partial S \partial Z)^2 (\partial Z)^{2g-4} \right]$$

  where $\tilde{F}_g(X)$ depends on the hypermultiplets only.
- In contrast to the previous case, nothing prevents contributions from higher genus or instantons. Little is known about this quaternionic version of the topological string.

The attractor mechanism, to all orders

- In the presence of these F-term corrections, it was shown that the attractor mechanism still holds, upon replacing the tree-level prepotential $F_0(X)$ by the complete topological amplitude $F(X, W^2)$ and imposing an additional attractor equation:

$$\Re X^I = p^I \,, \quad \Re F_I = q^I \,, \quad W^2 = 2^8$$

where now, $F_I = \partial F(X, W)/\partial X^I$. In particular, the near horizon geometry is still $AdS^2 \times S^2$, and the vector multiplet scalars are fixed to a value $t^A(p, q)$, independent of their value at infinity.

Cardoso de Wit Kappeli Mohaupt

- Furthermore, the Bekenstein-Hawking-Wald entropy, typically

$$S_{BHW} = 2\pi \int_{\Sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{h} d\Omega \sim \frac{1}{4} A + \dots$$

where $\epsilon^{\mu\nu}$ is the binormal on the horizon Σ, takes the simple form

$$S_{BHW} = \frac{i\pi}{4} \left[\bar{X}^I F_I - X^I \bar{F}_I \right] - \frac{\pi}{2} \Im \left[W \partial_W F \right]$$

evaluated at the attractor point.

Legendre transform and the OSV fact

- The Legendre trick continues to hold: setting $X^I = p^I + i\phi^I$,

$$S_{BHW} = \frac{i\pi}{4}\left[(X^I - 2i\phi^I)F_I - (\bar{X}^I + 2i\phi^I)\bar{F}_I\right] + \frac{i\pi}{4}\left[W\partial_W F - \bar{W}\partial_{\bar{W}}\bar{F}\right]$$

and using the homogeneity relation $X^I F_I + W\partial_W F = 2F$,

$$S_{BHW} = \frac{i\pi}{2}(F - \bar{F}) + \frac{\pi}{2}\phi^I(F_I + \bar{F}_I) = \mathcal{F}(p^I, \phi^I) + \pi\phi^I q_I$$

where

$$\mathcal{F}(p^I, \phi^I) = -\pi\, \text{Im}\left[F(X^I = p^I + i\phi^I; W^2 = 2^8)\right]$$

Thus, to leading order, the entropy $S_{BHW}(p, q)$ is the Legendre transform of the free energy $\mathcal{F}(p, \phi)$ of a thermodynamical ensemble of black holes with fixed magnetic charge p^I and electric potential ϕ_I.

The OSV conjecture for black hole degeneracies

- This suggests that the thermodynamical ensemble implicit in the BHW entropy is a “mixed” ensemble, where magnetic charges are treated micro-canonically but electric charges are treated canonically:

$$Z(p^I,\phi^I) \stackrel{?}{=} e^{\mathcal{F}(p^I,\phi^I)} \quad \text{where} \quad Z(p^I,\phi^I) := \sum_{q_I\in\Lambda_{el}} \Omega(p^I,q_I) e^{-\phi^I q_I}$$

- Using the relation between the free energy and the topological string, this leads to the Ooguri-Strominger-Vafa conjecture:

$$\sum_{q_I\in\Lambda_{el}} \Omega(p^I,q_I) e^{-\phi^I q_I} \stackrel{?}{=} |\Psi|^2 \ , \qquad \Psi_{top} := \exp\left(\frac{i\pi}{2} F(p^I + i\phi^I)\right)$$

Conversely, by inverse Laplace transform,

$$\Omega(p^I,q_I) \stackrel{?}{=} \int d\phi^I \ |\Psi_{top}|^2 e^{\phi^I q_I}$$

Comments on the OSV conjecture

- In its strongest form, the conjecture provides a way to compute the exact microscopic degeneracies $\Omega(p^I, q_I)$ from the topological string amplitude $F(X, W)$.
- In its weaker form, the conjecture is supposed to hold only asymptotically to all orders in inverse charges.
- For the strongest form to have a chance to hold, one should extend the definition of $F(X, W)$ to include non-perturbative effects in W. Conversely, one may hope to understand the non-perturbative completion of the topological string from a detailed knowledge of black hole degeneracies.

Comments on the OSV conjecture

- The conjecture encounters some immediate problems: the sum in $Z(p,\phi)$ does not converge, the mixed ensemble is thermodynamically unstable. The integration contour is not specified.
- Barring this issue, due to charge quantization $Z(p,\phi)$ is formally periodic under imaginary shifts $\phi^I \to \phi^I + 2ik^I$, $k^I \in Z$. This is not the case of $|\Psi_{top}|^2$, in fact one rather expects

$$\sum_{q_I \in \Lambda_{el}} \Omega(p^I, q_I) e^{-\pi \phi^I q_I} \stackrel{?}{=} \sum_{k^I \in \Lambda^*_{el}} \Psi^* \left(p^I - 2k^I - i\phi^I\right) \, \Psi \left(p^I + 2k^I + i\phi^I\right)$$

- What about symplectic invariance ? holomorphic anomalies ?

What does $\Omega(p, q)$ really count ?

- If $\log \Omega$ is to satisfy the second principle, it should probably be the absolute number of states. But if so, is it legal to keep only the F-term interactions ? Moreover, the actual number of states may change at lines of marginal stability.
- Alternatively, $\Omega(p, q)$ may be a particular supersymmetric index, which happens to be insensitive to D-terms. But the index is sometimes much smaller than the absolute number, so how about thermodynamics ? Moreover, the index sometimes jumps at special points on the hypermultiplet space...

Tests and elaborations of the OSV conjecture

- The proposal has been tested in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. At large N, this "factorizes" into $\sum_l \Psi_{top}(t + mlg_s)\Psi_{top}(\bar{t} - mlg_s)$.

  Vafa

- This was generalized for $O(-m) \oplus O(2g - 2 + m) \to \Sigma_g$, whose topological amplitude is related to q-deformed 2D Yang-Mills. The agreement with OSV for genus $g > 1$ however requires modular properties which are less than obvious.

  Aganagic Ooguri Saulina Vafa

- Exponentially suppressed corrections in 2D Yang-Mills have been studied, and seem to require further powers of Ψ on the rhs...

  Dijkgraaf Gopakumar Ooguri Vafa

- Some checks have been made against conjecturally exact formulae for black hole degeneracies in $N = 4$ theories.

  Cardoso de Wit Mohaupt Kappeli; Shih Yin

- Extensive checks have been conducted on small black holes in $N = 2, 4$

  Dabholkar Denef Moore BP

Large Black Hole degeneracies from OSV

- Recall that the A-model topological string amplitude $F(X^I, W^2)$ is an homogeneous function of degree 2 in (X^I, W): at large volume,

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} - \frac{W^2}{64 \cdot 24} \frac{c_A X^A}{X^0} - \frac{X_0^2}{(2\pi i)^3} \sum_{h=0}^{\infty} \sum_{\beta} \left(\frac{\pi W}{4 X^0} \right)^{2h} N_{h,\beta} e^{2\pi i \beta_A X^A / X^0}$$

  where $A = 1, \ldots n_V - 1$ runs over a base of 2-cycles of Y, $C_{ABC} = \int_Y J_A J_B J_C$ are triple intersection numbers, $X^A/X^0 = B^A + iV^A$ are the Kähler moduli; $c_A = \int_Y J_A c_2(T^{1,0}(X))$ and $N_{h,\beta}$ are rational numbers known as the Gromov-Witten invariants. Recall that terms with $\beta = 0$ are not exponentially suppressed.
- Assume $p^0 = 0$ for simplicity: the topological free energy is then

$$\mathcal{F}(p, \phi) = -\frac{\pi}{6} \frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2} \frac{C_{AB}(p) \phi^A \phi^B}{\phi^0} + 2\text{Re}(F_{GW})$$

$$\hat{C}(p) = C(p) + c_A p^A \,, \quad C(p) = C_{ABC} p^A p^B p^C \,, \quad C_{AB}(p) = C_{ABC} p^C$$

Large Black Hole degeneracies from OSV

- LET US DROP F_{GW} and compute the Laplace transform

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 d\phi^A \exp\left(\mathcal{F}(p,\phi) + \pi\phi^A q_A\right)$$

The ϕ^A integral is Gaussian, with saddle at $\phi^A_* = -C^{AB}(p) q_B \phi^0$:

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 \phi_0^{(n_V-1)/2} \det[C_{AB}(p)]^{-1/2} \exp\left(-\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \pi\phi^0 \hat{q}_0\right)$$

with $\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$.

- The ϕ^0 integral is now of Bessel type, with saddle at $\phi^0_* = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$:

$$\Omega_{OSV}(p^A, q_A) = \det[C_{AB}(p)]^{-1/2} [\hat{C}(p)]^{(n_V+1)/2} \hat{I}_{(n_V+1)/2}\left[2\pi\sqrt{-\hat{C}(p)\hat{q}_0/6}\right]$$

where $I_\nu(z)$ is a modified Bessel function, with asymptotics

$$\hat{I}_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z \left(1 + a/z + b/z^2 + \dots\right)$$

- Thus, taken literally, the OSV formula predicts the micro-canonical entropy

$$S_{OSV}(p^A, q_A) \sim 2\pi\sqrt{-\hat{C}(p)\hat{q}_0/6} - \frac{n_V + 2}{2}\log[-\hat{C}(p)\hat{q}_0] + \dots$$

- The leading square-root term reproduces the tree-level Bekenstein-Hawking entropy $S_{BH} = 2\pi\sqrt{C(p)\hat{q}_0}$ at large charges.
- The replacement $C(p) \to \hat{C}(p) = C(p) + C_A p^A$ takes into account the leading effect of R^2 corrections to tree-level supergravity. This is consistent with the microscopic counting based on $M5$ brane:

$$2\pi\sqrt{\hat{C}(p)\hat{q}_0/6} = 2\pi\sqrt{(D_{ABC}p^A p^B p^C + c_{2A}p^A/6)q_0}$$

  As a result, “small black holes” which were singular at tree-level ($C(p) = 0$) acquire a smooth horizon due to R^2 interaction.
- The logarithmic correction is purely an effect of changing from mixed to micro-canonical ensemble. The infinite series of power corrections is determined unambiguously.

The fine print

- Integrals have been carried out formally. Since $C_{AB}(p)$ in general has signature $(1, n_V - 2)$, the gaussian integral needs to be computed by rotating the contour for ϕ^A to the imaginary axis. We have assumed that the ϕ^0 integral picked the correct saddle point.
- In what regime was it correct to neglect the Gromov-Witten contribution ? If one scales all charges at the same rate, the topological coupling $\lambda \sim 1/\phi^0_*$ goes to 0 but the Kahler classes $\Im t^A = p^A/\phi^0_*$ stay of order 1. Thus the GW instantons cannot be neglected.
- It is possible to scale $\hat{q}_0$ faster than p^A, but slower than $(p^A)^3$, so that the Kahler classes go to infinity. In this case, the leading correction comes from the tree-level $\chi\zeta(3)(X^0)^2$, which perturbs the saddle point. This predicts the leading correction to the entropy

$$S(p,q) = 2\pi\sqrt{-\hat{C}\hat{q}_0/6} + \frac{\zeta(3)\chi}{96\pi^2}\frac{\hat{C}(p)}{\hat{q}_0}$$

  which still grows like a power of the charges. Unfortunately, this cannot be checked against the microscopic counting, since the Cardy formula does not apply.

The finer print

- Rather, the regime of interest is the one where the Cardy formula applies: $N \gg c$, i.e. $\hat{q}_0 \gg \hat{C}(p)$. In this regime, provided $p^A \neq 0$, the Kahler classes are large, so that non-degenerate instantons can be consistently neglected.
- However, in this regime $\phi^0_* \sim 1/\lambda \ll 1$, so that the topological coupling is strong. Fortunately, the degenerate instantons contributions can be resummed into the Mac-Mahon function,

$$\frac{\zeta(3)}{\lambda^2} + \sum_{h=2}^{\infty} \lambda^{2h-2} \frac{(2h-1) B_{2h} B_{2h-2}}{(2h-2)(2h)!} = -\sum_{k=1}^{\infty} k \log(1-e^{-k\lambda}) - \frac{1}{12} \log \lambda + \text{cte}$$

The Mac-Mahon function is exponentially suppressed at strong coupling, while the log term can be reabsorbed into a redefinition $e^{F_{top}} \longrightarrow \lambda^{\chi/24} e^{F_{top}}$.

Testing OSV: small black holes

- Our goal is to test the OSV conjecture in cases where black holes degeneracies are exactly known. For this, restrict to K_3-fibered CY, which admit a dual description as heterotic / $K^3 \times T^2$.
- The heterotic string admits a class of perturbative BPS states, known as Dabholkar-Harvey states:

$$|osc, N\rangle_L \otimes |osc, 0\rangle_R \otimes |n_i, w^i\rangle$$

satisfying the matching condition $N - 1 = n_i w^i$. They preserve 8 SUSY, and carry purely electric charge, in the natural heterotic polarization. It is easy to count them exactly by using simple modular forms.

- At strong coupling, these states remain stable and become black holes, carrying both electric and magnetic charges, in the natural type II polarization. In contrast to the general "4-charge" black holes, they are singular at tree-level, but acquire a smooth horizon due to R^2 interactions.

Sen 95; Dabholkar 04; Kallosh Maloney Dabholkar; Hubeny Maloney Rangamani; Bak Kim Rey

OSV prediction for small black holes

- For a K3-fibered CY 3-fold, the Kähler moduli split into the modulus X^1/X^0 of the base, and the moduli X^a/X^0 of the fiber ($a = 2, \dots n_V - 1$). The intersection form decomposes into

$$C_{ABC} X^A X^B X^C = X^1 C_{ab} X^a X^b + C_{abc} X^a X^b X^c$$

- Further consider a state whose only non-vanishing magnetic charge is p^1 (D4/K3):

$$C(p) = 0 \,, \quad \hat{C}(p) = 24 p^1 \,, \quad C_{AB}(p) = \begin{pmatrix} 0 & 0 \\ 0 & p^1 C_{ab} \end{pmatrix} \,, \quad \hat{q}_0 = q_0 + \frac{1}{2} C^{ab} q_a q_b / p_1$$

- The dependence on ϕ^1 now disappears from the integrand. Since F_{top} is invariant under monodromies $\phi_1 \to \phi_1 + \phi_0$, it is natural to restrict the integration range to $[0, \phi_0]$:

$$\Omega_{OSV}(p^1, q_A) = \int d\phi^0 \, \phi_0^{n_V/2} \exp\left(-\frac{4\pi p_1}{\phi^0} + \pi \phi^0 \hat{q}_0 \right) \sim \hat{I}_{(n_V+2)/2} \left[4\pi \sqrt{p^1 \hat{q}_0} \right]$$

- Caveat: when $p^a = 0$, the Kähler classes vanish at the saddle point. Strictly speaking, for such states the OSV formula is meaningless...

A benchmark case: $II/K3 \times T^2$ vs Het/T^6

- On the macroscopic side: thanks to $N = 4$, $F_{h>1} = 0$. F_1 can be extracted from R^2 coupling,

$$f_{R^2} \sim \log T_2 |\eta(T)|^4 \Rightarrow F_1 = \log \eta^{24}(T) , \quad T = X_1/X_0$$

- The gauge group is $U(1)^6 \times U(1)^{22}$, but 4 $U(1)$ are part of $N = 2$ gravitino multiplets, hence $n_V = 24$. Accordingly,the OSV prediction for small BH degeneracies is

$$\Omega_{OSV}(p^1, q_0) = \hat{I}_{13} \left[4\pi \sqrt{p^1 \hat{q}_0} \right]$$

- On the heterotic side, these small BPS BH are dual to Dabholkar Harvey states, enumerated by

$$\frac{1}{\eta^{24}} := \frac{1}{q \prod_{k=1}^{\infty} (1 - q^k)} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} , \quad N - 1 = p^1 q_0$$

- The leading exponential behavior is given by Cardy's formula $\log p_{24} = 2\pi \sqrt{24(N-1)/6}$. Subleading corrections can be extracted using the Rademacher formula...

The Rademacher expansion

Consider a vector-valued modular form $f_{\mu=1..r}(\tau)$ of weight $w<0$,

$$f_\mu(\tau+1)=e^{2\pi i\Delta_\mu}f_\mu(\tau)\ ,\quad f_\mu(-1/\tau)=(-i\tau)^w S_{\mu\nu}f_\nu(\tau)$$

with Fourier expansion $f_\mu(\tau)=q^{\Delta_\mu}\sum_{m=0}^\infty \Omega_\mu(m)q^m$

Theorem: the Fourier coefficients can be expressed as a convergent series

$$\Omega_\nu(n)=\sum_{s=1}^\infty\sum_{\mu=1}^r\sum_{m+\Delta_\mu<0} s^{w-2}\,\mathrm{Kl}(n,\nu;m,\mu;s)|m+\Delta_\mu|^{1-w}$$

$$\times\Omega_\mu(m)\times\hat{I}_{1-w}\left[\frac{4\pi}{s}\sqrt{|m+\Delta_\mu|(n+\Delta_\nu)}\right]$$

where $\mathrm{Kl}(n,\nu;m,\mu;s)$ are generalized Kloosterman sums, equal to $S^{-1}_{\nu\mu}$ for $s=1$ and $\hat{I}_\nu(z)$ is a modified, modified Bessel function of the 1st kind,

$$\hat{I}_\nu(z)=2\pi\left(\frac{z}{4\pi}\right)^{-\nu}I_\nu(z)\sim z^{-\nu-\frac{1}{2}}e^z(1+a/z+b/z^2+\dots)$$

The Rademacher expansion (cont.)

- All $s > 1$ contributions are exponentially suppressed wrt to $s = 1$, yet they are exponentially large in an absolute sense.
- The Hardy-Ramanujan-Cardy formula emerges by keeping the leading term $s = 1, m = 0$, using $\Delta = c/24, w = -c/2$:

$$\begin{aligned} \log \Omega(n) &\sim 4\pi\sqrt{|\Delta|(n+\Delta)} + \frac{1}{2}(w - \frac{3}{2}) \log(n+\Delta) + \dots \\ &= 2\pi\sqrt{\frac{c(n+\Delta)}{6}} - \frac{1}{4}(c+3)\log(n+\Delta) + \dots \end{aligned}$$

- The Rademacher expansion depends only on the polar part

$$f_\mu^- = \sum_{m+\Delta_\mu<0} \Omega_\mu(m) q^{m+\Delta_\mu}$$

(and modular data). Indeed, one proof is to represent $f_\mu(\tau)$ (or rather its Farey transform $q\partial_q^{1-w} f$) as the Poincaré series (i.e. sum over $Sl(2,Z)$ images) of f_μ^-.

Back to the bench

- In particular, for the inverse of the Dedekind function, $w = -12$, $\Delta = -1$,$\Omega(0) = 1$ hence

$$p_{24}(N) = \hat{I}_{13}\left[4\pi\sqrt{p^1\hat{q}_0}\right] + 2^{-14}\hat{I}_{13}\left[2\pi\sqrt{p^1\hat{q}_0}\right] + \dots$$

- Comparing to the OSV prediction, we find agreement to ALL orders in $1/(p^1 q_0)$! However, the OSV formula fails to reproduce subleading corrections which grow like $e^{2\pi\sqrt{p^1 q_0}}$.
- In order to obtain matching, we must drop non-holomorphic contributions from f_{R^2}, and consider the degeneracies of states with arbitrary angular momentum J.
- Note also that the matching relies on little data: only the large volume limit of 1-loop f_{R^2} (which is a universal tree-level term on heterotic side), the number of vector multiplets and the modular weight.
- This is NOT another test of het/type II duality: we did not really need the heterotic string to count 1/2 BPS states in type II on K_3...

$N = 4$ CHL strings

- More general $N = 4$ models with $0 \leq k \leq 22$ vector multiplets of $N = 4$ can be constructed, either as orbifolds of type II/ $K3 \times T^2$ by an Enriques involution, or as freely acting asymmetric orbifolds of Het/T^6.
- In the untwisted sector of the orbifold, the BPS states are a projection of the DH states in the Het/T^6 model. Their degeneracies are now counted by a modular form

$$Z_{untw} = \frac{1}{2}(\frac{\theta}{\eta^{24}} + \psi)$$

where θ is a partition function for the lattice of electric charges under the $22 - k$ gauge fields which have been projected out, and ψ enforces the projection. Modular weight:

$$w = \frac{1}{2}(22 - k) - 12 = -1 - k/2 \qquad \Rightarrow 1 - w = (k + 4)/2 = (n_V + 2)/2$$

Degeneracies are dominated by θ/η^{24}, and are in agreement with the OSV prediction.

- In addition, there are BPS states in the twisted sectors, which are counted by modular forms related to ψ by modular transformation. Their asymptotics appear to be equal to that of the untwisted, unprojected sector, again vindicating OSV.

$N = 4$ CHL strings (a case study)

- Consider the simplest case:

$$\Gamma_{6,22} = E_8(-1) \oplus E_8(-1) \oplus II^{1,1} \oplus II^{5,5}$$

orbifolded by $g|P_1, P_2, P_3, P_4\rangle = e^{2\pi i \delta \cdot P_3}|P_2, P_1, P_3, P_4\rangle$ This projects out the $U(1)$ associated to $P_1 - P_2$, leaving only the physical electric charges $Q = (P_1 + P_2, P_3, P_4)$.

- DH states arise in the untwisted sector by taking the ground state on the right, an arbitrary, orbifold invariant excitation of the 24 oscillators on the left, and level-matched internal momentum:

$$Z_{untw} = \frac{1}{2}\left(\frac{Z_{6,6}\left[{0 \atop 0}\right]\theta^2_{E_8[1]}(\tau)}{\eta^{24}(\tau)} + \frac{Z_{6,6}\left[{0 \atop 1}\right]\theta_{E_8[1]}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)} \right)$$

- From this we need to extract the number of states with given $Q = (P_1 + P_2, P_3, P_4)$. For this, change basis from (P_1, P_2) to

$$P_1 + P_2 = 2\Sigma + \wp \,, \quad P_1 - P_2 = 2\Delta - \wp$$

where S, Δ take values in the E_8 root lattice, and $\mathcal{P}$ is an element of the finite group $Z = \Lambda_r(E_8)/2\Lambda_r(E_8)$.

$N = 4$ CHL strings (cont)

- In order to sum over the "unphysical charges" Δ, introduce E_8 level-2 theta functions with characteristics:

$$\Theta_{E_8[2],\wp}(\tau) := \sum_{\Delta \in E_8(1)} e^{2\pi i \tau (\Delta - \frac{1}{2}\wp)^2}$$

and use

$$\theta^2_{E_8[1]}(\tau) = \sum_{\mathcal{P} \in E_8/2E_8} \theta_{E_8[2],\mathcal{P}}(\tau)\theta_{E_8[2],\mathcal{P}}(\tau) \ , \quad \theta_{E_8[1]}(2\tau) = \theta_{E_8[2],0}(\tau)$$

hence

$$Z_u = \frac{\theta^2_{E_8[2],\mathcal{P}}(\tau)}{\eta^{24}(\tau)} \pm \frac{1}{\eta^8(\tau)\eta^8(2\tau)} := q^{\Delta \pm} \sum_{N=0}^{\infty} d^u_{\pm}(N) q^N$$

CHL strings, cont.

- In the twisted sector, the situation is simpler:

$$Z_t = \frac{1}{2}\left(\frac{1}{\eta^{12}\theta_4^4} \pm \frac{1}{\eta^{12}\theta_3^4}\right) := q^{\Delta_\pm} \sum_{N=0}^{\infty} d_\pm^t(N) q^N$$

- Using the Rademacher formula, we find

$$\begin{aligned} \dim \mathcal{H}_{BPS}(Q) \quad &= \quad 2^{-5} \hat{I}_9 \left(4\pi\sqrt{Q^2/2}\right) \\ &+ \hat{I}_9 \left(4\pi\sqrt{Q^2/4}\right) \begin{cases} 15 \cdot 2^{-10} + 2^{-6} e^{2\pi i P\cdot\delta}\,, & \wp \in \mathcal{O}_1 \\ 2^{-10}\,, & \wp \in \mathcal{O}_{248} \\ -2^{-10}\,, & \wp \in \mathcal{O}_{3875} \\ 2^{-10} e^{i\pi Q^2}\,, & Q \in \Lambda_1 \end{cases} + \dots \end{aligned}$$

Hence we have agreement to all orders with OSV in all sectors. Subleading terms however are not captured by OSV, and depend crucially on the sector.

Absolute degeneracies vs. helicity supertraces

- We obtained agreement to all orders between the OSV prediction (at strong gravitational coupling) and the absolute degeneracy of DH states (at weak coupling). In general however, we expect that only a suitable index can be trusted in comparing weak and strong coupling results.
- The natural indexes to invoke are helicity supertraces:

$$\Omega_n = \mathrm{Tr}(-1)^F J_3^n$$

  where F is the target space fermion number, and J_3 one generator of the little group of a massive particle in D=3+1. For low n, and large supersymmetry, this index receives only contributions from short multiplets, while long (non BPS) multiplets cancel out.
- For $N = 4$ SUSY, the natural index for $1/2$ (resp. $1/4$) BPS states is Ω_4 (resp. Ω_6). In heterotic orbifold constructions, Ω_4 is in fact equal to the absolute degeneracy of 1/2-BPS states, “explaining” agreement.
- For $N = 2$ SUSY, the natural index is $\Omega_2 \sim N_V - N_H$. As we shall see, in heterotic orbifolds this can be much smaller than the absolute degeneracy !

A few words on $N = 2$ models

- A number of type II/CY - Het/$K3 \times T^2$ dual pairs are known, where OSV can be tested. While $F_{h>1}$ are now $\neq 0$, the degeneracies of small BH predicted by OSV, to all orders in $1/p^1 q_0$, at small p^1/q_0 are universally given by

$$\Omega_{OSV} = \hat{I}_{(n_V+2)/2}(4\pi\sqrt{Q^2/2})$$

- For heterotic asymmetric orbifolds with $N = 2$ supersymmetry, the DH states can be counted as before. In contrast to $N = 4$, in the untwisted sector DH states typically come in vector/hyper pairs, and the helicity supertrace Ω_2 is exponentially smaller than the OSV prediction. The absolute degeneracies agree with Ω_{OSV} at leading order only.
- In contrast, twisted states are all hypers, and have $\Omega_{abs} = \Omega_2$ in agreement to Ω_{OSV} to all orders in 1/Q.
- In a class of models such as Het/K3 with standard embedding, untwisted and twisted states cannot be distinguished, hence OSV gives the correct result to all orders.
- In other models such as FHSV, untwisted and twisted states can be distinguished by the modding of their charges, and OSV appears to fail in reproducing either Ω_{abs} or Ω_2, unless some coarse-graining is made.

An $N = 2$ example: the FHSV model

- Consider a Z_2 orbifold of type II/$K_3 \times T^2$, by an Enriques involution of K_3 times a shift of T^2. This is dual to a Z_2 orbifold of Het/T^6 by a reversal of T^4 times an exchange of the two E_8.
- The electric charges untwisted/twisted states take value in the lattices

$$M_0 = E_8(-1/2) \oplus II^{2,2}\ , \quad M_1 = E_8(-1/2) \oplus (II^{2,2} + \delta)$$

Define M_0' the sublattice of vectors $2P_1 \oplus P_2$ in M_0.

- Absolute degeneracies go as

$$\Omega_{abs}(Q) = \begin{cases} \hat{I}_\nu(4\pi\sqrt{\frac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ \hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_1 \end{cases}$$

$\nu = 13$ for generic moduli, but can vary.

An $N = 2$ example: the FHSV model (cont)

- Helicity supertraces are counted by

$$Z_u = \frac{2^6}{\eta^6 \vartheta_2^6}\,, \qquad Z_t^{\pm} = \frac{1}{2}\left(\frac{2^6}{\eta^6 \vartheta_4^6} \pm \frac{2^6}{\eta^6 \vartheta_3^6} \right)$$

- Using the Rademacher formula, they grow as

$$\Omega_2(Q) = \begin{cases} 2^{-8} e^{2\pi i Q\cdot\delta}(1 - e^{i\pi Q^2/2})\hat{I}_7(2\pi\sqrt{\frac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ -2^{-3}\hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2}) + 2^{-11} i e^{i\pi Q^2}\hat{I}_7(2\pi\sqrt{\frac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_1 \end{cases}$$

- Compare to the OSV prediction ($\chi = 0$):

$$I_7\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) \qquad \forall Q$$

Could the OSV formula have been exact ?

- Go back to the benchmark case: exact degeneracies can be extracted by a contour integral:

$$p_{24}(N) = \frac{1}{2\pi i} \oint q^{-N} dq/\Delta(q) = \int dt\ t^{-14} \frac{\exp\left(\frac{\pi(N-1)}{t}\right)}{\Delta\left(e^{-4\pi t}\right)}$$

- By contrast, the OSV formula can be rewritten as

$$\Omega_{OSV}(p^1, q_0) \sim \int d\tau_1\ d\tau_2\ \tau_2^{-14} \frac{\exp\left(\frac{\pi(N-1)}{\tau_2}\right)}{|\Delta\left(e^{-2\pi\tau_2 + 2\pi i \tau_1}\right)|^2}$$

- The two agree asymptotically when $\Delta(q) \sim q$, but the OSV formula does not appear to make sense non-perturbatively ! Furthermore the fact that Δ appears on both sides is a peculiarity of this model !

An $N = 4$ exception to OSV

- Let us consider Type $IIA/K_3 \times T^2$ at the Z_2 orbifold point, and perform a further orbifold by the "quantum symmetry" acting as -1 on each twisted sector, combined with a shift along T^2: this gives a type II $N = 4$ model with 6+6 gauge fields.
- The heterotic dual is unclear; however, another dual description can be obtained by making a Z_2 orbifold of type II/$T^4 \times T^2$ by $(-1)^{F_L}$ times a shift on T^2 This projects out all RR fields, leaving 6+6 vectors. In constrast to the previous (2,2) case, SUSY is realized as (4,0) on the worldsheet.

Vafa Sen

- The amplitude F_1 can be computed at one-loop on the (2,2) case: one finds $F_1 \sim \log \theta_4(T)$, which has no perturbative part but only instantons: thus small black holes remain small, even with R^2 corrections !

Kounnas Gregori Obers Pioline Petropoulos

- Just as in the heterotic case, the (4,0) model admits a spectrum of DH states, enumerated by θ_i^4/η^{12}. The microscopic degeneracies thus grow as $\hat{I}_5(2\pi\sqrt{2p^1 q_0})$, not matched by OSV !

Issues with duality

- We have seen that taken literally, the OSV formula predicts logarithmic corrections with depend on magnetic charges only. In our small black hole tests, we got around this by considering ratios $\Omega(p,q)/\Omega(p,q')$.
- When all p and q are taken not to vanish, this problem cannot be dismissed so casually. Consider e.g. the very special supergravities introduced in Lecture 1. Taking OSV literally, we find at the semi-classical level

$$\Omega(p^I, q_I) \sim \left[I_3(p^\sharp_A + p^0 q_A) \right]^{(n_v+2)/6} I_4(p,q)^{-(n_v+2)/2} e^{\pi I_4(p,q)}$$

where I_4 is the quartic invariant of the 4D U-duality group $Conf(J)$, while I_3 is the cubic invariant of the 5D U-duality group $Str_0(J)$. The logarithmic correction to the entropy breaks 4D U-duality !

A better variational principle

- de Wit and coll. have proposed an alternative ensemble where electric-magnetic duality is manifest: they note that the attractor equations follow from the variational principle

$$S_{BHW}(p,q) = \langle \Sigma(X,\bar{X},p,q) \rangle_{X^I,\bar{X}^I}$$

$$\Sigma(X,\bar{X},p,q) := \frac{i\pi}{4}(\bar{X}^I F_I - X^I \bar{F}_I) + 2iW_{p,q}(X) - 2i\bar{W}_{p,q}(\bar{X})$$

and $W_{p,q} = q_I X^I - p^I F_I$. Hence they propose

$$\Omega(p,q) = \int dX^I d\bar{X}^I \, |\det \Im\tau_I J| e^{\Sigma}$$

- Performing the integral over $\Re(X^I)$ semi-classically, one finds a saddle point at $\Re(X^I) = p^I$. The remaining integral over $\phi^I = \Im(X^I)$ reproduces OSV, except for a measure factor,

$$\Omega(p,q) \sim |\det \Im\tau_I J| \, e^{\mathcal{F}(p,\phi)+\pi q_I \phi^I}$$

This cures the problem in the semi-classical approximation – in fact the prefactor completely disappears ! However, corrections to the saddle point are bound to spoil this success.

Reverse engineering

- Rather than extracting BH degeneracies from the topological amplitude, one may try to construct the BH partition function from our partial knowledge of exact degeneracies.
- In type II/$K3 \times T^2$, the lattices of electric charges are

$$\begin{aligned}\Lambda^{IIA}_{elec} &= D0(q_0) \oplus D2/T2(q_1) \oplus D2/\gamma_2(q_a) \oplus \dots \\ \Lambda^{IIA}_{mag} &= D6/K3 \times T^2(p^0) \oplus D4/K3(p^1) \oplus D4/T_2 \times \gamma_2(p^a) \oplus \dots\end{aligned}$$

  Exact degeneracies are known for purely electric heterotic states , i.e. for vanishing $D2/T2, D4/T^2 \times \gamma^2, D6/K3 \times T^2$.
- Setting $p^0 = p^a = 0$, the BH partition function includes terms with $q^1 = 0$:

$$Z'_{BH} = \sum_{q^0, q^a \in II^{3,19}} p_{24} \left(1 + p^1 q_0 + \frac{1}{2} q_a C^{ab} q_b\right) e^{-\pi(q_0 \phi^0 + q_a \phi^a)}$$

Reverse engineering (cont.)

- Inserting the unity

$$1 = \sum_N \delta\left[N - 1 - \frac{1}{2} q_a C^{ab} q_b\right] = \sum_N \sum_{k^0=0}^{p^1-1} \frac{1}{p^1} e^{2\pi i k^0 (N-1-\frac{1}{2}q_a C^{ab} q_b)/p^1}$$

inside the sum, the sum over N reconstructs the Dedekind function

$$Z'_{BH} = \frac{1}{p^1} \sum_{k^0=0}^{p^1-1} \frac{e^{-2\pi i \tau q_a C^{ab} q_b - \pi \phi^a q_a}}{\Delta(\tau)}, \quad \tau = \frac{i\phi^0 + 2k^0}{2p^1}$$

Doing a modular transformation on τ and a Poisson resummation on q_a gives

$$Z'_{BH} = \sum_{k_0=0}^{p^1-1} \sum_{k^a \in II^{19,3}} Z_0(\phi^A + 2ik^A), \quad Z_0(\phi^A) = \frac{\exp\left[-\frac{\pi}{2} \frac{p^1 C_{ab} \phi^a \phi^b}{\phi^0}\right]}{(p^1)^2 \, \Delta\left(\frac{2ip_1}{\phi^0}\right)}$$

Reverse engineering (cont.)

- While Z_0 looks close to the topological string amplitude, it is in fact different: no $|\Delta|^2$, and the argument has no ϕ^1 dependence !
- The sum over translations $\phi^A \to \phi^A + 2ik^A$ guarantees that the BH partition function has the expected periodicity due to the charge quantization. Yet much of the information in the topological string amplitude could be lost in the process of averaging !
- This procedure has been systematized and shows that subleading corrections to the entropy can be obtained by counting open string vacua for D4-branes on CY.

  Strominger Gaiotto; Denef Moore
- It is tempting to conjecture that the exact black hole partition function is a theta series whose general term is the topological string amplitude. This resonates well with Kontsevitch's “very wild guess”
- In an seemingly unrelated development, non-Gaussian theta series have been constructed based the same Jordan algebras that govern the very special supergravities. It would be very interesting if invariance under monodromies in the CY moduli space could be realized in a similar fashion.

  Kazhdan Pioline Waldron

Discussion

- The OSV conjecture for the partition function of BPS black holes has passed several non-trivial tests, leading to agreement with microscopic degeneracies to all orders in $1/Q^2$.
- For this to hold, a number of ambiguities had to be lifted: integration contour, holomorphic anomalies, identification of Ω_{OSV} with helicity supertraces, count states with arbitrary J.
- OSV is very successful in $N = 4$ models, less so in some $N = 2$ models. When $\chi \neq 0$, the saddle point lies at strong coupling of the pointlike instanton series, requiring a non-perturbative completion of the topological amplitude in this sector.
- At the non-perturbative level, a relation like "$Z_{BH} = |e^F|^2$" cannot hold, if only because the rhs is not periodic in ϕ modulo $2i$. This suggests that the BH partition function may instead be a theta series built on e^F, possibly with interesting automorphic properties.
- Combining a recent analysis of the relation between 4D and 5D BH, the Gopakumar-Vafa relation between entropy of 5D BH and F_{top} and the OSV formula in 4D, leads to a bizarre relation $|e^{F(1/g_s)}|^2 = e^{F(g_s)}$ also suggestive of automorphicity.

Open problems

- In principle, one expects the infinite series of tree-level higher derivative corrections to become important in the singular geometry of Het DH states. The amazing agreement at leading order suggests some kind of non-renormalization theorem.
- There are also DH states in type IIA/$K_3 \times T^2$, with a large entropy $2\pi\sqrt{2nw}$. In contrast to the Het DH states, they are 1/4-BPS, have zero helicity supertrace, but do not seem to be resolved by R^2 corrections.
- Similarly, there are 1/4-BPS DH states in type II/T^6, with the same entropy. The leading higher derivative corrections are the famous $\zeta(3)R^4$, but those are unlikely to give the correct entropy !
- In a rather orthogonal approach, Sen was able to reproduce the BH entropy to all orders using a different ensemble, with a chemical potential μ for Q^2 rather than Q, and keeping non-holomorphic corrections. It would be interesting to relate the two approaches...
- An outstanding challenge is to understand subleading corrections to large black holes. A somewhat naive analysis of the elliptic genus almost gives the right Bessel function...