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ICTP lectures on instantons, wall-crossing and hypermultiplet moduli spaces

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Abstract: Motivated by precision counting of BPS black holes in $\mathcal{N}=2$ string vacua, I survey recent advances in understanding quantum corrections to the vector multiplet moduli space in $\mathcal{N}=2$ super Yang-Mills field theories and Calabi-Yau compactifications of type II string theories on $\mathbb{R}^3 \times S^1$. By T-duality and decompactification, the latter is identical to the hypermultiplet moduli space of the dual type II string theory on the same Calabi-Yau threefold. In either case, the hyperkähler (respectively, quaternion-Kähler) metric is regular across lines of marginal stability, the one-instanton effects on one side being reproduced by multi-instanton effects on the other side, by courtesy of the Kontsevich-Soibelman wall-crossing formula. I also review an elementary derivation of this wall-crossing formula based on the quantum mechanics of multi-centered black hole solutions. Finally, I describe some recent progress in understanding the topology of the moduli space in presence of Kaluza-Klein monopole / NS5-brane instanton corrections.

Contents

1. Introduction and overview

These are PRELIMINARY notes for a set of 4 one-hour lectures to be given at the "ICTP School on D-brane Instantons, Wall Crossing and Microstate Counting", Trieste, Dec. 15-20, 2011. This is based primarily on a series of works [1, 2, 3, 4, 5, 6, 7, 8] with my collaborators S. Alexandrov, J. Manschot, D. Persson, F. Saueressig, A. Sen and S. Vandoren, whom I wish to thank for very enjoyable collaborations. The material in Section §2 also relies in part on the work of D. Gaiotto, G. Moore and A. Neitzke [9].

1.1 Precision counting of black hole micro-states

In searching for a consistent quantum theory of gravity, our most valuable clue is perhaps the Bekenstein-Hawking area law for black holes, ascribing an entropy $S_{BH} = A/4$ to an event horizon of area $A \gg 1$ in Planck units. Reproducing this universal law from a statistical description of the black hole is a necessary requirement on any model of quantum gravity. A stronger consistency requirement is that the macroscopic and microscopic descriptions should also agree for finite size systems, where the black hole entropy becomes sensitive to non-universal quantum corrections to General Relativity.

String theory famously provides a microscopic explanation for the area law, including the numerical coefficient 1/4, for a large class of supersymmetric (BPS) black hole solutions in extended supergravity [10]. The reason for restricting to BPS solutions is, as usual, that the microscopic counting is reliable at weak string coupling, while the macroscopic description is valid at strong coupling. While some of the BPS states may pair up into long multiplets and decay as the coupling is varied, the index is immune to long multiplets and can be meaningfully compared. The agreement between the two descriptions has been verified to subleading order [11], and in special cases, to all orders in an asymptotic expansion in a regime of large electromagnetic charges [12] (see e.g. [13, 14] for further discussion and references).

For vacua with $\mathcal{N} = 4$ supersymmetry, one can go beyond this asymptotic expansion and, using duality arguments, compute the index for all values of the charges and moduli exactly, in terms of Fourier coefficients of a certain Siegel modula form [15, 16, 17]. The dependence on the asymptotic value of the moduli enters as a choice of contour in the Fourier integral, and the corresponding discontinuity of the index, exponentially suppressed at large charges, matches the contribution of two-centered 1/2-BPS black hole configurations which are gained or lost across the wall. The chamber independent part of the index can be assembled into a Mock modular partition function, and matches the Bekenstein-Hawking entropy of single-centered black hole configurations, including subleading corrections [**?**]. The spectrum of 1/8-BPS states in $\mathcal{N} = 8$ string theory is also known in considerable detail (see e.g. [18] and references therein).

In view of this impressive success for BPS black holes in $\mathcal{N} \geq 4$ vacua, it is natural to try and achieve the same precision counting for BPS black holes in vacua with $\mathcal{N} = 2$ unbroken supersymmetries, e.g. in type II string theories compactified on a Calabi-Yau three-fold $\mathcal{X}$. Unlike the previous case however, the answer is less constrained by duality symmetries, the only obvious symmetries being the monodromy group of $\mathcal{X}$, and when available, the modular group of the "black string" CFT [11]. Wall-crossing phenomena in $\mathcal{N} = 2$ vacua are also far more severe, the entropy of multi-centered solutions sometimes dwarfing that of single centered black holes with the same charges [19]. Finally, the micro-states depend on the detailed geometry of the internal space $\mathcal{X}$, specifically on the number of stable supersymmetric cycles in a given homology class, which is in general hard to compute. Yet, the Bekenstein-Hawking-Wald entropy of single-centered black holes [20] suggests a tantalizing relation between the microscopic index and the topological string amplitude [21], which remains to be fully uncovered. Recents attempts to establish this relation [22, 23, 24] were based on partition functions in mixed ensembles, which, despite being well-suited for modular invariance, failed to properly take into account chamber dependence and monodromy invariance. It is therefore highly desirable to identify an observable which is sensitive to BPS black holes and which incorporates duality symmetries and wall-crossing phenomena in a natural way.

1.2 Black holes vs. instantons

One such observable is offered by the low-energy effective action in 2+1 dimensions, after compactifying the original four-dimensional vacua on a circle of radius R. In order to be sensitive to BPS black holes only, each carrying 4 fermionic zero-modes, one should restrict to four-fermion couplings, or equivalently to the two-derivative action.

In three dimensions, all gauge fields can be dualized into scalars, and the effective action consists of a non-linear supersymmetric sigma model, whose target space $\mathcal{M}_3$ includes the moduli space $\mathcal{M}_4$ of the 4D scalars fields, the electric and magnetic Wilson lines $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ of the 4D gauge fields around the circle, the radius R and the NUT potential σ (dual to the Kaluza-Klein gauge field around the circle). For a fixed, large radius R, the fields $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ live in a torus $\mathcal{T}$ fibered over $\mathcal{M}_4$, while σ parametrizes a circle bundle S^1_σ over $\mathcal{T}$. The metric on $\mathcal{M}_3$ is schematically

$$\mathrm{d}s^2_{\mathcal{M}_3} \sim \frac{\mathrm{d}R^2}{R^2} + \mathrm{d}s^2_{\mathcal{M}_4} + \frac{\mathrm{d}\zeta^2 + \mathrm{d}\tilde{\zeta}^2}{R^2} + \frac{(\mathrm{d}\sigma + \zeta \mathrm{d}\tilde{\zeta} - \tilde{\zeta}\mathrm{d}\zeta)^2}{R^4} \ . \tag{1.1}$$

In particular, to all orders in $1/R$, the metric on $\mathcal{M}_3$ has continuous translational isometries along the torus $\mathcal{T}$ and circle S^1_σ.

At finite R however, the metric receives exponentially suppressed instanton corrections, from 4D BPS black holes whose Euclidean worldline winds around the circle [25]. These corrections break the translational isometries along $\mathcal{T}$ to a discrete subgroup and are of the form

$$\delta\, \mathrm{d}s^2_{\mathcal{M}_3} \sim \bar{\Omega}(\gamma, z^a)\, \mathcal{R}(\gamma, z^a)\, e^{-2\pi R|Z(\gamma,z^a)| + 2\pi\mathrm{i}(p^\Lambda \tilde{\zeta}_\Lambda - q_\Lambda \zeta^\Lambda)} \ , \tag{1.2}$$

where $\gamma = (p^\Lambda, q_\Lambda)$ are the electromagnetic charges carried by the black hole, and $|Z(\gamma, z^a)|$ is the mass of the black hole in 4D Planck units. The prefactor $\mathcal{R}(\gamma, z^a)$ originates from the fluctuation determinant in the instanton background, and the "instanton mesure" $\bar{\Omega}(\gamma, z^a)$ is closely related to the black hole index $\Omega(\gamma, z^a)$. There are also additional corrections from k Kaluza-Klein monopoles (KKM), schematically of the form

$$\delta\, \mathrm{d}s^2_{\mathcal{M}_3} \sim e^{-|k|R^2 + \mathrm{i}\pi k \sigma} \ , \tag{1.3}$$

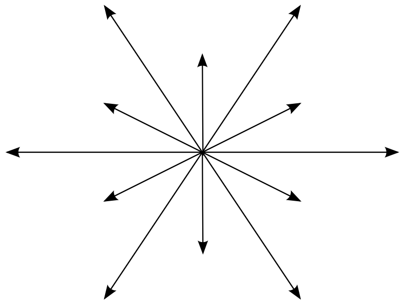

Figure 1: Two-dimensional projection of the root diagram of the discrete symmetry group obtained by combining monodromy, Heisenberg and S-duality.

which break translations along S^1_σ to a discrete subgroup. These effects do not admit a black hole interpretation (the corresponding Lorentzian configurations have closed time-like curves). It should be stressed that the black hole corrections (1.2) form a divergent asymptotic series, due to the exponential growth of the instanton measure $\bar{\Omega}(\gamma, z^a)$. In fact, one may argue by Borel-type resummation that the ambiguity of this asymptotic series is on the order of the KKM corrections (1.3) [5].

By construction, the metric on $\mathcal{M}_3$ must be consistent with all dualities of string theory. In particular, it should be invariant under monodromies in $\mathcal{M}_4$ and under the $SL(2,\mathbb{Z})$ type IIB S-duality (or equivalently, the modular symmetry in M-theory on $\mathcal{X} \times T^2$). Moreover, the metric should also be smooth across walls of marginal stability: just as single particle states turn into the continuum of multi-particle states across the wall, single-instanton contributions turn into multi-instanton contributions. Thus, the metric on $\mathcal{M}_3$ offers a convenient packaging of the micro-state degeneracies which naturally incorporates duality symmetries and chamber dependence.

Even better, the inclusion of Kaluza-Klein monopoles (may) lead to enhanced duality symmetries, analogous to the genus 2 modular group $Sp(2,\mathbb{Z})$ in $\mathcal{N} = 4$ vacua (see Figure 1). In fact, similar ideas as above suggest that the F^6 couplings in $\mathcal{N} = 4$ string vacua in 3 dimensions should encode the index of 1/4-BPS states in 4 dimensions, and should be given by an automorphic form of $SO(8, n, \mathbb{Z})$. Similarly, the $\nabla^6 R^4$ couplings in M-theory on T^8 should encode the index of 1/8-BPS states in 4 dimensions, and be given by an automorphic form of $E_{8(8)}(\mathbb{Z})$.

Finally, we should mention that the same strategy also works for $D = 4$, $\mathcal{N} = 2$ Yang-Mills theories with rigid supersymmetry: the exact moduli space $\mathcal{M}_4$ of the uncompactified theory (including four-dimensional instantons) can be obtained using techniques introduced by Seiberg-Witten [26]. Upon compactification on a circle, the low-energy physics can be described by a non-linear sigma model with target space $\mathcal{M}_3$, given by a torus bundle over $\mathcal{M}_4$ [27]. In the large radius limit, the metric on $\mathcal{M}_3$ (schematically of the form (1.1), with the first and last term omitted) is invariant under continuous translations along the torus.

but instanton effects from $D = 4$ dyons whose worldline winds along the circle break these continuous translations to a discrete Abelian subgroup (in fact, similar instanton effects explain confinement in non-supersymmetric QCD in $D = 2 + 1$ [28]). The corrections are qualitatively of the same form as (1.2), where $\bar{\Omega}(\gamma, z^a)$ are closely related to the BPS index $\Omega(\gamma, z^a)$ of $D = 4$ dyons. In these lectures, we shall discuss the rigid case as well, mainly as a warm-up for the gravitational set-up.

1.3 Twistors, symplectomorphisms and wall-crossing

For $\mathcal{N} = 2$ theories, the advantages of the observable $\mathrm{d}s^2_{\mathcal{M}_3}$ come at a price: the metric on $\mathcal{M}_3$ is a tensorial object, and the metric components γ_{ij} are not gauge invariant. Moreover, supersymmetry requires that the exact metric on $\mathcal{M}_3$ is quaternion-Kähler, i.e. has reduced holonomy $USp(1) \times USp(n) \subset SO(4n)$ (for $\mathcal{N} = 2$ gauge theories discussed at the end of the last paragraph, $\mathcal{M}_3$ is hyperkähler, which makes little difference for what we are about to say). Unlike e.g. special Kähler manifolds, which can be described by a single holomorphic function (the prepotential), such metrics cannot be described by a simple unconstrained holomorphic function (in particular, $\mathcal{M}_3$ does not admit a global complex structure). Twistor techniques nevertheless allow to describe quaternion-Kähler manifolds analytically [29, 30, 31, 32]. The trick is to consider the twistor space $\mathcal{Z}$, a non-trivial $S^2 = \mathbb{C}P^1$ bundle over $\mathcal{M}_3$ which carries a canonical complex structure, and indeed a canonical complex contact structure. The latter can be represented by (the kernel of) the local one-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2 \ , \qquad p_\pm = -\frac{1}{2}(p_1 \mp \mathrm{i}p_2) \tag{1.4}$$

where $\vec{p}$ is the $SU(2) = USp(1)$ part of the Levici-Civita connection on $\mathcal{M}_3$ (the reader is encouraged to check that Dt transforms homogeneously under $SU(2)$ frame rotations). Locally, one can always choose Darboux coordinates $\xi^\Lambda_{[i]}, \tilde{\xi}_\Lambda^{[i]}, \alpha^{[i]}$ ($\Lambda = 0, \ldots, n-1$) on $\mathcal{U}_i \subset \mathcal{Z}_{\mathcal{M}}$ such that

$$2\, e^{\Phi_{[i]}} \frac{Dt}{\mathrm{i}t} = \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]}\, \mathrm{d}\tilde{\xi}_\Lambda^{[i]} \, . \tag{1.5}$$

The global complex structure on $\mathcal{Z}$ is encoded in transition functions between different Darboux coordinate systems on the overlap $\mathcal{U}_i \cap U_j$ which preserve (1.5), i.e. complex contact transformations. In fact, for the σ-independent corrections (1.2), one can show that the contact transformations are symplectomorphisms of the complexified torus $\mathcal{T}_\mathbb{C}$ parametrized by $\xi^\Lambda_{[i]}, \tilde{\xi}_\Lambda^{[i]}$. In particular, the instanton corrections (1.2) on $\mathcal{M}_3$ imply a set of symplectomorphisms on $\mathcal{Z}$ governed by the BPS invariants $\Omega(\gamma, z^a)$. The consistency of the complex contact structure on $\mathcal{Z}$ across walls of marginal stability leads to a wall-crossing formula for $\Omega(\gamma, z^a)$, initially obtained from purely mathematical reasoning by Kontsevich and Soibelman [9, 33]. Incorporating Kaluza-Klein monopole effects in this twistorial description is an outstanding open problem, to which we shall return below in a slight different set-up.

For $\mathcal{N} = 2$ gauge theories, the above discussion goes through by replacing quaternion-Kähler by hyperkähler, and complex contact structure by complex symplectic structure. The twistor space is now a trivial $\mathbb{C}P^1$ bundle over $\mathcal{M}_3$, and there are no Kaluza-Klein monopoles to worry about. In this case, the exact metric on $\mathcal{M}_3$ including all instanton effects is known [9], as we shall review in Section §2.

1.4 Wall-crossing from black hole halos

While the moduli space $\mathcal{M}_3$ of the reduced theory gives a beautiful physical realization of the Kontsevich-Soibelman (KS) wall-crossing formula, it is perhaps not the most economic derivation of the latter. The authors of [34] have given a new derivation of the KS formula based on the idea of supersymmetric galaxies (which furnish a supergravity analog of the framed BPS states of [35], which does not refer to the moduli space $\mathcal{M}_3$, but still uses much of the same underlying mathematics. A more elementary derivation, based on the quantum mechanics of multi-centered configurations, is as follows [8].

Recall that a wall of marginal stability is characterized by the existence of two charge vectors γ_1, γ_2 whose central charge align, $\arg Z(\gamma_1) = \arg Z(\gamma_2)$. As a result, multi-centered configurations whose constituents carry charge in the two-dimensional lattice spanned by γ_1 and γ_2 exist only on one side of the wall, and decay on the side. The jump in the index $\Delta\Omega(M\gamma_1 + N\gamma_2)$ across the wall is entirely due to the loss (or gain) of such multi-centered configurations, and so is equal to the index of such configurations. Near the wall, the centers are far apart, and this index naively reduces to the index of the supersymmetric quantum mechanics of the centers (interacting by Coulomb and Newton forces, as well as scalar exchange), multiplied by the product of the indices $\Omega(\gamma_i)$ carried by each center. When some of the charges of the constituents coincide however, this prescription is not quite correct, since the centers obey Bose or Fermi statistics (depending on the sign of their index) and their wave-functions must be appropriately (anti)symmetrized. As we shall see in §3, it turns out that one can treat all centers as distinguishable (hence obeying Boltzmann statistics) provided one replaces the integer-valued index $\Omega(\gamma_i)$ carried by each center by an effective, rational valied index $\bar{\Omega}(\gamma_i)$ defined by

$$\bar{\Omega}(\gamma) = \sum_{m|\gamma} \Omega(\gamma/d)/m^2 \ . \tag{1.6}$$

This rational invariant is fact the same that determines the instanton corrections (1.2), and also enters in constructions of modular invariant black hole partition functions [36]. It is also reminiscent of the multi-covering formula for Gromov-Witten instantons [37], and is indeed related to it by S-duality (see below).

Having taken care of statistics by this replacement, one may now compute the jump in the index by quantizing the classical phase space of multi-centered solutions, in the spirit of [38, 39]. We shall present two equivalent ways of performing this quantization: the first one, based on quiver quantum mechanics with Abelian gauge groups, leads to the "Higgs branch" (HB) formula for $\Delta\bar{\Omega}(M\gamma_1 + N\gamma_2)$. The second, based on evaluating the symplectic volume of the classical phase by localization, leads to the "Coulomb branch" (CB) formula. Both formulae are completely explicit, and allow to express the jump $\Delta\bar{\Omega}(M\gamma_1 + N\gamma_2)$ in terms of $\bar{\Omega}(M\gamma_1 + N\gamma_2)$ on one side of the wall. They are also similar in form to the formula given by Joyce and Song (JS) for the variation of generalized Donaldson-Thomas invariants of coherent sheaves [40]. The KS, JS, HB and CB formulae agree in all cases that we have checked, though a rigorous proof of their equivalence would be highly desirable.

1.5 Hypermultiplet moduli spaces

We now return to type II string vacua on $\mathbb{R}^3 \times S^1 \times \mathcal{X}$. Due to the decoupling of vectors and hypermultiplets in $\mathcal{N} = 2$ supergravity, the moduli space of scalar fields in four dimensions is actually a product $\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ is a special Kähler manifold parametrized by the vector multiplet scalars, while $\mathcal{M}_H$ is a quaternion-Kähler manifold parametrized by the hypermultiplet scalars. In type IIA, $\mathcal{M}_V$ describes the complexified Kähler structure on $\mathcal{X}$, while $\mathcal{M}_H$ describes the complex structure of $\mathcal{X}$, together with the periods of the RR 3-form field C on 3-cycles $\gamma \in H_3(\mathcal{X}, \mathbb{Z})$, the 4D string coupling g_4 and the NS axion ψ. In type IIB, the situation is reversed.

Upon compactification on a circle, $\mathcal{M}_H$ goes along for the ride, and the moduli space $\mathcal{M}_3$ also decomposes as

$$\mathcal{M}_3 = \widetilde{\mathcal{M}}_V \times \mathcal{M}_H \ . \tag{1.7}$$

In particular, the scalars $R, \zeta, \tilde{\zeta}, \sigma$ appearing in (1.1) are all part of $\widetilde{\mathcal{M}}_V$, and similarly the instanton corrections (1.2),(1.3) only affect the metric on $\widetilde{\mathcal{M}}_V$. As is well known however, T-duality along the circle exchanges the two factors in (1.7), while mapping type IIA to type IIB [25]. Upon decompactifying the dual circle, we therefore conclude that the VM moduli space $\widetilde{\mathcal{M}}_V$ in type IIA (respectively, type IIB) on $\mathbb{R}^3 \times S^1 \times \mathcal{X}$ is isomorphic to the HM moduli space $\mathcal{M}_H$ in type IIB (respectively, type IIA) on $\mathbb{R}^4 \times \mathcal{X}$. Under this equivalence, the radius R on the VM side is mapped to the 4D string coupling $1/g_4$, while the NUT potential σ is mapped to the NS axion ψ. In particular, the instanton corrections (1.2), originally interpreted as Dp-branes wrapped on a p-cycle γ times S^1, are now interpreted as 4D instantons, obtained by wrapping a D$(p-1)$-brane on the same cycle γ. Thus, the problem of computing the BPS spectrum in type IIA (say) on a Calabi-Yau threefold $\mathcal{X}$ is equivalent to that of computing D-instanton corrections to the HM moduli space in type IIB on the same Calabi-Yau threefold $\mathcal{X}$! In particular, the $SL(2,\mathbb{Z})$ modular symmetry of type IIA/S^1=M-theory/T^2 is reinterpreted as the S-duality of type IIB string theory. Moreover, Kaluza-Klein monopole effects on the type IIA side are mapped to NS5-brane instanton corrections on the type IIB side. S-duality mixes these effects with D5-brane instantons, in the same way as modular invariance mixes D6-branes with KK monopoles.

At this point, it may appear that the most appropriate way of computing the BPS spectrum in $\mathcal{N} = 2$ string vacua would be to use heterotic-type IIA duality (whenever available, i.e. when $\mathcal{X}$ is a K3-fibration), and determine the HM moduli space exactly. Indeed, on the heterotic side, the HM moduli space is in principle entirely computable in terms of the $(0,4)$ superconformal sigma model on the heterotic worldsheet, since the heterotic dilaton is, unlike in type II theories, a vector multiplet. Unfortunately, our current understanding of this SCFT is rather limited, and the type II (VM or HM) description remains the most useful one, especially in view of the power of mirror symmetry for CY compactifications. Using insights from mirror symmetry, S-duality and twistor techniques, we shall make some headway in obtaining the NS5-brane corrections to the HM moduli space in $D = 4$, or equivalently, the Kaluza-Klein monopole corrections to the VM moduli space in $D = 3$.

1.6 Outline

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2. Seiberg-Witten theories on $\mathbb{R}^3 \times S^1$

2.1 The 4D effective action and BPS spectrum

We consider a gauge theory on $\mathbb{R}^4$ with $\mathcal{N} = 2$ supersymmetry, gauge group G of rank r, and a dynamical scale Λ. The space of vacua admits a Coulomb branch, where the gauge group is broken to $U(1)^r$ (and possibly Higgs or mixed branches, depending on the matter content). The low-energy dynamics on the Coulomb branch is described by a non-linear sigma model on a complex r-dimensional manifold $\mathcal{M}_4$, which parametrizes the vevs of the vector multiplet (VM) scalars, coupled to r Abelian gauge fields and fermions. We shall denote by $u^a, a = 1 \dots r$ a generic system of complex coordinates on $\mathcal{M}_4$. Supersymmetry requires that $\mathcal{M}_4$ is a (rigid) special Kähler manifold, i.e. a complex manifold equipped with a rank $2r$ vector bundle with structure group $Sp(r, \mathbb{Z})$ and a holomorphic symplectic *Lagrangian* section $\Omega(u) = (X^\Lambda(u), F_\Lambda(u))$ such that the Kähler metric on $\mathcal{M}_4$ is given by the norm of the section,

$$\mathrm{d}s^2_{\mathcal{M}_4} = \partial_{u^a} \bar{\partial}_{\bar{u}^{\bar{b}}} K \, \mathrm{d}u^a \, \mathrm{d}\bar{u}^{\bar{b}} \ , \qquad K = \mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) \ . \tag{2.1}$$

The adjective *Lagrangian* above means that the section $\Omega(u)$ must embed $\mathcal{M}_4$ into a Lagrangian subspace of $\mathbb{C}^{2r}$, i.e. that the two-form $\mathrm{d}X^\Lambda \wedge \mathrm{d}F_\Lambda$ on $\mathcal{M}_4$ must vanish. This means that, locally, possibly after a symplectic rotation, the entries X^Λ of Ω can be taken as complex coordinates on $\mathcal{M}_4$, while the entries F_Λ can be obtained as derivatives $F_\Lambda = \partial_{X^\Lambda} F$ of a local function $F(X^\Lambda)$, called the prepotential. The complete $\mathcal{N} = 2$ supersymmetric effective action at two-derivative order is encoded in the section Ω, in particular the bosonic part is

$$S = \int \partial_{u^a} \bar{\partial}_{\bar{u}^{\bar{b}}} K \, \mathrm{d}u^a \ \wedge \star \mathrm{d}\bar{u}^{\bar{b}} + \frac{1}{4\pi} \operatorname{Im} \tau_{\Lambda\Sigma} \mathcal{F}^\Lambda \wedge \star \mathcal{F}^\Sigma + \frac{1}{4\pi} \operatorname{Re} \tau_{\Lambda\Sigma} \mathcal{F}^\Lambda \wedge \mathcal{F}^\Sigma \tag{2.2}$$

where $\mathcal{F}^\Lambda$ are the r Maxwell fields and $\tau_{\Lambda\Sigma} = \partial_{X^\Lambda} \partial_{X^\Sigma} F$. A symplectic rotation of the section Ω amounts to an electric-magnetic duality rotation of the gauge fields $(\mathcal{F}^\Lambda, \star \operatorname{Im} \tau_{\Lambda\Sigma} \mathcal{F}^\Sigma)$.

The prepotential F can be computed perturbatively at weak coupling (i.e. when all the vev's are much larger than Λ), and receives no perturbative correction beyond one-loop []. This perturbative result is clearly not exact, since it leads to non-positive definite kinetic terms $\operatorname{Im} \tau_{\Lambda\Sigma}$ for vev's of order Λ. In [26, 41], it was shown how to compute the exact two-derivative action (2.2), including all instanton corrections. The answer involves a family of Riemann surfaces Σ_u, fibered over $\mathcal{M}_4$, and a certain holomorphic one-form λ, such that the section Ω is given by periods of λ on a symplectic basis of $H_1(\Sigma_u, \mathbb{Z})$:

$$X^\Lambda = \int_{A^\Lambda} \lambda \ , \qquad F_\Lambda = \int_{B^\Lambda} \lambda \ . \tag{2.3}$$

For example, for $G = SU(2)$ with no flavor, Σ_u is the family of elliptic curves[1]

$$y^2 = (x^2 - u)^2 - \Lambda^4 \ , \qquad \lambda = \mathrm{d}x/y \ . \tag{2.4}$$

Setting $\Lambda = 1$ and denoting $a = X^1$, $a_D = F_1$, for brevity, one arrives at [42]

$$a = \frac{1}{1+i}(1-u^2)^{1/4}\, F\left(-\frac{1}{4}, \frac{3}{4}, 1; \frac{1}{1-u^2}\right) \ , \quad a_D = \frac{\mathrm{i}}{4}(u^2-1)\, F\left(\frac{3}{4}, \frac{3}{4}, 2; 1-u^2\right) \ , \tag{2.5}$$

leading to the prepotential

$$F = \frac{\mathrm{i}a^2}{2\pi}\left(2\log\frac{a^2}{\Lambda^2} - \sum_{k=0}^{\infty} F_k\, a^{-4k}\right) \tag{2.6}$$

with $F_1 = 1/2^5, F_2 = 5/2^{14}, \ldots$. The terms F_k, $k \geq 1$ correspond to non-perturbative corrections from k 't Hooft-Polyakov instantons.

The moduli space $\mathcal{M}_4$ is singular at $u = \pm 1$ and $u = \infty$, where the Riemann surface degenerates. Around these points, the section Ω experiences monodromies

$$M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \ , \qquad M_{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \ , \qquad M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \ . \tag{2.7}$$

These are elements in $\Gamma_0(4) \subset SL(2, \mathbb{Z}) = Sp(1, \mathbb{Z})$, satisfying $M_1 M_{-1} M_\infty = 1$. The singularity at $u = \infty$ originates from the one-loop logarithm, while the singularities at $u = 1$ and $u = -1$ correspond to BPS particles of charges $(q, p) = \pm(0, 1)$ and $\pm(2, -1)$ becoming massless. Indeed, the mass of a BPS particle of charge $(q, p \in \Gamma)$, where $\Gamma = H_1(\Sigma_u, \mathbb{Z})$ is the charge lattice, is given by the absolute value of the central charge,

$$\mathcal{M} = |Z(\gamma; u)| \ , \qquad Z(\gamma; u) = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda \ , \tag{2.8}$$

which vanishes at $u = \pm 1$ for the above choice of charges. The monodromy around a singularity where a particle of charge (q, p) becomes massless is, in general,

$$M = \begin{pmatrix} 1 + pq & q^2 \\ -p^2 & 1 - pq \end{pmatrix} \ . \tag{2.9}$$

In general, the BPS spectrum may be a complicated function of the moduli and parameters of the Lagrangian. For one thing, short multiplets of bosonic and fermionic type may pair up and leave the BPS spectrum (an example of this is the Higgs mechanism, where a vector multiplet and charged hypermultiplet pair up and leave the spectrum). This problem may be evaded by consider the index

$$\Omega(\gamma) = -\frac{1}{2}\,\mathrm{Tr}\,(-1)^{2J_3}\, J_3^2 \ . \tag{2.10}$$

which is immune to this phenomenon, since a bosonic (half-hypermultiplet) BPS state contributes $\Omega = +1$ while a fermionic (vector-multiplet) BPS state contributes $\Omega = -2$. In

[1] Here we use the family of curves constructed in [41], which differs from that of [26] by an isogeny.

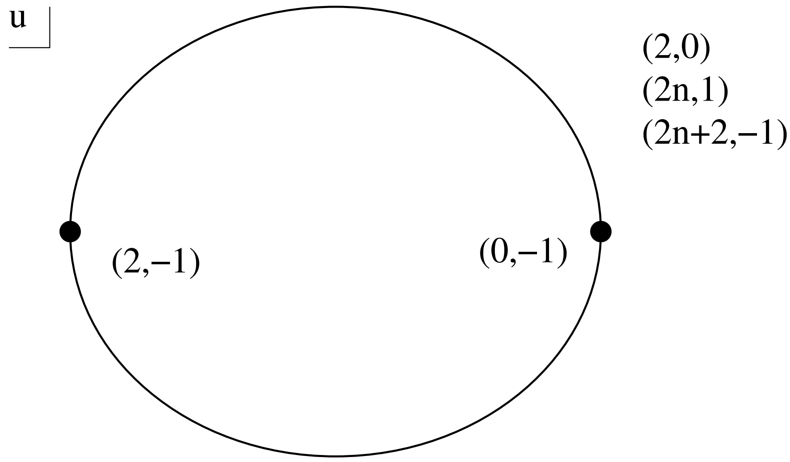

Figure 2: Chamber structure of the u-plane and BPS spectrum in $\mathcal{N}=2, D=4$ SYM theory with $SU(2)$ gauge group. The line $\operatorname{Im}(a/a_D)=0$ separates the strong and weak coupling chambers. The only stable BPS states in the strong coupling chamber are the monopole and dyons with charges $(q,p)=\pm(0,1)$, $\pm(2,-1)$.

addition, a BPS bound state of charge γ may decay into n unbound BPS states with charges γ_i, such that $\gamma=\sum_i \gamma_i$. Since Z is linear in the charges, this is energetically allowed only when $Z(\gamma_i)$ all have the same phase as $Z(\gamma)$. In particular, for $n=2$ the locus

$$W(\gamma_1,\gamma_2)=\{u \;/\; \arg(Z(\gamma_1;u)=\arg(Z(\gamma_2;u)\} \tag{2.11}$$

defines a codimension-one wall in the space of moduli and parameters, known as a wall of marginal stability. Note that $W(\gamma_1,\gamma_2)$ depends only on the two-plane spanned by γ_1 and γ_2, and is invariant under a change of basis in that plane. Across the wall $W(\gamma_1,\gamma_2)$, the index $\Omega(\gamma)$ for all $\gamma=M\gamma_1+N\gamma_2$ will in general jump, due to the gain/loss of BPS bound states made of constituents with charges in the two-dimensional sublattice spanned by γ_1,γ_2.

In the rank one case, the answer is quite simple. There is only one wall of marginal stability, given by the equation $\operatorname{Im}(a/a_D)=0$. This defines a closed (approximately elliptic) curve in the u plane, which passes through the singularities $u=1$ and $u=-1$, and separates two chambers, the 'weak coupling' (outer) and 'strong coupling' (inner) regions (see Figure 2). Since the monopole $\pm(0,1)$ and dyon $\pm(2,-1)$ become massless on the curve, they must be part of the BPS spectrum in both chambers. In the weak coupling region, the images $\pm(2n,1)$,$\pm(2n+2,-1)$, $n\in\mathbb{Z}$ of these states under the monodromy M_∞ must also belong to the BPS spectrum. Finally, the W-boson, a fermionic multiplet with $\Omega=-2$ and charge $\pm(2,0)$, must also belong to the weak coupling spectrum. It may be shown that these are in fact the only BPS states in the weak coupling region, while only BPS states in the strong coupling region are the monopole and dyon [26, 43]. We shall see in (2.56) below that this is consistent with the Kontsevich-Soibelman wall-crossing formula.

2.2 Circle reduction, semi-flat metric and rigid c-map

Under compactification on a circle of radius R, the geometry of the moduli space $\mathcal{M}_4$ and the BPS spectrum become parts of a more elaborate geometrical structure, the hyperkähler moduli space $\mathcal{M}_3$.

To see how this arises, consider first the dimensional reduction of the action (2.2) on $\mathbb{R}^3 \times S^1$: Decomposing the four-dimensional gauge potentials into $A^\Lambda = \zeta^\Lambda \mathrm{d}x^3 + A_3^\Lambda$, where x^3 is the coordinate along S^1 and dualizing the three-dimensional gauge potential A_3^Λ into a scalar field $\tilde\zeta_\Lambda$, we arrive at a non-linear sigma model on a space $\mathcal{M}_3$ of real dimension $4r$, with metric

$$\mathrm{d}s^2 = R\,\mathrm{d}s^2_{\mathcal{M}_4} + \frac{1}{R}\left(\mathrm{d}\tilde\zeta_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Sigma\right)[\operatorname{Im}\tau]^{\Lambda\Lambda'}\left(\mathrm{d}\tilde\zeta_{\Lambda'} - \tau_{\Lambda'\Sigma'}\zeta^{\Sigma'}\right)\,, \tag{2.12}$$

where $[\operatorname{Im}\tau]^{\Lambda\Lambda'}$ denotes the inverse of the matrix $[\operatorname{Im}\tau]_{\Lambda\Lambda'}$. Large gauge transformations along the circle imply that the coordinates $(\zeta^\Lambda, \tilde\zeta_\Sigma)$ are periodic with unit period. In fact, the second term in (2.12) is recognized, up to an overall factor, as the Kähler metric on the Jacobian torus $\mathcal{T}_u = H^1(\Sigma,\mathbb{R})/H^1(\Sigma_u,\mathbb{Z})$ of the Riemann surface Σ_u. Due to the flatness of the metric on this torus, the metric (2.12) is sometimes known as the 'semi-flat' metric.

The metric (2.12) is an accurate description of the dynamics of the compactified gauge theory at large radius R. In particular, it is hyperkähler, consistently with $\mathcal{N}=4$ supersymmetry in three-dimensions. In fact, it is an example of the "rigid c-map" construction [44], which produces a hyperkähler manifold of real dimension $4r$ out of a rigid special Kähler manifold of real dimension $2r$. To see this, note that the metric (2.12) derives from the Kähler potential

$$K_C = \mathrm{i}R(X^\Lambda \bar F_\Lambda - \bar X^\Lambda F_\Lambda) + \frac{1}{R}(W_\Lambda + \bar W_\Lambda)[\operatorname{Im}\tau]^{\Lambda\Sigma}(W_\Sigma + \bar W_\Sigma) \tag{2.13}$$

in complex coordinates X^Λ and

$$W_\Lambda = \mathrm{i}\left(\tilde\zeta_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Lambda\right)\,. \tag{2.14}$$

Moreover, (2.12) admits the holomorphic symplectic form

$$\Omega_C = \mathrm{d}W_\Lambda\,\mathrm{d}X^\Lambda \tag{2.15}$$

It is straightforward to check that the complex structures J_1, J_2, J_3 obtained from the two-forms $\operatorname{Re}(\Omega)$, $\operatorname{Im}(\Omega)$ and $\omega_3 \equiv \partial\bar\partial K_C$ by raising one index using the metric (2.12), satisfy the quaternion algebra

$$J_i\,J_j = -\delta_{ij} + \epsilon_{ijk}J_k\,. \tag{2.16}$$

This proves that the semi-flat metric (2.12) is hyperkähler. We shall elaborate on the hyperkähler structure of (2.12) in §2.5 below.

2.3 Circle compactification and electric instantons

At finite radius however, the semi-flat metric (2.12) must be corrected by instanton corrections, coming from BPS states in the four-dimensional theory whose Euclidean worldline winds around the torus [27]. These instanton corrections to the two-derivative action are the analog of the instanton corrections to the potential in pure (non-supersymmetric) Yang-Mills theory in 2+1 dimensions, responsible for confinement [28]. Qualitatively, these corrections are expected to be of the form

$$\delta\mathrm{d}s^2 \sim \Omega(\gamma;u)\,e^{-2\pi R|Z(\gamma,z^a)| + 2\pi\mathrm{i}(p^\Lambda\tilde\zeta_\Lambda - q_\Lambda\zeta^\Lambda)}\,. \tag{2.17}$$

To see this, we start by considering instanton corrections from purely electric BPS states in $D = 4$, namely from the W-boson.

For a $U(1)$ theory on $\mathbb{R}^3 \times S^1$ with a single hypermultiplet of electric charge q, the metric on $\mathcal{M}_3$ can be determined exactly using symmetry considerations. The metric falls into the Gibbons-Hawking ansatz for 4D hyperkähler spaces with one tri-holomorphic isometry,

$$ds^2_{\mathcal{M}_3} = V(\vec{x})\,\mathrm{d}\vec{x}^2 + \frac{1}{V(\vec{x})}\left(\mathrm{d}\tilde{\zeta} + A\right)^2 , \tag{2.18}$$

where $\vec{x} = (\,\mathrm{Re}\,a,\, \mathrm{Im}\,a, \zeta/R)$ are coordinates on $\mathbb{R}^2 \times S^1$ with flat metric. The metric (2.18) is hyperkähler provided V is harmonic (possibly with singularities) and $\mathrm{d}A = \star \mathrm{d}V$. The potential V is given by a one-loop integral of the hypermultiplet on S^1 [45, 25],

$$V \sim 2q^2 \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\left(\vec{k}^2 + \frac{1}{R^2}(m+q\zeta)^2 + q^2|a|^2\right)^2} , \tag{2.19}$$

up to an irrelevant infinite additive constant. Thus,

$$V = \frac{q^2 R}{4\pi} \sum_{m=-\infty}^{\infty} \left(\frac{1}{\sqrt{q^2R^2|a|^2 + (q\zeta+m)^2}} - \kappa_m \right) , \tag{2.20}$$

where the constants κ_m are chosen such that the sum is convergent, $\kappa_m \sim 1/|m|$ as $|m| \to \infty$. To study the behavior in the large $|a|$ regime, one must Poisson-resum over m, leading to

$$V = -\frac{q^2 R}{4\pi} \log|a/\Lambda|^2 + \frac{q^2R}{2\pi} \sum_{n\neq 0} K_0(2\pi R|nqa|)\, e^{2\pi \mathrm{i} n q\zeta} , \tag{2.21}$$

where Λ depends on the constant in (2.20). The connection A is

$$A = \frac{\mathrm{i}q^2}{4\pi}\left(\log\frac{a\bar{\Lambda}}{\bar{a}\Lambda}\right)\mathrm{d}\zeta - \frac{q^2R}{4\pi}\left(\mathrm{d}\log\frac{a}{\bar{a}}\right) \sum_{n\neq 0} \mathrm{sgn}(n) K_1(2\pi R|nqa|)\, e^{2\pi \mathrm{i} n q\zeta} \tag{2.22}$$

Keeping only the first term in (2.21) and (2.22), one recovers the semi-flat metric (2.12), with

$$\tau(a) = \frac{q^2}{2\pi \mathrm{i}} \log(a/\Lambda) . \tag{2.23}$$

Since the modified Bessel function behaves as $K_s(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$ as $z \to 0$, the terms with $n \neq 0$ in (2.21) and (2.22) are non-perturbative corrections of the form (2.17), with $p^\Lambda = 0$. Note that unlike the semi-flat metric, which was singular in codimension 2 (namely at $a = \bar{a} = 0$), for $|q| \neq 1$ the exact metric has singularities in codimension 3, at $a = \bar{a} = 0, q\zeta \in \mathbb{Z}$, locally of the form $\mathbb{C}^2/\mathbb{Z}_q$. For $q = 1$, the exact metric is completely regular.

In general, the metric on $\mathcal{M}_3$ involves corrections from all solitons with arbitrary electric and magnetic charges. It is difficult to compute these corrections from first principles, since there is in general no choice of frame where all charges would be purely electric. Nevertheless, we shall see that wall-crossing constraints suggest a natural prescription to incorporate these corrections, which reduces to the above in the case where all charges are electric.

2.4 Twistor techniques for hyperkähler manifolds

- By definition, a HK manifold is a Riemannian manifold whose metric is hermitian with respect to three complex structures J^i satisfying the quaternion algebra (2.16). It follows that $\mathcal{M}$ is $4n$-dimensional, has reduced holonomy $USp(n) \subset SO(4n)$, and carries a family of complex structures

$$J(t,\bar t) = \frac{1-t\bar t}{1+t\bar t}\,J^3 + \frac{t+\bar t}{1+t\bar t}\,J^2 + \mathrm{i}\frac{t-\bar t}{1+t\bar t}\,J^1 \tag{2.24}$$

parametrized by $t \in \mathbb{C}P^1 = S^2$. This complex structure extends to a complex structure on the twistor space $\mathcal{Z}_{\mathcal{M}} = \mathbb{C}P^1 \times \mathcal{M}$.

- Using the hyperkähler metric on $\mathcal{Z}$, one obtains a triplet of Kähler forms ω^i. The complex two-form

$$\Omega^{[0]}(t) = \omega^+ - \mathrm{i}t\,\omega^3 + t^2\,\omega^- \,, \tag{2.25}$$

with $\omega^\pm = -\frac{1}{2}(\omega^1 \mp \mathrm{i}\omega^2)$ is of holomorphic w.r.t. to the complex structure $J(t,\bar t)$. $\Omega^{[0]}$ is regular at $t=0$, but has a pole at $t=\infty$. Since it is only defined up to overall factor, one may instead consider

$$\Omega^{[\infty]}(t) \equiv t^{-2}\,\Omega^{[0]}(t) = \omega^- - \mathrm{i}\omega^3/t + \omega^+/t^2 \ , \tag{2.26}$$

Ω is then real w.r.t. to the antipodal map $t \mapsto -1/\bar t$, in the sense that

$$\Omega^{[\infty]}(t) = \overline{\Omega^{[0]}(-1/\bar t)} \tag{2.27}$$

More generally, one may introduce a covering of $\mathbb{C}P^1$ by open sets U_i, and a complex two-form $\Omega^{[i]}$, holomorphic on U_i, such that

$$\Omega^{[i]} = f_{ij}^2 \Omega^{[j]} \mod dt \ , \qquad \Omega^{[\bar i]}(t) = \overline{\Omega^{[i]}(-1/\bar t)} \tag{2.28}$$

where f_{ij} are the transition functions of the $\mathcal{O}(1)$ bundle on $\mathbb{C}P^1$. The knowledge of $\Omega(\zeta)$ allows to reconstruct the HK metric by expanding around $t=0$ (or any other point) [46].

- Locally, one can choose **complex Darboux coordinates** $\nu^\Lambda_{[i]}(t)$ and $\mu_\Lambda^{[i]}(t)$ on $\mathcal{Z}_{\mathcal{S}}$, **regular** in patch U_i, such that $\Omega^{[i]} = d\mu_\Lambda^{[i]} \wedge d\nu^\Lambda_{[i]}$. On the overlap of two patches $U_i \cap U_j$, they must be related by a complex symplectomorphism [1].,

$$\mu_\Lambda^{[i]} = \partial_{\nu^\Lambda_{[i]}} S^{[ij]} \ , \quad \nu^\Lambda_{[j]} = \partial_{\mu_\Lambda^{[j]}} S^{[ij]} \ , \quad S^{[ij]} = S^{[ij]}(\nu^\Lambda_{[i]}, \mu_\Lambda^{[j]}, t) \ . \tag{2.29}$$

On the overlap $U_i \cap U_j \cap U_k$, the symplectomorphisms $S^{[ij]}, S^{[jk]}, S^{[ik]}$ must of course compose in the appropriate way. The symplectomorphism $S^{[ij]}$ must also be conjugate to $S^{[\bar j \bar i]}$ under real conjugation. Finally, a set of symplectomorphisms $S^{[ij]}$ on $U_i \cap U_j$ related by local complex symplectomorphisms in each U_k lead to the same complex symplectic structure, therefore to the same HK metric.

- Any triholomorphic isometry of $\mathcal{S}$ yields a **triplet of moment maps** $\vec{\mu}_\kappa = (v, \bar{v}, x)$, such that $\kappa \cdot \vec{\omega} = d\vec{\mu}_\kappa$, or better real global holomorphic section $\eta \in H^0(\mathcal{Z}, \mathcal{O}(2))$:

$$\eta = \frac{v}{t} + x - \bar{v}t \tag{2.30}$$

- A $4d$-dimensional hyperkähler manifold $\mathcal{M}$ admitting d commuting tri-holomorphic isometries κ^Λ is called "toric hyperkähler ". In this case one may choose the d moment maps $\nu_{[i]} = f_{i0}^2 t\eta^\Lambda$ as "position" coordinates. $S^{[ij]}$ must now be of the form

$$S^{[ij]} = \nu^\Lambda_{[i]}\mu^{[j]}_\Lambda + H^{[ij]}(\eta^\Lambda_{[i]}, t) \tag{2.31}$$

such that, on $U_i \cap U_j$,

$$\mu^{[i]}_\Lambda - \mu^{[j]}_\Lambda = \partial_{\eta^\Lambda} H^{[ij]} \tag{2.32}$$

The general solution to these gluing conditions is

$$\mu^{[i]}_\Lambda(t) = \rho_\Lambda + \sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}\, t'}\, \frac{t+t'}{2(t'-t)}\, \partial_{\eta^\Lambda} H^{[0j]}(t'), \tag{2.33}$$

where $\kappa^\Lambda = \partial_{\rho_\Lambda}$ generate the tri-holomorphic isometries. The Kähler potential K of the hyperkähler metric then arises by Legendre transform from the "tensor Lagrangian" $\mathcal{L}$, a function of $3d$ variables [47, 46],

$$\mathcal{L}(x^\Lambda, v^\Lambda, \bar{v}^\Lambda) = \oint \frac{dt}{2\pi\mathrm{i}t} H(\eta^\Lambda, t)\,, \qquad K(v^\Lambda, \bar{v}^\Lambda, w_\Lambda, \bar{w}_\Lambda) = \langle \mathcal{L} - x^\Lambda(w_\Lambda + \bar{w}_\Lambda)\rangle_{x^\Lambda} \tag{2.34}$$

The local holomorphic function $H^{[ij]}$ is sometimes called the **generalized prepotential**. E.g. $H = \eta^2/t + m\,\eta\log\eta$ produces Taub-NUT space with mass parameter m.

- Perturbations away from toric hyperkähler metrics are described by the variations $H^{[ij]}_{(1)}(\nu_{[i]}, \mu^{[j]}, t)$ of the generating function $S^{[ij]}$ which preserve the consistency conditions on triple overlaps, reality conditions and are defined up to local symplectomorphisms. As such, they correspond to holomorphic sections of $H^1(\mathcal{Z}, \mathcal{O}(2))$. Eg: the Atiyah-Hitchin manifold is a deformation of Taub-NUT generated, at linear order in the large radius limit, by $H_{(1)} = \eta\, e^\mu$ [1].

2.5 The rigid c-map in twistor space

The semi-flat hyperkähler metric (2.12) admits a tri-holomorphic isometric action of the torus T^{2r}, corresponding to the vector fields ∂_{ζ^Λ} and $\partial_{\tilde{\zeta}_\Lambda}$. The corresponding moment maps are

$$\xi^\Lambda = \zeta^\Lambda + \frac{\mathrm{i}R}{2}\left(t\,\bar{X}^\Lambda - t^{-1}\,X^\Lambda\right) \tag{2.35}$$

$$\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + \frac{\mathrm{i}R}{2}\left(t\,\bar{F}_\Lambda - t^{-1}\,F_\Lambda\right) \tag{2.36}$$

Importantly, ξ^Λ and μ_Λ provide regular Darboux coordinates on an open patch $\mathcal{U}_1$ around the equator,

$$\mathrm{d}\xi^\Lambda \wedge \mathrm{d}\tilde{\xi}_\Lambda = t^{-1}\,\Omega^{[0]}(t)\,. \tag{2.37}$$

The combinations

$$\nu^\Lambda_{[0]} = t\xi^\Lambda \ , \qquad \tilde{\xi}^{[0]}_\Lambda = \tilde{\xi}_\Lambda + \frac{\mathrm{i}R}{2t} F_\Lambda \left(\frac{2t}{\mathrm{i}R} \xi^\Lambda \right) \tag{2.38}$$

$$\nu^\Lambda_{[\infty]} = \xi^\Lambda / t \ , \qquad \tilde{\xi}^{[\infty]}_\Lambda = \tilde{\xi}_\Lambda - \frac{\mathrm{i}t}{2} \bar{F}_\Lambda \left(-\frac{2}{\mathrm{i}tR} \xi^\Lambda \right) \tag{2.39}$$

are regular at the north and south pole, respectively, and can be chosen as Darboux coordinates in $\mathcal{U}_0$ and $\mathcal{U}_\infty$. The transition functions from $\mathcal{U}_1$ to $\mathcal{U}_0$ and $\mathcal{U}_\infty$ are given by

$$H^{[10]} = -\frac{R^2}{4t^2} F \left(\frac{2t}{\mathrm{i}R} \xi^\Lambda \right) \ , \qquad H^{[1\infty]} = \frac{R^2}{4t^2} \bar{F} \left(\frac{2}{\mathrm{i}tR} \xi^\Lambda \right) \ , \tag{2.40}$$

where $F(X)$ is the prepotential describing the special Kähler metric on $\mathcal{M}_4$. Indeed, the rigid c-map can be obtained using the Legendre transform construction from the tensor Lagrangian (after suitabler rescalings) [48, 49]

$$\mathcal{L} = \mathrm{Im} \oint \frac{dt}{2\pi \mathrm{i} t} \frac{F\left(t\xi^\Lambda\right)}{t^2} \ . \tag{2.41}$$

2.6 The Ooguri-Vafa metric in twistor space

While the semi-flat metric (2.12) admitted $2r$ tri-holomorphic isometries, the effect of electric instantons is to break r of those to a discrete subgroup. The remaining r commuting isometries nevertheless ensure that the metric can be obtained from the Legendre transform construction. The Darboux coordinates $\nu^\Lambda = t\xi^\Lambda$ can still be chosen as the moment maps for the $U(1)^r$ action generated by $\partial_{\tilde{\zeta}_\Lambda}$:

$$\xi^\Lambda = \zeta^\Lambda + \frac{\mathrm{i}R}{2} \left(t\, \bar{X}^\Lambda - t^{-1}\, X^\Lambda \right) \tag{2.42}$$

The coordinates $\tilde{\xi}_\Lambda$ around the equator however differ from the semi-flat result (2.35) by instanton corrections. By analyzing the Ooguri-Vafa metric (2.18), one finds [9]

$$\begin{aligned} \tilde{\xi}_\Lambda =& \tilde{\zeta}_\Lambda + \frac{\mathrm{i}}{2} \left(t\, \bar{F}_\Lambda - t^{-1}\, F_\Lambda \right) \\ &+ \frac{\mathrm{i}q_\Lambda}{4\pi} \int_{\ell_+} \frac{\mathrm{d}t'}{t'} \frac{t'+t}{t'-t} \log\left(1 - e^{2\pi \mathrm{i} q_\Lambda \xi^\Lambda}\right) - \frac{\mathrm{i}q_\Lambda}{4\pi} \int_{\ell_-} \frac{\mathrm{d}t'}{t'} \frac{t'+t}{t'-t} \log\left(1 - e^{-2\pi \mathrm{i} q_\Lambda \xi^\Lambda}\right) \ , \end{aligned} \tag{2.43}$$

where $\ell_\pm$ are the BPS rays $\ell_\pm = \{ t \ : \ a/t \in \pm \mathrm{i}\mathbb{R}^+ \}$ (or small deformations thereof). In particular, the coordinates $\tilde{\xi}_\Lambda$ are discontinuous across the BPS rays, with discontinuity

$$\Delta_{\ell\pm}\, \tilde{\xi}_\Lambda = \mathrm{i}q_\Lambda \, \log\left(1 - e^{\pm 2\pi \mathrm{i} q_\Lambda \xi^\Lambda}\right) \ . \tag{2.44}$$

This is a symplectic transformation generated by

$$H^{[\ell\pm]} = \frac{1}{2\pi} \, \mathrm{Li}_2 \left(e^{\pm 2\pi \mathrm{i} q_\Lambda \xi^\Lambda} \right) \tag{2.45}$$

where $\mathrm{Li}_2(z) \equiv \sum_{n=1}^\infty z^n/n^2$ is the dilogarithm function. Eq. (2.43) is then a special case of the general formula (2.33) obtained earlier in [1].

2.7 Instanton corrections from mutually non-local solitons

In general, the metric on $\mathcal{M}_3$ involves corrections from all BPS solitons in $D = 4$ with arbitrary electric and magnetic charges. It is difficult to compute these corrections from first principles, since there is in general no choice of frame where all charges would be purely electric (see however the next subsection). Nevertheless, one may try to covariantize the twistor construction for electric instantons given above under electric-magnetic duality, as follows. For a given charge vector $\gamma = (p^\Lambda, q_\Lambda)$, at a fixed point in $\mathcal{M}_4$, consider the BPS ray

$$\ell(\gamma) = \{t \ : Z(\gamma; u^a)/t \in \mathrm{i}\mathbb{R}^-\} \ . \tag{2.46}$$

Following [9], we postulate that across the BPS ray $\ell(\gamma)$, the Darboux coordinates ξ^Λ and $\tilde{\xi}_\Lambda$ are related by a symplectomorphism generated by

$$H^{[\ell(\gamma)]} = \frac{1}{2\pi}\sigma(\gamma)\,\Omega(\gamma, u)\ \mathrm{Li}_2\left(\mathcal{X}_\gamma\right)\ , \qquad \mathcal{X}_\gamma \equiv e^{2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde{\xi}_\Lambda)}\ , \tag{2.47}$$

where $\sigma(\gamma)$ is a quadratic refinement of the symplectic pairing on the charge lattice, i.e. a function $\sigma : \Gamma \to U(1)$ which satisfies

$$\sigma(\gamma + \gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\,\sigma(\gamma)\,\sigma(\gamma')\,. \tag{2.48}$$

The most general solution to this condition is

$$\sigma(\gamma) = e^{-\mathrm{i}\pi p^\Lambda q_\Lambda + 2\pi\mathrm{i}(q_\Lambda\theta^\Lambda - p^\Lambda\phi_\Lambda)} \tag{2.49}$$

where $(\theta^\Lambda, \phi_\Lambda) \in (\Gamma\otimes\mathbb{R})/\Gamma$ are "characteristics". They can be absorbed in a shift $(\zeta, \tilde{\zeta}) \mapsto (\zeta - \theta, \tilde{\zeta} - \phi)$, however the sign $e^{-\mathrm{i}\pi p^\Lambda q_\Lambda} = \pm 1$ cannot be removed and plays a crucial role in ensuring consistency with wall-crossing, as we discuss momentarily. For $p^\Lambda = 0$ and $\theta^\Lambda = 0$, (2.47) reduces to (2.45). As a result, the functions $\mathcal{X}_\gamma$ satisfy the integral equations, for all γ,

$$\mathcal{X}_\gamma = \mathcal{X}^{\mathrm{sf}}_\gamma \exp\left[-\frac{1}{2\pi\mathrm{i}}\sum_{\gamma'}\Omega(\gamma'; u)\,\langle\gamma, \gamma'\rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'}\frac{t'+t}{t'-t}\log\left(1 - \sigma(\gamma')\,\mathcal{X}_{\gamma'}(t')\right)\right] \tag{2.50}$$

where

$$\mathcal{X}^{\mathrm{sf}}_\gamma = \exp\left[2\pi\mathrm{i}(q_\Lambda\zeta^\Lambda - p^\Lambda\tilde{\zeta}_\Lambda) + \frac{\mathrm{i}R}{2}\left(t\,\bar{Z}_\gamma - t^{-1}\,Z_\gamma\right)\right] \tag{2.51}$$

These integral equations are similar to equations arising in the context of the Thermodynamic Bethe Ansatz [9, 50].

Now, as the point u^a in $\mathcal{M}_4$ is varied, the location of the BPS rays on the $\mathbb{C}P^1$ fiber changes. In particular, across a wall of marginal stability two BPS rays through each other. The metric on $\mathcal{M}_3$ should be smooth however, since singularities are only expected when some state becomes massless. In particular, the total discontinuity of the Darboux coordinates across the two colliding BPS rays should be the same on both sides of the wall. This is in turn determines the jump of BPS index $\Omega(\gamma; u^a)$ across the wall.

To express this in mathematical terms, note that the functions $\mathcal{X}_\gamma$ generate Hamiltonian vector fields (i.e. infinitesimal symplectomorphisms) $\delta_\gamma f = \{\mathcal{X}_\gamma, f\}$ of the complex torus

$\mathcal{T}_{\mathbb{C}}$ parametrized by $\xi^\Lambda, \tilde{\xi}_\Lambda$, where $\{\cdot,\cdot\}$ is the (complex) Poisson bracket associated to the (complex) symplectic form Ω on $\mathcal{T}_{\mathbb{C}}$. The Poisson algebra of the functions $\mathcal{X}_\gamma$

$$\{\mathcal{X}_\gamma, \mathcal{X}_{\gamma'}\} = \langle\gamma,\gamma'\rangle\, \mathcal{X}_{\gamma+\gamma'} \tag{2.52}$$

translates into the Lie algebra of the infinitesimal symplectomorphisms δ_γ,

$$[\delta_\gamma, \delta_{\gamma'}] = \langle\gamma,\gamma'\rangle\, \delta_{\gamma+\gamma'} \tag{2.53}$$

The symplectomorphism U_γ across the BPS ray $\ell(\gamma)$ is then

$$U_\gamma^{(\Omega(\gamma;t^a))}(t^a) \equiv \exp\left(\sigma(\gamma)\,\Omega(\gamma;t^a)\sum_{d=1}^{\infty}\frac{\delta_{d\gamma}}{d^2}\right) . \tag{2.54}$$

The consistency of the twistor construction across a wall of marginal stability then requires that the product

$$A_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} U_\gamma\ , \tag{2.55}$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. This wall-crossing formula was first established by Kontsevich and Soibelman in the context of Donaldson-Thomas invariants of Calabi-Yau categories. Indeed, it can be checked that the KS formula reproduces the change of the BPS spectrum in pure Seiberg-Witten $SU(2)$ theory (as first observed by Denef),

$$U_{2,-1}^{(1)}\cdot U_{0,1}^{(1)} = U_{0,1}^{(1)}\cdot U_{2,1}^{(1)}\cdot U_{4,1}^{(1)}\dots U_{2,0}^{(-2)}\dots U_{6,-1}^{(1)}\cdot U_{4,-1}^{(1)} U_{2,-1}^{(1)} \tag{2.56}$$

as well as in $SU(2)$ theory with $N_f = 1,2,3$ flavors [9, 51]. More generally, truncating the KS formula to the algebra spanned by the generators δ_{γ_1}, δ_{γ_2}, $\delta_{\gamma_1+\gamma_2}$, where γ_1 and γ_2 are primitive vectors, and using the BCH formula $e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$, the identity

$$U_{\gamma_1}^{(\Omega^+(\gamma_1))}\cdot U_{\gamma_1+\gamma_2}^{(\Omega^+(\gamma_1+\gamma_2))}\cdot U_{\gamma_2}^{(\Omega^+(\gamma_2))} = U_{\gamma_2}^{(\Omega^-(\gamma_2))}\cdot U_{\gamma_1+\gamma_2}^{(\Omega^-(\gamma_1+\gamma_2))}\cdot U_{\gamma_1}^{(\Omega^-(\gamma_1))} \tag{2.57}$$

implies the primitive wall-crossing formula

$$\Delta\Omega(\gamma\to\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\,\langle\gamma_1,\gamma_2\rangle\,\Omega^+(\gamma_1)\,\Omega^+(\gamma_2)\ , \tag{2.58}$$

where the prefactor on the r.h.s. is the index of the angular momentum degrees of freedom of a 2-centered black hole configuration. In order to obtain the correct sign in this relation, it is crucial to include the quadratic refinement $\sigma(\gamma)$.

Since the operators $U_{k\gamma}$ associated to multiples of a given charge vector γ commute with each other, they may be combined into a single factor

$$V_\gamma(t^a) \equiv \exp\left(\sum_k \sigma(k\gamma)\,\bar{\Omega}(k\gamma;t^a)\,\delta_{k\gamma}\right) \tag{2.59}$$

or, in terms of the transition functions (2.47),

$$H^{[\ell(\gamma)]} = \frac{1}{2\pi}\sigma(\gamma)\,\bar{\Omega}(\gamma,u)\,e^{2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde{\xi}_\Lambda)}\ , \tag{2.60}$$

for each (non-necessarily primitive) vector γ in the charge lattice. In that sense, the rational invariants are the ones which most naturally govern the instanton corrections.

2.8 A weak coupling test

To further test the validity of the above construction, one may compute the leading instanton correction to $\mathcal{M}_3$ in the weak coupling chamber [52, 53]. As usual, it is convenient to compute corrections to the four-fermion vertex, equal by supersymmetry to the Riemann tensor of the metric on $\mathcal{M}_3$. The instanton configuration of interest is a BPS monopole whose worldline winds around the circle. Its collective coordinates are the center of motion and angular "electric" coordinate ϕ (not to be confused with the circle coordinate x^3 !), parametrizing $\mathbb{R}^3 \times S^1$, and four fermions ψ corresponding to the supersymmetries broken by the monopole, with world-line action

$$\mathcal{L} = M + \frac{1}{2} M \left(\dot{\vec{x}}^2 + \frac{(\dot{\phi})^2}{|a|^2} + \psi\dot{\psi} \right) , \qquad M = \frac{4\pi}{g^2}|a| \ . \tag{2.61}$$

The path integral of the collective modes on a circle of radius R is then

$$\left[\sqrt{\frac{M}{2\pi(2\pi R)}} \right]^{-4+3} \sum_{n_e \in \mathbb{Z}} \exp\left[-\frac{1}{2}\frac{|a|^2}{M} \left(n_e + \frac{\theta}{2\pi} \right)^2 + \mathrm{i} n_e \zeta \right] , \tag{2.62}$$

coming from $\psi, \vec{x}, \phi$, respectively. The fluctuation determinant $\mathcal{R}$ in the monopole background can be computed using Callias index theorem, as first performed by Kaul in the context of the one-loop correction to the monopole mass [54]. We quote the result and refer to [52] for the derivation,

$$\log \mathcal{R} = -4R|a| \cosh^{-1}\frac{\Lambda}{|a|} - 2\log(2\pi R) + \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}t}{\cosh t} \log \left\| 1 - e^{-2\pi R|a|\cosh t + \mathrm{i}\zeta} \right\|^2 \ . \tag{2.63}$$

The divergent term can be absorbed in a renormalization of the bare monopole mass and θ angle, $M \to M_{\text{eff}} = 4\pi|a|/g^2_{\text{eff}}$ and $\theta \to \theta_{\text{eff}}$ where

$$\frac{\theta_{\text{eff}}}{2\pi} + \frac{4\pi \mathrm{i}}{g^2_{\text{eff}}} = \tau = F''(a) \ . \tag{2.64}$$

In total, the coefficient of the four-fermion coupling is given by

$$\frac{2^{9/2}\pi}{R|a|^{1/2}} \left(\frac{2\pi R}{g^2_{\text{eff}}} \right)^{7/2} \exp\left(\frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}t}{\cosh t} \log \left\| 1 - e^{-2\pi R|a|\cosh t + \mathrm{i}\zeta} \right\|^2 \right) \sum_{n_e} e^{-S} \tag{2.65}$$

where

$$S = 2\pi R \left[\frac{4\pi}{g^2_{\text{eff}}}|a| + \frac{g^2_{\text{eff}}|a|}{8\pi} \left(n_e + \frac{\theta_{\text{eff}}}{2\pi} \right)^2 \right] - \mathrm{i}\, n_e \zeta - \mathrm{i}\tilde{\zeta} \tag{2.66}$$

Remarkably, this agrees exactly with the curvature $R_{a\zeta\bar{a}\tilde{\zeta}}$ of the metric on $\mathcal{M}_3$ which follows from the twistor construction ! Note also that S agrees with the weak coupling limit of the action appearing in (2.17), for $p = 1$ and $q = n_e$.

3. Wall-crossing in $N = 2$ gauge theories / string vacua

3.1 Boltzmannian view of the wall-crossing

- We consider $\mathcal{N} = 2$ supergravity in 4 dimensions (this includes field theories with rigid $\mathcal{N} = 2$ as a special case). Let $\Gamma = \Gamma_e \oplus \Gamma_m$ be the lattice of electric and magnetic charges, with symplectic pairing

$$\langle \gamma, \gamma' \rangle = \langle (p^\Lambda, q_\Lambda), (p'^\Lambda, q'_\Lambda) \rangle \equiv q_\Lambda p'^\Lambda - q'_\Lambda p_\Lambda \in \mathbb{Z} \tag{3.1}$$

- BPS states preserve 4 out of 8 supercharges, and saturate the bound $\mathcal{M} \geq |Z(\gamma, t^a)|$ where $Z(\gamma, t^a) = e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge/stability data.

- We are interested in the index $\Omega(\gamma; t^a) = \mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of stable states with charge $\gamma \in \Gamma$, with center of motion and fermionic zero modes factored out.

- The BPS invariants $\Omega(\gamma; t^a)$ are locally constant functions of t^a, but may jump across codimension-one subspaces

$$W(\gamma_1, \gamma_2) = \{t^a \,/\, \arg[Z(\gamma_1)] = \arg[Z(\gamma_2)]\} \tag{3.2}$$

  where γ_1 and γ_2 are two primitive (non-zero) vectors such that $\gamma = M\gamma_1 + N\gamma_2$, $M, N \geq 1$.

- We choose γ_1, γ_2 such that $\Omega(\gamma; t^a)$ has support only on the positive cone (root basis property)

$$\tilde{\Gamma}: \quad \{M\gamma_1 + N\gamma_2, \quad M, N \geq 0, \quad (M, N) \neq (0, 0)\} \ . \tag{3.3}$$

- Let $c_\pm$ be the chamber in which $\arg(Z_{\gamma_1}) \gtrless \arg(Z_{\gamma_2})$. Our aim is to compute $\Delta\Omega(\gamma) \equiv \Omega^-(\gamma) - \Omega^+(\gamma)$ as a function of $\Omega^+(\gamma)$ (say).

- Assume that $M(\gamma_1), M(\gamma_2)$ are much greater than the dynamical scale (Λ or m_P). In this limit, those single-particle states which are potentially unstable across W) can be described by *classical configurations* with n centers of charge $M_i\gamma_1 + N_i\gamma_2 \in \tilde{\Gamma}$, satisfying $(M, N) = \sum_i (M_i, N_i)$.

- *In addition, in either chamber, there may be multi-centered configurations whose charge vectors do not lie in $\tilde{\Gamma}$. However, they remain bound across W and do not contribute to $\Delta\Omega(\gamma)$.*

- Assume for definiteness that $\gamma_{12} < 0$. Then multi-centered solutions with charges in $\tilde{\Gamma}$ *exist only in chamber c_-*, **not** c_+. E.g. two-centered solutions can only exist when [55]

$$r_{12} = \frac{1}{2} \frac{\langle \alpha_1, \alpha_2 \rangle \, |Z(\alpha_1) + Z(\alpha_2)|}{\mathrm{Im}\,[Z(\alpha_1)\bar{Z}(\alpha_2)]} > 0 \ . \tag{3.4}$$

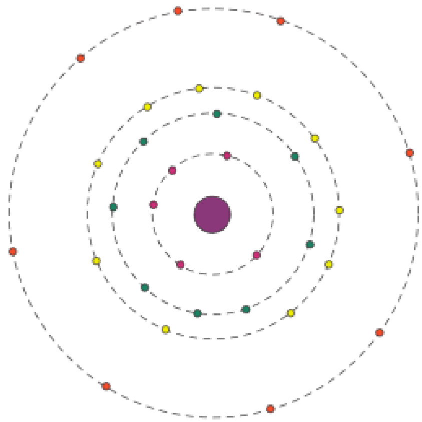

Figure 3: Onion-like structure of the black hole halo configurations relevant for semi-primitive wall crossing [24]

- At the wall, r_{ij} diverges : the single-particle bound state decays into the continuum of multi-particle states.

- $\Delta\Omega(\gamma)$ is equal to the index of the **SUSY quantum mechanics** of n *point-like particles*, each carrying its own set of degrees of freedom with index $\Omega(\gamma_i)$, interacting via Newtonian and Coulomb forces.

- For primitive decay $\gamma \to \gamma_1 + \gamma_2$, the quantization of the phase space of two-centered configuration reproduces the primitive WCF

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}+1}\, |\gamma_{12}|\, \Omega^+(\gamma_1)\, \Omega^+(\gamma_2) \ , \tag{3.5}$$

  where $(-1)^{\gamma_{12}+1}\, |\gamma_{12}|$ is the index of **Landau states** on a sphere of radius r_{12} threaded by a magnetic flux $\gamma_{1,2}$, or equivalently the angular momentum degeneracy.

- This generalizes to *semi-primitive wall-crossing* $\gamma \to \gamma_1 + N\gamma_2$: the potentially unstable configurations consist of of a "halo" of m_s particles of charge $s\gamma_2$, $\sum s m_s = N - m$, orbiting around a "core" of charge $\gamma_1 + m\gamma_2$, see figure 3. This leads to [24]

$$\frac{\sum_{N\geq 0} \Omega^-(1,N)\, q^N}{\sum_{N\geq 0} \Omega^+(1,N)\, q^N} = \prod_{k>0} \left(1 - (-1)^{k\gamma_{12}} q^k\right)^{k\,|\gamma_{12}|\ \Omega^+(k\gamma_2)} \ . \tag{3.6}$$

- E.g. for $\gamma \mapsto \gamma_1 + 2\gamma_2$,

$$\begin{aligned} \Delta\Omega(1,2) =& \mathit{\Omega}^+(\mathit{1},\mathit{0}) \left[2\gamma_{12}\, \Omega^+(0,2) + \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\gamma_{12}}\, \boldsymbol{\Omega}^+(\mathbf{0},\mathbf{1}) \left(\boldsymbol{\gamma_{12}} \boldsymbol{\Omega}^+(\mathbf{0},\mathbf{1}) + \mathbf{1}\right) \right] \\ &+ \mathit{\Omega}^+(\mathit{1},\mathit{1}) \left[(-1)^{\gamma_{12}} \gamma_{12} \Omega^+(0,1) \right] \ . \end{aligned} \tag{3.7}$$

- The term $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{d}(\mathbf{d}+\mathbf{1})$ with $d = \gamma_{12}\Omega^+(0,1)$, reflects the Bose/Fermi statistics of identical particles, i.e. the projection on **(anti)symmetric wave functions**.

- It is instructive to rewrite the semi-primitive wcf using the **rational BPS invariants**

$$\bar{\Omega}(\gamma) \equiv \sum\nolimits_{d|\gamma} \Omega(\gamma/d)/d^2 \ , \tag{3.8}$$

- By the Möbius inversion formula,

$$\Omega(\gamma) = \sum\nolimits_{d|\gamma} \mu(d)\, \bar{\Omega}(\gamma/d)/d^2 \tag{3.9}$$

where $\mu(d)$ is the Möbius function (i.e. 1 if d is a product of an even number of distinct primes, -1 if d is a product of an odd number of primes, or 0 otherwise).

- The rational DT invariants $\bar{\Omega}(\gamma)$ appear in the JS formula, in constructions of *modular invariant black hole partition functions*, and in instanton corrections to hypermultiplet moduli spaces.

- In the (1,2) example,

$$\begin{aligned}\Delta\bar{\Omega}(1,2) =& \bar{\Omega}^+(1,0)\left[2\gamma_{12}\,\bar{\Omega}^+(0,2) + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\gamma_{12}}\,\bar{\boldsymbol{\Omega}}^+(\mathbf{0},\mathbf{1})^{\mathbf{2}}\right] \\ &+ \bar{\Omega}^+(1,1)\left[(-1)^{\gamma_{12}}\gamma_{12}\bar{\Omega}^+(0,1)\right] \ .\end{aligned} \tag{3.10}$$

is simpler, and manifestly consistent with charge conservation.

- More generally, using the identity $\prod_{d=1}^{\infty}(1-q^d)^{\mu(d)/d} = e^{-q}$, or working backwards, the semi-primitive wcf can be rewritten as

$$\frac{\sum_{N\geq 0}\bar{\Omega}^-(1,N)\,q^N}{\sum_{N\geq 0}\bar{\Omega}^+(1,N)\,q^N} = \exp\left[\sum_{s=1}^{\infty} q^s(-1)^{\langle\gamma_1,s\gamma_2\rangle}\langle\gamma_1,s\gamma_2\rangle\bar{\boldsymbol{\Omega}}^+(\mathbf{s}\boldsymbol{\gamma_2})\right] \ . \tag{3.11}$$

- Physically, this follows by treating the particles in the halo as *distinguishable*, each carrying an effective index $\bar{\Omega}(s\gamma_2)$, and applying **Boltzmann** statistics !

- In general, we expect that the WCF is given by a sum

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2}\ \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma}\\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|}\prod\nolimits_{i=1}^{n}\bar{\Omega}^+(\alpha_i) \ , \tag{3.12}$$

over all *unordered* decompositions of the total charge vector γ into a sum of n vectors $\alpha_i \in \tilde{\Gamma}$. The symmetry factor $|\mathrm{Aut}(\{\alpha_i\})|$ is conventional, but natural in *Boltzmannian statistics.*

- The KS and JS formulae give a mathematical (implicit/explicit) prediction for the coefficients $g(\{\alpha_i\})$. After reviewing these formulae, we shall check them against a physical derivation based on black hole halo picture.

3.2 The Kontsevich-Soibelman formula

- Consider the Lie algebra $\mathcal{A}$ spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, e_{\gamma_1+\gamma_2} \ , \qquad \kappa(x) = (-1)^x\, x \ . \tag{3.13}$$

For a given charge vector γ and value of the VM moduli t^a, consider the operator $U_\gamma(t^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2}\right) \tag{3.14}$$

The operators e_γ / U_γ can be realized as *Hamiltonian vector fields* / **symplectomorphisms** of a twisted torus.

- The KS wall-crossing formula states that the product

$$A_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} U_\gamma \ , \tag{3.15}$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As t^a crosses W, $\Omega(\gamma; t^a)$ *jumps and the order of the factors is reversed*, **but the operator A_{γ_1,γ_2} stays constant.** Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ M/N\downarrow}} U^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ M/N\uparrow}} U^-_{M\gamma_1+N\gamma_2} \ , \tag{3.16}$$

- The algebra $\mathcal{A}$ is infinite dimensional but filtered. The KS formula may be projected to any finite-dimensional algebra

$$\mathcal{A}_{M,N} = \mathcal{A}/\{\sum_{m>M \,\mathrm{or}\, n>N} \mathbb{R}\cdot e_{m\gamma_1+n\gamma_2}\} \ . \tag{3.17}$$

This projection is sufficient to infer $\Delta\Omega(m\gamma_1 + n\gamma_2)$ for any $m \leq M, n \leq N$, e.g. using the Baker-Campbell-Hausdorff formula.

- For example, the primitive wcf follows in $\mathcal{A}_{1,1}$ from

$$\begin{aligned} &\exp(\bar\Omega^+(\gamma_1)e_{\gamma_1})\, \exp(\bar\Omega^+(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2})\, \exp(\bar\Omega^+(\gamma_2)e_{\gamma_2}) \\ = &\exp(\bar\Omega^-(\gamma_2)e_{\gamma_2})\, \exp(\bar\Omega^-(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2})\, \exp(\bar\Omega^-(\gamma_1)e_{\gamma_1}) \end{aligned}$$

and the order 2 truncation of the BCH formula

$$e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \ . \tag{3.18}$$

- Noting that the operators $U_{k\gamma}$ for different $k \geq 1$ commute, one may combine them into a single factor

$$V_\gamma \equiv \prod_{k=1}^{\infty} U_{k\gamma} = \exp\left(\sum_{\ell=1}^{\infty} \bar{\boldsymbol{\Omega}}(\ell\gamma)\, e_{\ell\gamma}\right), \qquad \bar{\Omega}(\gamma) = \sum_{m|\gamma} m^{-2}\Omega(\gamma/m)\,. \tag{3.19}$$

and rewrite the KS formula as a product over **primitive** charge vectors only,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\downarrow}} V^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\uparrow}} V^-_{M\gamma_1+N\gamma_2}\,, \tag{3.20}$$

- The semi-primitive formula can be derived similarly by projecting the KS formula to $\mathcal{A}_{1,\infty}$,

$$V^+_{\gamma_1}\, V^+_{\gamma_1+\gamma_2}\, V^+_{\gamma_1+2\gamma_2} \cdots V^+_{\gamma_2} = V^-_{\gamma_2} \cdots V^-_{\gamma_1+2\gamma_2}\, V^-_{\gamma_1+\gamma_2}\, V^-_{\gamma_1} \tag{3.21}$$

and combining on either side the factors $V^+_{\gamma_1+N\gamma_2}$ in a single exponential using the order-2 BCH formula:

$$e^{X_1^+}\, V^+_{\gamma_2} = V^-_{\gamma_2} e^{X_1^-} \tag{3.22}$$

The Hadamard lemma for $e^Y = V^+_{\gamma_2} = V^-_{\gamma_2}, X = e^{X_1^+}$

$$e^Y\, X\, e^{-Y} = X + [Y,X] + \frac{1}{2!}[Y,[Y,X]] + +\frac{1}{3!}[Y,[Y,[Y,X]]] + \ldots \tag{3.23}$$

then leads directly to $Z^-(1,q) = Z^+(1,q)\, Z_{\rm halo}(\gamma_1,q)$, where $Z^\pm(1,q) = \sum_{N\geq 0} \Omega^\pm(\gamma_1 + N\gamma_2)\, q^N$.

- By projecting the KS formula to $\mathcal{A}_{M,\infty}$, one can obtain "order M" generalizations of the semi-primitive WCF, e.g. for $M=2$

$$\widetilde{Z}_2^-(q) = \widetilde{Z}_2^+(q)\, Z_{\rm halo}(2\gamma_1, q) \tag{3.24}$$

where

$$\begin{aligned} \widetilde{Z}_2^\pm(q) \equiv & \sum_{N\geq 0}^{\infty} \bar{\Omega}^\pm(2\gamma_1 + N\gamma_2)\, q^N \\ & \pm\frac{1}{4} \sum_{N_1,N_2\geq 0} \kappa(|N_1-N_2|\gamma_{12})\, \bar{\Omega}^\pm(\gamma_1+N_1\gamma_2)\, \bar{\Omega}^+(\gamma_1+N_2\gamma_2)\, q^{N_1+N_2}\,. \end{aligned} \tag{3.25}$$

and $Z_{\rm halo}(2\gamma_1, q)$ is the same factor which appeared in the semi-primitive wcf, after replacing $\gamma_1 \mapsto 2\gamma_1$.

- E.g for D6-D0 bound states (i.e. dimension zero sheaves on $\mathcal{X}$): at large volume, zero B-field,

$$\begin{array}{c|c|c|c|c|c} D6\backslash D0 & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \cdot & -\chi & -\chi & -\chi & -\chi \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 0 & 0 & 0 & 0 & \ldots \\ 3 & 0 & 0 & 0 & 0 & \ldots \end{array} \tag{3.26}$$

$$\Omega^+(1,0) = 1\,, \qquad \Omega^+(0,n) = -\chi \quad (n>0)\,. \tag{3.27}$$

- As the B-field is increased, one enters the DT chamber, wherein

$$\begin{array}{c|c|c|c|c|c} D6\backslash D0 & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \cdot & -\chi & -\chi & -\chi & -\chi \\ 1 & 1 & -\chi & \frac{1}{2}(\chi^2+5\chi) & -\frac{1}{6}(\chi^3+15\chi^2+20\chi) & \dots \\ 2 & 0 & 0 & -\chi & -\frac{1}{6}(\chi^3+15\chi^2+20\chi) & \dots \\ 3 & 0 & 0 & 0 & -\chi & \dots \end{array} \tag{3.28}$$

- The partition function of rank 1 DT invariants is

$$Z^-(1,q) = [M(-q)]^\chi \ , \qquad M(q) = \prod_{n\geq 1} 1-q^n)^n \tag{3.29}$$

- The partition function of rank 2 DT invariants is

$$\begin{aligned} Z^-(2,q) =& \frac{1}{4}\left([M(q)]^{2\chi} - [M(-q^2)]^\chi\right) \\ & - \frac{1}{4}\sum_{n_1,n_2} \kappa(|n_1-n_2|)\Omega^-(1,n_1)\,\Omega^-(1,n_2)\,q^{n_1+n_2} \end{aligned} \tag{3.30}$$

- When α_i have generic phases, $g(\{\alpha_i\})$ can be computed by projecting the KS formula to the subalgebra spanned by $e_{\sum \alpha_j}$ where $\{\alpha_j\}$ runs over all subsets of $\{\alpha_i\}$.

- E.g., for $n=3$, assuming that the phase of the charges are ordered according to

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2,\ \alpha_2+\alpha_3,\ \alpha_3\,, \tag{3.31}$$

we find

$$g(\{\alpha_1,\alpha_2,\alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\,\alpha_{12}\,(\alpha_{13}+\alpha_{23}) \tag{3.32}$$

As we shall see later, this fits the macroscopic index of 3-centered configurations !

- Similarly, for $n=4$, assuming the clockwise ordering

$$\begin{aligned} &\alpha_1\,,(\alpha_1+\alpha_2\,,\ \alpha_1+\alpha_3\,,\ \alpha_1+\alpha_2+\alpha_3)\,,\ \alpha_2\,, \\ &\quad(\alpha_2+\alpha_3\,,\ \alpha_1+\alpha_2+\alpha_4)\,,\alpha_1+\alpha_2+\alpha_3+\alpha_4\,,\ \alpha_1+\alpha_3+\alpha_4\,, \\ &\quad\alpha_3\,,\ (\alpha_1+\alpha_4\,,\ \alpha_2+\alpha_4\,,\ \alpha_2+\alpha_3+\alpha_4\,,\ \alpha_3+\alpha_4)\,,\ \alpha_4\ , \end{aligned} \tag{3.33}$$

we find

$$\begin{aligned} g(\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}) =& (-1)^{1+\sum_{i<j}\alpha_{ij}} \times \\ & [\langle\alpha_1,\alpha_2\rangle\,\langle\alpha_1+\alpha_2,\alpha_3\rangle\,\langle\alpha_1+\alpha_2+\alpha_3,\alpha_4\rangle \\ & +\langle\alpha_1,\alpha_3\rangle\,\langle\alpha_1+\alpha_3,\alpha_4\rangle\,\langle\alpha_2,\alpha_1+\alpha_3,\alpha_4\rangle \\ & +\langle\alpha_2,\alpha_3\rangle\,\langle\alpha_1,\alpha_4\rangle\,\langle\alpha_2+\alpha_3,\alpha_1+\alpha_4\rangle] \end{aligned} \tag{3.34}$$

which is a prediction for the index of the 4-body SUSY quantum mechanics.

- KS have proposed a *quantum deformation* of their formula, which governs wall-crossing properties of **motivic DT invariants** $\Omega_{\rm ref}(\gamma; y, t)$. Physically, these correspond to the "refined index"

$$\Omega_{\rm ref}(\gamma, y) = \ {\rm Tr}'_{\mathcal{H}(\gamma)} (-y)^{2J_3} \equiv \sum_{n\in\mathbb{Z}} (-y)^n \, \Omega_{{\rm ref},n}(\gamma) \,, \tag{3.35}$$

where J_3 is the angular momentum in 3 dimensions along the z axis (more accurately, a combination of angular momentum and $SU(2)_R$ quantum numbers). As $y \to 1$, $\Omega_{\rm ref}(\gamma; y, t) \to \Omega(\gamma; t)$.

- Caution: this index is protected in $\mathcal{N} = 2, D = 4$ field theories, but not in supergravity/string theory, where $SU(2)_R$ is generically broken.

- To state the formula, consider the Lie algebra $\mathcal{A}(y)$ spanned by generators $\{\tilde{e}_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[\tilde{e}_{\gamma_1}, \tilde{e}_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, \tilde{e}_{\gamma_1+\gamma_2} \ , \qquad \kappa(x) = \frac{(-y)^x - (-y)^{-x}}{y - 1/y} \ . \tag{3.36}$$

- To any charge vector γ, attach the operator

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \mathbf{E}\left(\frac{y^n\, \tilde{e}_\gamma}{y - 1/y}\right)^{-(-1)^n \Omega_{{\rm ref},n}(\gamma)} \ , \quad \mathbf{E}(x) \equiv \exp\left[\sum_{k=1}^{\infty} \frac{(xy)^k}{k(1 - y^{2k})}\right] \ , \tag{3.37}$$

where $\mathbf{E}$ is the *quantum dilogarithm function.*

- The motivic version of the KS wall-crossing formula again states that the ordered product

$$\hat{A}_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\ge 0, N\ge 0}} \hat{U}_\gamma \ , \tag{3.38}$$

is constant across the wall.

- As before, one may combine the $\hat{U}_{k\gamma}$ into a single factor

$$\hat{V}_\gamma = \prod_{\ell\ge 1} \hat{U}_{\ell\gamma} = \exp\left[\sum_{N=1}^{\infty} \bar{\Omega}_{\rm ref}(N\gamma, y)\, \tilde{e}_{N\gamma}\right] \tag{3.39}$$

where $\bar{\Omega}_{\rm ref}(N\gamma, y)$ are the **"rational motivic invariants"**, defined by

$$\bar{\Omega}^+_{\rm ref}(\gamma, y) \equiv \sum_{m|\gamma} \frac{(y - y^{-1})}{m(y^m - y^{-m})} \Omega^+_{\rm ref}(\gamma/m, y^m) \,. \tag{3.40}$$

- The motivic KS formula becomes

$$\prod_{\substack{M\ge 0, N\ge 0 > 0,\\ \gcd(M,N)=1, M/N\downarrow}} \hat{V}^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\ge 0, N\ge 0 > 0,\\ \gcd(M,N)=1, M/N\uparrow}} \hat{V}^-_{M\gamma_1+N\gamma_2} \,, \tag{3.41}$$

- $\Delta\bar{\Omega}_{\rm ref}(\gamma,y)$ can be computed using the same techniques as before, e.g. the primitive wcf read

$$\Delta\Omega_{\rm ref}(\gamma_1+\gamma_2,y)=\frac{(-y)^{\langle\gamma_1,\gamma_2\rangle}-(-y)^{-\langle\gamma_1,\gamma_2\rangle}}{y-1/y}\,\Omega_{\rm ref}(\gamma_1,y)\,\Omega_{\rm ref}(\gamma_2,y) \tag{3.42}$$

- The refined semi-primitive wall-crossing formula is given by

$$Z^-(1,q,y)=Z^+(1,q,y)\,Z_{\rm halo}(\gamma_1,q,y) \tag{3.43}$$

where

$$Z_{\rm halo}(\gamma_1,q,y)\equiv\exp\left(\sum_{\ell=1}^{\infty}\frac{(-y)^{\langle\gamma_1,\ell\gamma_2\rangle}-(-y)^{-\langle\gamma_1,\ell\gamma_2\rangle}}{y-y^{-1}}\,\bar{\Omega}_{\rm ref}(\ell\gamma_2,y)\,q^{\ell}\right)\ , \tag{3.44}$$

or in terms of the integer motivic invariants,

$$Z_{\rm halo}(\gamma_1,q,y)=\prod_{\substack{k\geq1,n\in\mathbb{Z}\\ 1\leq j\leq k|\gamma_{12}|}}\left(1-(-1)^{k|\gamma_{12}|}q^k y^{n+2j-1-k|\gamma_{12}|}\right)^{(-1)^n\,\Omega_{{\rm ref},n}(k\gamma_2)} \tag{3.45}$$

3.3 The Joyce-Song formula

- In the context of the **Abelian category of coherent sheaves** on a Calabi-Yau threefold, Joyce & Song have shown that the jump of (generalized, rational) DT invariants across the wall is given by

$$\Delta\bar{\Omega}(\gamma)=\sum_{n\geq2}\sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma}\\ \gamma=\alpha_1+\dots+\alpha_n}}\frac{g(\{\alpha_i\})}{|{\rm Aut}(\{\alpha_i\})|}\prod\nolimits_{i=1}^{n}\bar{\Omega}^+(\alpha_i)\ . \tag{3.46}$$

where the coefficient g is given by

$$\begin{aligned}g(\{\alpha_i\})\ =&\frac{1}{2^{n-1}}\,(-1)^{n-1+\sum_{i<j}\langle\alpha_i,\alpha_j\rangle}\sum_{\sigma\in\Sigma_n}\\ &\mathcal{L}\left(\alpha_{\sigma(1)},\dots\alpha_{\sigma(n)}\right)\,U\left(\alpha_{\sigma(1)},\dots\alpha_{\sigma(n)}\right)\end{aligned} \tag{3.47}$$

- To formulate the JS formula, we need to introduce S, U and $\mathcal{L}$ factors, which are functions of an ordered list of charge vectors $\alpha_i\in\tilde{\Gamma}, i=1\dots n$.

- We define $S(\alpha_1,\dots,\alpha_n)\in\{0,\pm1\}$ as follows. If $n=1$, set $S(\alpha_1)=1$. If $n>1$ and, for every $i=1\dots n-1$, either

$$\begin{aligned}&(a)\qquad\langle\alpha_i,\alpha_{i+1}\rangle\leq0\quad\text{and}\quad\langle\alpha_1+\dots+\alpha_i,\alpha_{i+1}+\dots+\alpha_n\rangle<0,\quad\text{or}\\ &(b)\qquad\langle\alpha_i,\alpha_{i+1}\rangle>0\quad\text{and}\quad\langle\alpha_1+\dots+\alpha_i,\alpha_{i+1}+\dots+\alpha_n\rangle\geq0\ ,\end{aligned} \tag{3.48}$$

let $S(\alpha_1,\dots,\alpha_n)=(-1)^r$, where r is the number of times option (a) is realized; otherwise, $S(\alpha_1,\dots,\alpha_n)=0$.

- To define the U factor, consider all ordered partitions of the n vectors α_i into $1 \le m \le n$ packets $\{\alpha_{a_{j-1}+1}, \cdots, \alpha_{a_j}\}$, $j = 1 \dots m$, with $0 = a_0 < a_1 < \cdots < a_m = n$, such that all vectors in each packet have the same phase $\arg Z(\alpha_i)$. Let

$$\beta_j = \alpha_{a_{j-1}+1} + \cdots + \alpha_{a_j}, \qquad j = 1 \dots m \tag{3.49}$$

  be the sum of the charge vectors in each packet.

- Next, consider all ordered partitions of the m vectors β_j into $1 \le l \le m$ packets $\{\beta_{b_{k-1}+1}, \cdots, \beta_{b_k}\}$, with $0 = b_0 < b_1 < \cdots < b_l = m$, $k = 1 \dots l$, such that the total charge vectors $\delta_k = \beta_{b_{k-1}+1} + \cdots + \beta_{b_k}, k = 1 \dots l$ in each packets all have the same phase $\arg Z(\delta_k)$.

- Define the U-factor as the sum

$$\begin{aligned} U(\alpha_1, \dots, \alpha_n) \equiv \sum_l \frac{(-1)^{l-1}}{l} \cdot \prod\nolimits_{k=1}^{l} \\ \prod_{j=1}^{m} \frac{1}{(a_j - a_{j-1})!} \; S(\beta_{b_{k-1}+1}, \beta_{b_{k-1}+2}, \dots, \beta_{b_k}) \; . \end{aligned} \tag{3.50}$$

  over all partitions of α_i and β_j satisfying the conditions above.

- If none of the phases of the vectors α_i coincide, $S = U$. Contributions with $l > 1$ arise only when $\{\alpha_i\}$ can be split into two (or more) packets with the same total charge, e.g.

$$U[\gamma_1, \gamma_2, \gamma_1, \gamma_2] = S[\gamma_1, \gamma_2, \gamma_1, \gamma_2] - \frac{1}{2} S[\gamma_1, \gamma_2]^2 = 1 - \frac{1}{2}(-1)^2 = \frac{1}{2} \tag{3.51}$$

- Finally (departing slightly from JS), define the (Landau) $\mathcal{L}$ factor Landau factor $\mathcal{L}$ is a

$$\mathcal{L}(\alpha_1, \dots, \alpha_n) = \sum_{\text{trees}} \prod_{\text{edges(i,j)}} \langle \alpha_i, \alpha_j \rangle \tag{3.52}$$

  where the sum runs over all *labeled trees* with n vertices labelled $\{1, \dots, n\}$, with edges oriented from i to j if $i < j$. There are n^{n-2} trees, labelled by their Prüfer code, a sequence of $n-2$ numbers in $\{1, \dots n\}$.

- To derive the primitive wcf, note that there is only one oriented tree with 2 nodes. Assuming $\gamma_{12} < 0$, the JS data is then

$\sigma(12)$	S	U	$\mathcal{L}$
12	a	-1	γ_{12}
21	b	1	$-\gamma_{12}$

(3.53)

  leading again to

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}} \gamma_{12} \, \Omega(\gamma_1) \, \Omega(\gamma_2) \; , \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle \tag{3.54}$$

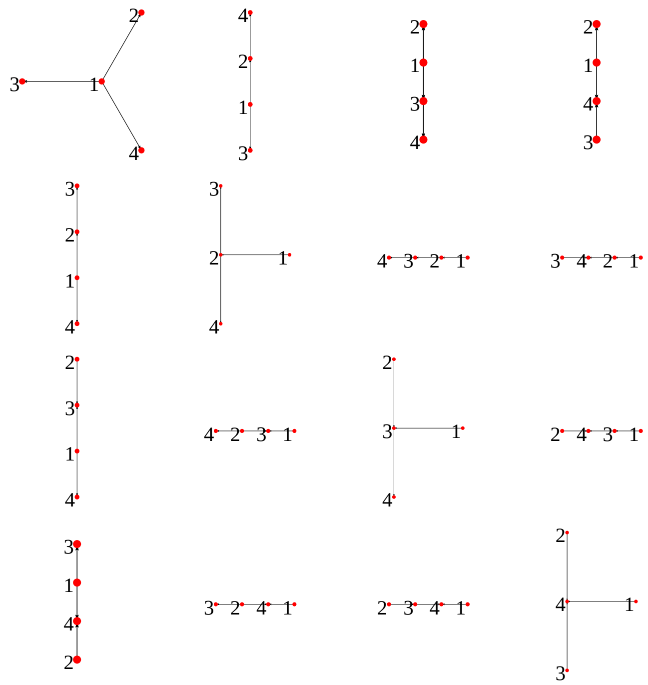

Figure 4: The 16 labelled trees contributing to four-body decay

- For generic 3-body decay, assuming the same phase ordering as before and taking into account the 3 possible oriented trees, the JS data

$$\begin{array}{|c|c|c|c|} \hline \sigma(123) & S & U & \mathcal{L} \\ \hline 123 & \text{bb} & 1 & \alpha_{12}\alpha_{13}+\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{23} \\ 132 & \text{b-} & 0 & \alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23} \\ 213 & \text{ab} & -1 & -\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{13} \\ 231 & \text{-a} & 0 & \alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23} \\ 312 & \text{ab} & -1 & \alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{12} \\ 321 & \text{aa} & 1 & \alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23} \\ \hline \end{array} \tag{3.55}$$

  leads to the same answer as KS,

$$g(\{\alpha_1,\alpha_2,\alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\,\alpha_{12}\,(\alpha_{13}+\alpha_{23}) \tag{3.56}$$

- We have checked that JS and KS also agree for generic 4-body decay (involving 16 trees), 5-body decay (125 trees) and for special cases (2,3), (2,4) (up to 1296 graphs !).
- While there is no general proof yet, it seems that the JS formula (derived for Abelian categories) is equivalent to the classical KS formula (stated for triangulated categories).
- Note that the JS formula involves large denominators and leads to many cancellations. We shall find a more economic formula which also works at the motivic level.

3.4 The "Higgs branch" formula

- An alternative formula can be given using the *Higgs branch* description of the multi-centered configuration, namely the **quiver** with n nodes $\{1\dots n\}$ of dimension 1 and α_{ij} arrows from i to j.

- Since α_i lie on a 2-dimensional sublattice $\tilde{\Gamma}$, the quiver has no oriented closed loop. **Reineke's formula** gives

$$g_{\text{ref}} = \frac{(-y)^{-\sum_{i<j}\alpha_{ij}}}{(y-1/y)^{n-1}} \sum_{\text{partitions}} (-1)^{s-1} y^{2\sum_{a\leq b}\sum_{j<i}\alpha_{ji}\, m_i^{(a)}\, m_j^{(b)}}\,, \tag{3.57}$$

where $\sum$ runs over all ordered partitions of $\gamma = \alpha_1 + \cdots + \alpha_n$ into s vectors $\beta^{(a)}$ ($1 \leq a \leq s$, $1 \leq s \leq n$) such that

1. $\beta^{(a)} = \sum_i m_i^{(a)} \alpha_i$ with $m_i^{(a)} \in \{0,1\}$, $\sum_a \beta^{(a)} = \gamma$
2. $\langle \sum_{a=1}^{b} \beta^{(a)}, \gamma \rangle > 0 \quad \forall \quad b \quad \text{with} \quad 1 \leq b \leq s-1$

- The formula agrees with KS/JS/Coulomb for $n = 2, 3, 4, 5$!

3.5 The "Coulomb branch" formula

- The moduli space $\mathcal{M}_n$ of BPS configurations with n centers in $\mathcal{N} = 2$ SUGRA is described by solutions to Denef's equations [55]

$$\sum_{j=1\ldots n, j\neq i}^{n} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i\,, \qquad \begin{cases} c_i = 2\,\mathrm{Im}\,[e^{-i\alpha} Z(\alpha_i)] \\ \alpha = \arg[Z(\alpha_1 + \cdots \alpha_n)] \end{cases}. \tag{3.58}$$

- $\mathcal{M}_n$ is a **compact symplectic manifold** of dimension $2n-2$, and carries an Hamiltonian action of $SU(2)$ [39]:

$$\omega = \frac{1}{2}\sum_{i<j} \alpha_{ij} \frac{\mathrm{d}\vec{r}_{ij} \wedge \mathrm{d}\vec{r}_{ij} \cdot \vec{r}_{ij}}{|r_{ij}|^3}\,, \qquad \vec{J} = \frac{1}{2}\sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|} \tag{3.59}$$

- Quantizing the internal degrees of freedom of the multi-centered configurations amounts to quantizing the symplectic space $\mathcal{M}_n$. The index is given, at least when $|\alpha_{ij}| \gg 1, y \to 1$, by

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij} - n+1}}{(2\pi \sinh \nu/\nu)^{n-1}} \int_{\mathcal{M}_n} \omega^{n-1}\, e^{2\nu J_3}\,, \qquad \nu \equiv \log y \tag{3.60}$$

We conjecture that this is exact for all α_{ij}, y.

- By the Duistermaat-Heckmann theorem, the integral **localizes** to the fixed points of the action of J_3, i.e. *collinear multi-centered configurations* along the z-axis, such that

$$\sum_{j=1\ldots n, j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i\,, \quad J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\, \mathrm{sign}(z_i - z_j)\,. \tag{3.61}$$

- These are classified by *permutations* σ describing the order of z_i along the axis. Let $\mathcal{S}(t)$ be the set of permutations allowed by Denef's equations. Localization leads to the **Coulomb branch formula**

$$g_{\text{ref}}(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij} + n-1}}{(y - y^{-1})^{n-1}} \sum_{\sigma \in \mathcal{S}(t)} s(\sigma)\, y^{\sum_{i<j}\alpha_{ij}\,\mathrm{sign}(\sigma(j)-\sigma(i))}\,. \tag{3.62}$$

where $s(\sigma) = (-1)^{\#\{i;\sigma(i+1)<\sigma(i)\}}$ originates from Hessian(J_3).

- For $n \leq 5$, we find perfect agreement with JS/KS !

$$g(\alpha_1, \alpha_2; y) = (-1)^{\alpha_{12}} \frac{\sinh(\nu\alpha_{12})}{\sinh \nu} \tag{3.63}$$

$$g(\alpha_1, \alpha_2, \alpha_3; y) = (-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}} \frac{\sinh(\nu(\alpha_{13} + \alpha_{23})) \sinh(\nu\alpha_{12})}{\sinh^2 \nu} , \tag{3.64}$$

4. Hypermultiplet moduli spaces in type II string vacua

4.1 Perturbative HM moduli space in type IIA and local c-map

- The HM moduli space in type IIA compactified on a CY 3-fold (family) $\mathcal{X}$ is a *quaternion-Kähler* manifold $\mathcal{M}$ of real dimension $2b_3(\mathcal{X}) = 4(h_{2,1} + 1)$. The same space arises as the VM moduli space in type IIB compactified on $\mathcal{X} \times S^1$. We shall mainly use the hypermultiplet language.

- $\mathcal{M} \equiv \mathcal{Q}_c(\mathcal{X})$ encodes

  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complex structure of the CY family $\mathcal{X}$,
  3. the periods of the RR 3-form C on $\mathcal{X}$,
  4. the NS axion σ, dual to the Kalb-Ramond B-field in 4D

- To write down the metric explicitly, let us choose a symplectic basis $\mathcal{A}^\Lambda, \mathcal{B}_\Lambda$, $\Lambda = 0 \dots h_{2,1}$ of $H_3(\mathcal{X}, \mathbb{Z})$.

- The *complex structure moduli space* $\mathcal{M}_c(\mathcal{X})$ may be parametrized by the periods $\Omega(z^a) = (X^\Lambda, F_\Lambda) \in H_3(\mathcal{X}, \mathbb{C})$ of the (3,0) form

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega_{3,0} , \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega_{3,0} , \tag{4.1}$$

  up to holomorphic rescalings $\Omega \mapsto e^f \Omega$.

- $\mathcal{M}_c(\mathcal{X})$ is endowed with a *special Kähler* metric

$$\mathrm{d}s^2_{\mathcal{SK}} = \partial\bar{\partial}\mathcal{K} , \qquad \mathcal{K} = -\log[\mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)] \tag{4.2}$$

  and a $\mathbb{C}^\times$ bundle $\mathcal{L}$ with connection $\mathcal{A}_K = \frac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar{a}} \mathrm{d}\bar{z}^{\bar{a}})$.

- Ω transforms as $\Omega \mapsto e^f \rho(M)\, \Omega$ under a monodromy M in $\mathcal{M}_c(\mathcal{X})$, where $\rho(M) \in Sp(b_3, \mathbb{Z})$.

- Topologically trivial harmonic C-fields on $\mathcal{X}$ may be parametrized by the real periods $C = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C \,. \tag{4.3}$$

- Large gauge transformations require that C lives in the *intermediate Jacobian torus*

$$C \in \mathcal{T} = H^3(\mathcal{X},\mathbb{R})/H^3(\mathcal{X},\mathbb{Z}) \tag{4.4}$$

i.e. that $(\zeta^\Lambda, \tilde\zeta_\Lambda)$ have unit periodicities. This is consistent with D-instanton charge quantization, as we shall discuss later.

- $\mathcal{T}$ carries a canonical *symplectic form* and complex structure induced by the Hodge $\star_{\mathcal{X}}$, hence a Kähler metric

$$\mathrm{d}s^2_{\mathcal{T}} = -\frac{1}{2}(\mathrm{d}\tilde\zeta_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\,\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde\zeta_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'}) \tag{4.5}$$

where $\mathcal{N}$ is the (Weil) period matrix ($\mathrm{Im}\,\mathcal{N}<0$),

$$\mathcal{N}_{\Lambda\Lambda'} = \bar\tau_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\,\mathrm{Im}\,\tau\cdot X]_\Lambda[\,\mathrm{Im}\,\tau\cdot X]_{\Lambda'}}{X^\Sigma\;\mathrm{Im}\,\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,. \tag{4.6}$$

while $\tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$ is the Griffiths period matrix.

- Under monodromies, $C \mapsto \rho(M)C$. We shall refer to the total space of the torus bundle $\mathcal{T} \to \mathcal{J}_{\mathbf{c}}(\mathcal{X}) \to \mathcal{M}_c(\mathcal{X})$ as the (Weil) **intermediate Jacobian of** $\mathcal{X}$.

- At **tree level**, i.e. in the strict weak coupling limit $R=\infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c-map metric* [44, 56]

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_{\mathcal{T}}}{R^2} + \frac{1}{16R^4}D\sigma^2\,. \tag{4.7}$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda\mathrm{d}\zeta^\Lambda - \zeta^\Lambda\mathrm{d}\tilde\zeta_\Lambda \tag{4.8}$$

This follows by circle reduction followed by T-duality.

- The c-map metric admits continuous isometries

$$T_{H,\kappa} : (C,\sigma) \mapsto (C+H, \sigma+\kappa+\langle C,H\rangle) \tag{4.9}$$

where $H \in H^3(\mathcal{X},\mathbb{R})$ and $\kappa\in\mathbb{R}$, satisfying the **Heisenberg group** relation

$$T_{H_1,\kappa_1}T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle}\,. \tag{4.10}$$

In addition, the metric is invariant under rescalings generated by the Euler vector field

$$R\partial_R + \zeta_\Lambda\partial_{\zeta^\Lambda} + \tilde\zeta_\Lambda\partial_{\tilde\zeta_\Lambda} + 2\sigma\partial_\sigma\,. \tag{4.11}$$

- The *one-loop correction* deforms the metric on $\mathcal{M}$ into [57, 58, 59, 60]

$$\begin{aligned} ds^2_{\mathcal{M}} =& 4\frac{R^2+2c}{R^2(R^2+c)}\,\mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_{\mathcal{T}}}{R^2} \\ &+ \frac{2\,c}{R^4}\,e^{\mathcal{K}}\,|X^\Lambda\mathrm{d}\tilde\zeta_\Lambda - F_\Lambda\mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16R^4(R^2+2c)}D\sigma^2\,. \end{aligned} \tag{4.12}$$

where

$$D\sigma = \mathrm{d}\sigma + \langle C,\mathrm{d}C\rangle + \mathbf{8c}\mathcal{A}_{\mathbf{K}}\,, \quad \mathbf{c} = -\frac{\chi(\mathcal{X})}{\mathbf{192}\pi} \tag{4.13}$$

- The one-loop correction to g_{rr} was computed by reducing the CP-even R^4 coupling in 10D on $\mathcal{X}$. The correction to $D\sigma$ can be obtained with less effort by reducing the topological coupling in $D = 10$ type IIA supergravity:

$$\int_{\mathcal{Y}} \left(\frac{1}{6} B \wedge \mathrm{d}C \wedge \mathrm{d}C - B \wedge I_8 \right), \quad I_8 = \frac{1}{48}(p_2 - \frac{1}{4} p_1^2) \tag{4.14}$$

On a complex 10-manifold,

$$B \wedge I_8 = \frac{1}{24} B \wedge \left[c_4 - c_1 \left(c_3 + \frac{1}{8} c_1^3 - \frac{1}{2} c_1 c_2 \right) \right]. \tag{4.15}$$

Integrating on $\mathcal{X}$ and using $c_4 = 0, c_3 = \chi(\mathcal{X}), c_1 = -\omega_c$ leads to

$$\int d^4x \left[\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}(\mathrm{d}C^\Lambda + \zeta^\Lambda \mathrm{d}B) \wedge \mathrm{d}\zeta^\Sigma - \frac{\chi(\mathcal{X})}{24\pi} B \wedge \omega_c \right] \tag{4.16}$$

where $C^\Lambda = \int_{\mathcal{A}^\Lambda} C$. Dualizing the two-forms C^Λ, B into $\tilde{\zeta}_\Lambda, \sigma$ produces the one-form $D\sigma$ indicated previously.

- The one-loop corrected metric is presumably *exact to all orders in* $1/R$. It will receive $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections from D-instantons and NS5-brane instantons, eventually breaking all continuous isometries.

- Note the **curvature singularity** at finite distance $R^2 = -2c$ when $\chi(\mathcal{X}) > 0$! This should hopefully be resolved by instanton corrections.

- The Heisenberg isometries continue to hold, but the Euler isometry is broken. Monodromies in $\mathcal{M}_c(\mathcal{X})$ now act non-trivially on σ. This has important implications for the topology of the HM moduli space, as we now discuss.

4.2 Topology of the HM moduli space

- At weak coupling, $\mathcal{M}$ is foliated by hypersurfaces $\mathcal{C}(R)$ of constant string coupling. We shall now discuss the topology of the leaves $\mathcal{C}(R)$, which is independent of R.

- Quotienting by translations along the NS axion σ, $\mathcal{C}/\partial_\sigma$ reduces to the **intermediate Jacobian** $\mathcal{J}_c(\mathcal{X})$, in particular C lives in the intermediate Jacobian torus $\mathcal{T} = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z})$, and its components $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ have integer periodicities. This is consistent with the fact that *Euclidean D2-branes* wrapping a special Lagrangian submanifold in **integer homology class** $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(\mathcal{X}, \mathbb{Z})$ induce corrections of the form

$$\delta \mathrm{d}s^2|_{\mathrm{D2}} \sim \exp\left(-8\pi \frac{|Z_\gamma|}{g_{(4)}} - 2\pi\mathrm{i}\langle \gamma, C \rangle \right), \tag{4.17}$$

where $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge.

- Continuous translations along σ will be broken by NS5-brane instantons to discrete shifts $\sigma \mapsto \sigma + 2$ (in our conventions). Thus $e^{\mathrm{i}\pi\sigma}$ **parametrizes the fiber of a circle bundle $\mathcal{C}$ over $\mathcal{J}_c(\mathcal{X})$**, to be determined.

- The horizontal one-form $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle - \frac{\chi(\mathcal{X})}{24\pi}\mathcal{A}_K$ implies that

$$c_1(\mathcal{C}) = \mathrm{d}\left(\frac{D\sigma}{2}\right) = \omega_{\mathcal{T}} + \frac{\chi(\mathcal{X})}{24}\,\omega_c \tag{4.18}$$

  where $\omega_{\mathcal{T}} = \mathrm{d}\tilde{\zeta}_\Lambda \wedge \mathrm{d}\zeta^\Lambda, \omega_c = -\frac{1}{2\pi}\mathrm{d}\mathcal{A}_K$ are the Kähler forms on $\mathcal{T}$ and $\mathcal{M}_c(\mathcal{X})$, respectively. The second term in (4.18) has non-integer periods, which indicates that the circle bundle $\mathcal{C}$ is in fact a twisted bundle. We shall return to this shortly.

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim \exp\left(-4\pi|k|/g_{(4)}^2 - \mathrm{i}k\pi\sigma\right)\, \mathcal{Z}^{(k)}(z^a, C)\ , \tag{4.19}$$

  where $\mathcal{Z}^{(k)} = \mathrm{Tr}\,(F^2(-1)^F)$ is the (twisted) partition function of the world-volume theory on a stack k five-branes. **For this to be globally well-defined, $\mathcal{Z}^{(k)}$ must be a section of $\mathcal{C}^k$.**

- Recall that the type IIA NS5-brane supports a *self-dual 3-form flux* $H = \mathrm{i}\star H$, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a **non-trivial line bundle** $\mathcal{L}^k_{\mathrm{NS5}}$ over the space of metrics and C fields [67, 68, 69, 70, 71, 72, 73, 74, 75]. Moreover, the restriction $\mathcal{L}_{\mathrm{NS5}}|_{\mathcal{T}}$ is known to be a line bundle with first Chern class $c_1 = \omega_{\mathcal{T}}$. To specify this bundle, one must choose holonomies $\sigma(H) \in U(1)$ around each cycle $H \in H_3(\mathcal{X}, \mathbb{Z})$, such that

$$\sigma(H + H') = (-1)^{\langle H, H'\rangle}\,\sigma(H)\,\sigma(H')\,. \tag{4.20}$$

  Thus, $\sigma(H)$ defines a *quadratic refinement of the intersection form on* $H^3(\mathcal{X}, \mathbb{Z})$. (not to be confused with the NS-axion σ !). The general solution can be parametrized by **characteristics** $\Theta \in H_3(\mathcal{X}, \mathbb{R})/H_3(\mathcal{X}, \mathbb{Z})$ (notation: $\mathbf{E}(x) \equiv e^{2\pi \mathrm{i}x}$)

$$\sigma(H) = \mathbf{E}\left(-\frac{1}{2}n^\Lambda m_\Lambda + \langle H, \Theta\rangle\right)\ ,\quad H = (n^\Lambda, m_\Lambda) \tag{4.21}$$

  The bundle $(\mathcal{L}_\Theta)^k$ is then defined by the twisted periodicity condition

$$\mathcal{Z}(\mathcal{N}, C + H) = \sigma^k_\Theta(H)\,\mathbf{E}\left(\frac{k}{2}\langle H, C\rangle\right)\mathcal{Z}(\mathcal{N}, C) \tag{4.22}$$

  Note that $\sigma(H)$ need not be ± 1, and Θ may depend on the metric of $\mathcal{X}$. It can be computed in principle from M-theory [72]. Note that $\sigma(H)$ may be a priori different from the quadratic refinement appearing in D-instantons corrections, though S-duality suggests that the two should be related.

- At weak coupling, the partition function of a chiral five-brane can be obtained by *holomorphic factorization* of the partition function of a non-chiral 3-form $H = \mathrm{d}\mathcal{B}$ on

$\mathcal{X}$, with **Gaussian** action. This leads to a **Siegel theta series** of rank $b_3(\mathcal{X})$, level $k/2$ satisfying the above periodicity property:

$$\mathcal{Z}^{(k)}_{\mu}(\mathcal{N},C) = N \sum_{n\in\Gamma_m+\mu+\theta} \mathbf{E}\left(\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde\zeta_\Lambda - \phi_\Lambda)n^\Lambda + \frac{k}{2}(\theta^\Lambda\phi_\Lambda - \zeta^\Lambda\tilde\zeta_\Lambda) \right) , \tag{4.23}$$

where Γ_m is a Lagrangian sublattice of $\Gamma = H^3(\mathcal{X},\mathbb{Z})$, N is a C-independent normalization factor, and μ runs over $(\Gamma_m/k)/\Gamma_m$, i.e. over the $|k|^{b_3/2}$ independent holomorphic sections of $\mathcal{L}^k_\Theta$. Plugging into (4.19), we arrive at the 5-brane instanton action in the weak coupling limit,

$$S = 4\pi\frac{|k|}{g^2_{(4)}} + \mathrm{i}k\pi\sigma + \mathrm{i}\pi k(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + 2\pi\mathrm{i}k(\tilde\zeta_\Lambda - \phi_\Lambda)n^\Lambda + \mathrm{i}\pi k(\theta^\Lambda\phi_\Lambda - \zeta^\Lambda\tilde\zeta_\Lambda) \tag{4.24}$$

- The Gaussian approximation breaks down when $|H| \sim 1/g_s$, and $S(H,C)$ should be replaced by the *non-linear* five-brane action. The partition function will no longer be holomorphic, but will satisfy the same transformation property under large gauge transformations.

- For the coupling $e^{-\mathrm{i}\pi k\sigma}\mathcal{Z}^{(k)}$ to be invariant under large gauge transformations, $e^{\mathrm{i}\pi\sigma}$ **must also transform as a section of $\mathcal{L}_\Theta$.** Therefore, σ must pick up additional shifts under discrete translations along $\mathcal{T}$,

$$T'_{H,\kappa} : (C,\sigma) \mapsto \left(C + H, \sigma + \kappa + \langle C, H\rangle - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{n^\Lambda m_\Lambda} + \langle \mathbf{H}, \mathbf{\Theta}\rangle\right) \tag{4.25}$$

where $H \equiv (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{b_3}$, $p \in \mathbb{Z}$. This is also necessary in order that large gauge transformations form a group,

$$T'_{H_2,\kappa_2}T'_{H_1,\kappa_1} = T'_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle + c(H_1)+c(H_2)-c(H_1+H_2)} . \tag{4.26}$$

where $\sigma(H) = (-1)^{2c(H)}$. The r.h.s. is of the form T'_{H_3,κ_3} with $\kappa_3 \in \mathbb{Z}$, thanks to the cocycle property of $\sigma(H)$.

- The second term in $c_1(\mathcal{C}) = \omega_\mathcal{T} + \frac{\chi_\mathcal{X}}{24}\omega_c$ implies that $e^{\mathrm{i}\pi\sigma}$ in addition transforms as a section of $\mathcal{L}^{\chi_\mathcal{X}/24}$ under *monodromies*. More specifically, under a monodromy M, σ must transform as

$$\sigma \mapsto \sigma + \frac{\chi(\mathcal{X})}{24\pi}\,\mathrm{Im}\, f_M + \mathbf{2}\kappa(\mathbf{M}) , \tag{4.27}$$

where f_M is a local holomorphic function on $\mathcal{M}_c(\mathcal{X})$ determined by the rescaling $\Omega_{3,0} \mapsto e^{f_M}\Omega_{3,0}$ of the holomorphic 3-form around the monodromy, and $\kappa(M)$ is again an undetermined constant defined modulo 1. This constant reflects the ambiguity in choosing the 24-th root of unity in the monodromy transformations, and further in picking its logarithm, which are presently not well understood. This is analogous to trying to find the transformation properties of $\log\eta$ under $SL(2,\mathbb{Z})$.

- To summarize, $\mathcal{M}$ is (at least at weak coupling) foliated by hypersurfaces $\mathcal{C}$ which are topologically a circle bundle $\mathcal{L}^{-\chi(\mathcal{X})/24} \otimes \mathcal{L}_\Theta$ over the *intermediate Jacobian* $\mathcal{J}_c(\mathcal{X})$.

4.3 Perturbative HM moduli space in type IIB

- The HM moduli space in type IIB compactified on a CY 3-fold $\hat{\mathcal{X}}$ is a QK manifold $\mathcal{M} \equiv \mathcal{Q}_K(\hat{\mathcal{X}})$ of real dimension $4(h_{1,1}+1)$

  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the **complexified Kähler moduli** $z^a = b^a + \mathrm{i}t^a = X^a/X^0$
  3. the periods of $C = C^{(0)} + C^{(2)} + C^{(4)} + C^{(6)} \in \mathbf{H}^{\mathrm{even}}(\hat{\mathcal{X}}, \mathbb{R})$
  4. the NS axion σ

- In type IIB/$\hat{\mathcal{X}}$, the perturbative metric on HM takes the same form as before, where are now the *Kähler moduli* of $\hat{\mathcal{X}}$, and $(\zeta^0, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \tilde{\zeta}_0)$ label the periods of the RR field .

- Near the infinite volume point, $\mathcal{M}_K(\hat{\mathcal{X}})$ is governed by

$$F(X) = -\frac{N(X^a)}{X^0} + \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma + \chi(\hat{\mathcal{X}}) \frac{\zeta(3)(X^0)^2}{2(2\pi\mathrm{i})^3} + F_{\mathrm{GW}}(X) \tag{4.28}$$

  where $N(X^a) \equiv \frac{1}{6}\kappa_{abc} X^a X^b X^c$, κ_{abc} is the cubic intersection form, $A_{\Lambda\Sigma}$ is a constant, real symmetric matrix, defined up to integer shifts and F_{GW} are *Gromov-Witten* instanton corrections:

$$F_{\mathrm{GW}}(X) = -\frac{(X^0)^2}{(2\pi\mathrm{i})^3} \sum_{k_a\gamma^a \in H_2^+(\hat{\mathcal{X}})} n^{(0)}_{k_a} \,\mathrm{Li}_3 \left[\mathbf{E}\left(k_a \frac{X^a}{X^0} \right) \right] \tag{4.29}$$

  The integers $n^{(0)}_{k_a}$ count the number of rational curves (i.e. genus 0 holomorphic curves) on $\hat{\mathcal{X}}$ in homology class $k_a\gamma^a$. The trilogarithm function $\mathrm{Li}_3(z) = \sum_{n\geq 1} z^n/n^3$ takes care of multicovering effects.

- The perturbative HM metric is as in IIA, with $c = \chi(\hat{\mathcal{X}})/192\pi$. Quantum mirror symmetry implies $\mathcal{Q}_c(\mathcal{X}) = \mathcal{Q}_K(\hat{\mathcal{X}})$. At the perturbative level, this reduces to classical mirror symmetry $\mathcal{SK}^{\mathrm{IIA}}(\mathcal{X}) = \mathcal{SK}^{\mathrm{IIB}}(\hat{\mathcal{X}})$.

- D-instantons are now Euclidean D5-D3-D1-D(-1), described mathematically by *coherent sheaves* E on $\hat{\mathcal{X}}$. Their charge vector γ is related to the Chern classes via the Mukai map

$$q_\Lambda X^\Lambda - p^\Lambda F_\Lambda = e^{-\mathcal{K}/2} Z_\gamma = \int_{\hat{\mathcal{X}}} e^{-(B+\mathrm{i}J)}\, \mathrm{ch}(E) \sqrt{\mathrm{Td}(\hat{\mathcal{X}})} \tag{4.30}$$

  or, in components,

$$p^0 = \mathrm{rk}(E)\ , \qquad p^a = \int_{\gamma^a} c_1(E) \tag{4.31}$$

$$\begin{aligned} q_a &= \int_{\gamma_a} \left[c_2(E) - \frac{1}{2} c_1^2(E) \right] + p^0 \left(A_{0a} - \frac{c_{2,a}}{24} \right) + A_{ab} p^b\ , \\ q_0 &= \int_{\hat{\mathcal{X}}} \mathrm{ch}(E)\, \mathrm{Td}(\hat{\mathcal{X}}) + p^a \left(A_{0a} - \frac{c_{2,a}}{24} \right) + A_{00} p^0. \end{aligned} \tag{4.32}$$

- Noting that $\int_{\hat{\mathcal{X}}} \mathrm{ch}(E)\,\mathrm{Td}(\hat{\mathcal{X}})$ is integer, being the index of the Dirac operator coupled to F, we see that the charges q_Λ are integer iff

$$A_{00} \in \mathbb{Z}\ , \quad A_{0a} \in \frac{c_{2,a}}{24} + \mathbb{Z}\ , \tag{4.33}$$

$$\frac{1}{2}\,\kappa_{abc} p^b p^c - A_{ab} p^b \in \mathbb{Z} \quad \text{for } \forall p^a \in \mathbb{Z}\ . \tag{4.34}$$

E.g. for the quintic, $\kappa_{aaa} = 5, A_{0a} = 25/12, A_{aa} = -11/2, A_{00} = 0$. Under these conditions, the D-instanton charge vector γ lies in $H^{\rm even}(\hat{\mathcal{X}}, \mathbb{Z})$, and the RR multiform C takes values in the **symplectic Jacobian** $T = H^{\rm even}(\hat{\mathcal{X}}, \mathbb{R})/H^{\rm even}(\hat{\mathcal{X}}, \mathbb{Z})$.

- It is often convenient to eliminate $A_{\Lambda\Sigma}$ by a non-integer symplectic transformation, leading to non-integer electric charges q'_Λ,

$$q'_\Lambda = q_\Lambda - A_{\Lambda\Sigma} p^\Sigma\ , \quad \tilde\zeta'_\Lambda = \tilde\zeta_\Lambda - A_{\Lambda\Sigma}\zeta^\Lambda\ , \quad F' = F - \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma \tag{4.35}$$

$$q'_a \in \mathbb{Z} - \frac{\mathbf{p^0}}{\mathbf{24}}\,\mathbf{c_{2,a}} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\kappa}_{\mathbf{abc}}\mathbf{p^b p^c}, \qquad q'_0 \in \mathbb{Z} - \frac{\mathbf{1}}{\mathbf{24}}\,\mathbf{p^a c_{2,a}}\,, \tag{4.36}$$

- In particular, monodromies $b^a \mapsto b^a + \epsilon^a$, $\epsilon^a \in \mathbb{Z}$ around the large volume point are most conveniently described in the primed frame: they act on the axions

$$\begin{aligned} &\zeta^0 \mapsto \zeta^0\ , \qquad \zeta^a \mapsto \zeta^a + \epsilon^a \zeta^0\ , \qquad \tilde\zeta'_a \mapsto \tilde\zeta'_a - \kappa_{abc}\zeta^b\epsilon^c - \frac{1}{2}\,\zeta^0 \kappa_{abc}\epsilon^b\epsilon^c\ , \\ &\tilde\zeta'_0 \mapsto \tilde\zeta'_0 - \epsilon^a\tilde\zeta'_a + \frac{1}{2}\,\kappa_{abc}\zeta^a\epsilon^b\epsilon^c + \frac{1}{6}\,\zeta^0\kappa_{abc}\epsilon^a\epsilon^b\epsilon^c\ , \qquad \sigma \mapsto \sigma + 2\kappa_a\epsilon^a \end{aligned} \tag{4.37}$$

and on the charge lattice as

$$\begin{aligned} &p^0 \mapsto p^0\ , \qquad p^a \mapsto p^a + \epsilon^a p^0\ , \qquad q'_a \mapsto q'_a - \kappa_{abc} p^b\epsilon^c - \frac{1}{2}\,p^0\kappa_{abc}\epsilon^b\epsilon^c\ , \\ &q'_0 \mapsto q'_0 - \epsilon^a q'_a + \frac{1}{2}\,\kappa_{abc} p^a\epsilon^b\epsilon^c + \frac{1}{6}\,p^0\kappa_{abc}\epsilon^a\epsilon^b\epsilon^c\ , \end{aligned} \tag{4.38}$$

in such a way that $q_\Lambda\zeta^\Lambda - p^\Lambda\tilde\zeta_\Lambda = q'_\Lambda\zeta^\Lambda - p^\Lambda\tilde\zeta'_\Lambda$ stays invariant. The constant shift of σ in (4.37) originates from the $\kappa(M)$ term in (4.27). For $p^0 \neq 0$, the combinations

$$\hat q_a = q'_a + \frac{1}{2}\,\kappa_{abc}\frac{p^b p^c}{p^0}\ , \qquad \hat q_0 = q'_0 + \frac{p^a q'_a}{p^0} + \frac{1}{3}\,\kappa_{abc}\frac{p^a p^b p^c}{(p^0)^2}\ , \tag{4.39}$$

are invariant under (4.38). In general, the monodromy may cross lines of marginal stability, and $\Omega(\gamma; z^a)$ is not necessarily invariant.

- In the absence of worldsheet instanton corrections, i.e. retaining only the first two terms in (4.28) and omitting the one-loop correction, the HM metric admits an **isometric action of** $SL(2,\mathbb{R})$, corresponding to type IIB S-duality in 10 dimensions. This action is most easily described in the "primed" frame, using coordinates

$$\begin{aligned} &\zeta^0 = \tau_1\ , \qquad \zeta^a = -(c^a - \tau_1 b^a)\ , \\ &\tilde\zeta'_a = c_a + \frac{1}{2}\,\kappa_{abc}\, b^b(c^c - \tau_1 b^c)\ , \quad \tilde\zeta'_0 = c_0 - \frac{1}{6}\,\kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c)\ , \\ &\sigma = -2(\psi + \frac{1}{2}\tau_1 c_0) + c_a(c^a - \tau_1 b^a) - \frac{1}{6}\,\kappa_{abc}\, b^a c^b (c^c - \tau_1 b^c)\ . \end{aligned} \tag{4.40}$$

In terms of these coordinates, a S-duality transformation $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ acts by

$$
\begin{gathered}
\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \qquad t^a \mapsto t^a|c\tau+d|\,, \qquad c_a \mapsto c_a + \varepsilon_a(\delta)\,, \\
\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}
\end{gathered}
\tag{4.41}
$$

where $\varepsilon_a(\delta)$ is an a priori unspecified constant. In order for the S-duality action $\tau_1 \mapsto \tau_1 + b$ to agree with the Heisenberg shift $\zeta^0 \mapsto \zeta^0 + b$, one must choose

$$
\varepsilon_a(\delta) = -c_{2,a}\,\varepsilon(\delta)\;, \tag{4.42}
$$

where $\varepsilon(\delta)$ is the multiplier system of the Dedekind eta function,

$$
\eta\left(\frac{a\tau+b}{c\tau+d}\right)/\eta(\tau) = \mathbf{E}(\varepsilon(\delta))\,(c\tau+d)^{1/2}\,. \tag{4.43}
$$

- This is consistent with the multiplier system $\mathbf{E}(-c_{2a}p^a\varepsilon(\delta))$ of the D4-D2-D0 partition function, which should describe D-instanton corrections to VM_3 with vanishing D6-brane charge in type IIA/$\mathcal{X}$ [24].

4.4 Twistor techniques for quaternion-Kähler manifolds

- Recall that a Riemannian manifold of real dimension $4n$ is **quaternion-Kähler** if its holonomy group is (exactly) $Sp(n) \times Sp(1)$. $\mathcal{M}$ is then Einstein. SUGRA requires *negative scalar curvature.* Let $\vec{p}$ be the $Sp(1)$ part of the Levi-Civita connection, $\mathrm{d}\vec{p} + \vec{p}\wedge\vec{p} = \frac{\nu}{2}\vec{\omega}$ the quaternionic 2-forms.

- $\mathcal{M}$ does not admit a (global) complex structure. Instead, it is more convenient to study its **twistor space** $\mathcal{Z}$. This is a *complex contact manifold* of real dimension $4n+2$, endowed with a (non-holomorphic) projection $\pi : \mathcal{Z} \to \mathcal{M}$ with $\mathbb{C}P^1$ fibers, and a real structure acting as the antipodal map on $\mathbb{C}P^1$ [31, 32].

- Equivalently, one may consider the hyperkähler cone $\mathcal{S}$, a $\mathbb{C}^2/\mathbb{Z}^2$ bundle over $\mathcal{M}$, or $\mathbb{C}^\times$ bundle over $\mathcal{Z}$, which carries a canonical hyperkähler metric with homothetic Killing vector and $SU(2)$ isometric action [61, 49, 2]. The complex contact structure on $\mathcal{Z}$ is the projectivization of the complex symplectic structure on $\mathcal{S}$.

- The complex contact structure on $\mathcal{Z}$ is given by the kernel of the $(1,0)$-form Dt (which transforms homogeneously under $Sp(1) = SU(2)$ frame rotations)

$$
Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2 \tag{4.44}
$$

Moreover, $\mathcal{M}$ carries a Kahler-Einstein metric

$$
\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dt|^2}{(1+t\bar{t})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}} \tag{4.45}
$$

- **Locally**, there exists a "*contact potential*" $\Phi(x^\mu, t)$ and Darboux complex coordinates $\alpha, \xi, \tilde{\xi}$ such that

$$\mathcal{X} = 2\, e^{\Phi}\, \frac{Dt}{t} = \mathrm{d}\alpha + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \ . \tag{4.46}$$

  Φ provides a Kähler potential K on $\mathcal{Z}$ via $e^K = (1+t\bar{t})e^{Re(\Phi)}/|t|$.

- The complex contact structure can be specified globally by providing **contactomorphisms** on the overlap of two Darboux coordinate patches. Those are conveniently specified by a *Hamilton function* $S^{[ij]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$:

$$\begin{aligned} \xi^\Lambda_{[j]} &= f_{ij}^{-2}\, \partial_{\tilde{\xi}^{[j]}_\Lambda} S^{[ij]}\,, & \tilde{\xi}^{[i]}_\Lambda &= \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\,, \\ \alpha^{[i]} &= S^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\ , & e^{\Phi_{[i]}} &= f_{ij}^2\, e^{\Phi_{[j]}}\,, \end{aligned} \tag{4.47}$$

  where $f_{ij}^2 \equiv \partial_{\alpha^{[j]}} S^{[ij]} = \mathcal{X}^{[i]}/\mathcal{X}^{[j]}$.

- $S^{[ij]}$ are subject to consistency conditions $S^{[ijk]}$, gauge equivalence under local contact transformations $S^{[i]}$, and reality constraints.

- For generic choices of $S^{[ij]}$, the **moduli space of solutions of the above gluing conditions**, regular in each patch, is finite dimensional, and equal to $\mathcal{M}$ itself.

- On each patch U_i, $u^{[i]}_m = (\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]})$ admit a Taylor expansion in t around ζ_i, whose coefficients are functions on $\mathcal{M}$. The functions $u^{[i]}_m(t, x^\mu)$ parametrize the "**twistor line**" over $x^\mu \in \mathcal{M}$.

- The metric on $\mathcal{M}$ can be obtained by expanding $\mathcal{X}^{[i]}$ and $\mathrm{d}u^{[i]}_m$ around t_i, extracting the $SU(2)$ connection $\vec{p}$ and a basis of $(1,0)$ forms on $\mathcal{M}$ in almost complex structure $J(t_i)$, and using $\mathrm{d}\vec{p} + \frac{1}{2}\,\vec{p}\times\vec{p} = \frac{\nu}{2}\,\vec{\omega}$.

- Deformations of $\mathcal{M}$ correspond to deformations of $S^{[ij]}$, so are parametrized by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

- Any (infinitesimal) isometry κ of $\mathcal{M}$ lifts to a **holomorphic** isometry $\kappa_{\mathcal{Z}}$ of $\mathcal{Z}$. The moment map construction [62] provides an element of $H^0(\mathcal{Z}, \mathcal{O}(2))$, given locally by holomorphic functions

$$\mu_\kappa = \kappa_{\mathcal{Z}} \cdot \mathcal{X} = e^{\Phi}\left(\mu_+\, t^{-1} - \mathrm{i}\mu_3 + \mu_- t\right)\ . \tag{4.48}$$

  The moment map of the Lie bracket $[\kappa_1, \kappa_2]$ is the contact-Poisson bracket $\{\mu_{\kappa_1}, \mu_{\kappa_2}\}_{PB}$. The zeros of μ canonically associate a (local) complex structure J_κ to κ.

- **Toric QK manifolds** are those which admit $d+1$ commuting isometries. In this case, one can choose $\mu_{[i]}$ as the position coordinates. The transition functions must then take the form

$$S^{[ij]} = \alpha^{[j]} + \xi^\Lambda_{[i]}\, \tilde{\xi}^{[j]}_\Lambda - H^{[ij]}\,, \tag{4.49}$$

  where $H^{[ij]}$ depends on $\xi^\Lambda_{[i]}$ only.

- More generally, one can consider **"nearly toric QK"**, where $H^{[ij]}$ is a general function but its derivatives wrt to $\tilde{\xi}_\Lambda^{[j]}, \alpha^{[j]}$ are taken to be infinitesimal. For one unbroken isometry κ, $\partial_{\alpha^{[j]}} H^{[ij]} = 0$.

- The twistor lines can then be obtained by Penrose-type integrals, e.g. (in case with one isometry, no "anomalous dimensions")

$$\xi^\Lambda_{[i]} = \zeta^\Lambda + \frac{Y^\Lambda}{t} - t\bar{Y}^\Lambda - \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi \mathrm{i}t'}\, \frac{t'+t}{t'-t}\, \partial_{\tilde{\xi}^{[j]}_\Lambda} H^{[+j]}(t') \tag{4.50}$$

$$e^{\Phi_{[i]}} = \frac{1}{4}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi \mathrm{i}t'} \left(t'^{-1}Y^\Lambda - t'\bar{Y}^\Lambda\right) \partial_{\xi^\Lambda_{[j]}} H^{[+j]}(\xi(t'), \tilde{\xi}(t')) \tag{4.51}$$

The locus $t=0$ defines the complex structure J_κ associated to the unbroken continuous isometry.

4.5 The local c-map in twistor space

- Let us now return to the HM moduli space $\mathcal{M}_H$ in type IIA compactified on X. The twistor space of the tree-level metric (4.7) is governed by the transition functions [63]

$$H^{[0+]}_{\rm pert} = \frac{i}{2} F(\xi^\Lambda)\ , \quad H^{[0-]}_{\rm tree} = \frac{i}{2} \bar{F}(\xi^\Lambda) \tag{4.52}$$

The one-loop corrected metric (4.12) is governed by the same transition function, but allowing for a logarithmic singularity in the Darboux coordinate α at the north and south pole: the Darboux coordinates around the equator are given [upon defining $\tilde{\xi}_\Lambda \equiv -2\mathrm{i}\tilde{\xi}^{[0]}_\Lambda, \tilde{\alpha} \equiv 4\mathrm{i}\alpha^{[0]} + 2\mathrm{i}\tilde{\xi}^{[0]}_\Lambda \xi^\Lambda, W(z) \equiv F_\Lambda \zeta^\Lambda - X^\Lambda \tilde{\zeta}_\Lambda$] by

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \left(t^{-1}X^\Lambda - t\,\bar{X}^\Lambda\right)/g_s^2\,, \\ \tilde{\xi}_\Lambda &= \tilde{\zeta}_\Lambda + \left(t^{-1}F_\Lambda - t\,\bar{F}_\Lambda\right)/g_s^2\,, \\ \tilde{\alpha} &= \sigma + \left(t^{-1}W - t\,\bar{W}\right)/g_s^2 - \tfrac{\mathrm{i}\chi_{\hat{\mathcal{X}}}}{24\pi}\,\log t\,, \end{aligned} \tag{4.53}$$

- The large volume monodromy $b^a \mapsto b^a + \epsilon^a$ on $\mathcal{M}$ lifts to a holormorphic action on the twistor space $\mathcal{Z}$,

$$\begin{aligned} &\xi^0 \mapsto \xi^0\,, \qquad \xi^a \mapsto \xi^a + \epsilon^a \xi^0\,, \qquad \tilde{\xi}'_a \mapsto \tilde{\xi}'_a - \kappa_{abc}\xi^b\epsilon^c - \frac{1}{2}\,\kappa_{abc}\epsilon^b\epsilon^c\xi^0\,, \\ &\tilde{\xi}'_0 \mapsto \tilde{\xi}'_0 - \tilde{\xi}'_a\epsilon^a + \frac{1}{2}\,\kappa_{abc}\xi^a\epsilon^b\epsilon^c + \frac{1}{6}\,\kappa_{abc}\epsilon^a\epsilon^b\epsilon^c\xi^0\,, \qquad \tilde{\alpha} \mapsto \tilde{\alpha} + 2\kappa_a\epsilon^a\,. \end{aligned} \tag{4.54}$$

Similarly, the Heisenberg action (4.25) lifts to a holomorphic action on $\mathcal{Z}_\mathcal{M}$ given by

$$\begin{aligned} T'_{(H,\kappa)}\ :\left(\xi^\Lambda,\ \tilde{\xi}_\Lambda,\ \tilde{\alpha}\right) \mapsto \Big(&\xi^\Lambda + \eta^\Lambda,\ \tilde{\xi}_\Lambda + \tilde{\eta}_\Lambda, \\ &\tilde{\alpha} + 2\kappa - \tilde{\eta}_\Lambda\xi^\Lambda + \eta^\Lambda\tilde{\xi}_\Lambda - \left(\eta^\Lambda\tilde{\eta}_\Lambda - 2\tilde{\eta}_\Lambda\theta^\Lambda + 2\eta^\Lambda\phi_\Lambda\right)\Big)\,, \end{aligned} \tag{4.55}$$

where $\eta^\Lambda, \tilde{\eta}_\Lambda, \kappa \in \mathbb{Z}$. Thus, the quotient of $\mathcal{Z}_\mathcal{M}$ by translations along $\partial_{\tilde{\alpha}}$ defines a complexified torus $\mathcal{T}_\mathbb{C}$, parametrized by the coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ and their complex conjugates, while $e^{\mathrm{i}\pi\tilde{\alpha}}$ parametrizes the fiber of a $\mathbb{C}^\times$-bundle $\mathcal{L}^\mathbb{C}_\Theta$ over $\mathcal{T}_\mathbb{C}$.

- The S-duality action (4.41) can be lifted to a holomorphic action on Z [64, 65, 3], e.g. in the patch U_0

$$
\begin{gathered}
\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}, \qquad \tilde{\xi}'_a \mapsto \tilde{\xi}'_a + \frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c - c_{2,a}\varepsilon(\delta)\,, \\
\begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} + \frac{1}{6}\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0+d) \\ -[c^2(a\xi^0+b)+2c]/(c\xi^0+d)^2 \end{pmatrix}.
\end{gathered}
\tag{4.56}
$$

where $\alpha' = (\tilde{\alpha} + \xi^\Lambda \rho'_\Lambda)/(4\mathrm{i})$. This reproduces the action (4.41) on the base together with a $SU(2)$ action along the fiber,

$$
t \mapsto \frac{c\tau_2 + t(c\tau_1+d) + t|c\tau+d|}{(c\tau_1+d)+|c\tau+d| - tc\tau_2}\,. \tag{4.57}
$$

- The transformation rule of ρ'_a can be summarized by saying that $\mathbf{E}(p^a\rho'_a)$ transforms like the automorphy factor of a multi-variable Jacobi form of index $m_{ab} = \frac{1}{2}\kappa_{abc}p^c$ and multiplier system $\mathbf{E}(-c_{2a}p^a\varepsilon(\delta))$.

- The tree-level, large volume contact potential $e^\Phi = \frac{\tau_2^2}{2}V(t^a)$, though not invariant, transforms so that $K_{\mathcal{Z}}$ undergoes a Kähler transformation,

$$
e^\Phi \mapsto \frac{e^\Phi}{|c\tau+d|}\,, \quad K_{\mathcal{Z}} \mapsto K_{\mathcal{Z}} - \log(|c\xi^0+d|)\,, \quad \mathcal{X}^{[i]} \to \frac{\mathcal{X}^{[i]}}{c\xi^0+d} \tag{4.58}
$$

4.6 D-instanton corrections

- Restoring the worldsheet instanton and one-loop corrections, the perturbative contact potential e^Φ no longer transforms homogeneously,

$$
\begin{aligned}
e^\Phi =& \frac{\tau_2^2}{2}V - \frac{\chi_Y\zeta(3)}{8(2\pi)^3}\tau_2^2 - \frac{\chi_Y}{192\pi} \\
&+ \frac{\tau_2^2}{4(2\pi)^3}\sum_{q_a\gamma^a\in H_2^+(Y)} n_{0,q_a}\,\mathrm{Re}\left[\mathrm{Li}_3\left(e^{2\pi\mathrm{i}q_a z^a}\right) + 2\pi q_a t^a\,\mathrm{Li}_2\left(e^{2\pi\mathrm{i}q_a z^a}\right)\right]
\end{aligned}
\tag{4.59}
$$

- $SL(2,\mathbb{Z})$ invariance can be restored by summing over images [65],:

$$
\tau_2^{k/2}\,\mathrm{Li}_k(e^{2\pi\mathrm{i}q_a z^a}) \to {\sum_{m,n}}' \frac{\tau_2^{k/2}}{|m\tau+n|^k}e^{-S_{m,n,q}}\,, \tag{4.60}
$$

where $S_{m,n,q} = 2\pi q_a|m\tau+n|\,t^a - 2\pi\mathrm{i}q_a(mc^a+nb^a)$ is the action of an (m,n)-string wrapped on $q_a\gamma^a$.

- After Poisson resummation on $n \to q_0$, we get a sum over D(-1)-D1 bound states

$$
\begin{aligned}
e^\Phi =& \frac{\tau_2^2}{2}V \\
&+ \frac{\tau_2}{8\pi^2}\sum_{q_0\in\mathbb{Z},q_a\gamma^a\in H_2^+(Y)} n_{q_a}^{(0)}\sum_{m=1}^\infty \frac{|q_\Lambda X^\Lambda|}{m}\cos\left(2\pi m\,q_\Lambda\zeta^\Lambda\right)K_1\left(2\pi m\,|q_\Lambda X^\Lambda|\tau_2\right)
\end{aligned}
\tag{4.61}
$$

- Going back to type IIA variables, these are interpreted as Euclidean $D2$ wrapped on SLAG in a Lagrangian subspace of $H_3(X, \mathbb{Z})$ (A-cycles only). These effects correct the mirror map into [66]

$$\tilde{\zeta}_a = \tilde{\zeta}_a^{(0)} + \frac{1}{8\pi^2} \sum_{q_a} n_{0,q} \sum_{n\in\mathbb{Z}, m\neq 0} \frac{m\tau_1 + n}{m|m\tau + n|^2} e^{-S_{m,n,q}}, \dots \tag{4.62}$$

- In the "one instanton" approximation, the contributions of B-cycles can be restored by **symplectic invariance** [3]:

$$e^{\Phi} = \dots + \frac{\tau_2}{8\pi^2} \sum_{\gamma} \sigma(\gamma)\, n_\gamma(z^a) \sum_{m=1}^{\infty} \frac{|W_\gamma|}{m} \cos\left(2\pi m \Theta_\gamma\right) K_1\left(2\pi m\, |W_\gamma|\right) \tag{4.63}$$

where $$W_\gamma \equiv \frac{1}{2}\tau_2 \left(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda\right), \quad \Theta_\gamma \equiv q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda , \tag{4.64}$$

and the quadratic refinement $\sigma(\gamma)$ is inserted for consistency with wall-crossing, as in §2.7.

- This can be lifted to an infinitesimal deformation of the contact structure on $\mathcal{Z}$, generated by

$$H = \frac{\mathrm{i}}{2(2\pi)^2} \sum_{\gamma;\, \mathrm{Re}\,(W_\gamma)>0} \sigma(\gamma)\, \sigma(\gamma)\, n_\gamma\, \mathrm{Li}_2\left(e^{-2\pi \mathrm{i}(q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)}\right) . \tag{4.65}$$

- Note that S-duality has turned Li_3 in the worldsheet instanton sum into Li_2 in the D-instanton sum.

- Beyond the "one-instanton" approximation (but still neglecting NS5-instantons), the twistor space can be described as follows. For fixed z^a, consider all "BPS rays" $\ell(\gamma) = \{t : \pm W_\gamma/t \in \mathrm{i}\mathbb{R}^-\}$ on S^2.

- The contact structure on $\mathcal{Z}$ is obtained by gluing Darboux coordinate patches on each sector, using a contact transformation U_γ generated by

$$S_\gamma^{[ij]} = \alpha^{[j]} + \xi_{[i]}^\Lambda\, \tilde{\xi}_\Lambda^{[j]} + \frac{\mathrm{i}}{2(2\pi)^2} \sigma(\gamma)\, n_\gamma\, \mathrm{Li}_2\left(e^{-2\pi\mathrm{i}(q_\Lambda \xi_{[i]}^\Lambda - p^\Lambda \tilde{\xi}_\Lambda^{[j]})}\right) . \tag{4.66}$$

- As $z^a \in \mathcal{M}_V$ is varied, the BPS rays may cross. The invariants n_γ should transform so as to leave the contact structure intact. By the same token as in §2.7, this will be the case provided $n_\gamma(z^a)$ satisfies the Kontsevich-Soibelman wall-crossing formula. By T-duality, $n_\gamma(z^a)$ should be equal to the generalized Donaldson-Thomas invariants $\Omega(\gamma, t)$. In terms of the rational invariants,

$$S_\gamma^{[ij]} = \alpha^{[j]} + \xi_{[i]}^\Lambda\, \tilde{\xi}_\Lambda^{[j]} + \frac{\mathrm{i}}{2(2\pi)^2} \sigma(\gamma)\, \bar{\Omega}(\gamma, t)\; e^{-2\pi\mathrm{i}(q_\Lambda \xi_{[i]}^\Lambda - p^\Lambda \tilde{\xi}_\Lambda^{[j]})} . \tag{4.67}$$

Since $\Omega(\gamma, t)$ grows exponentially fast, the D-instanton series must be treated as a divergent asymptotic series [5].

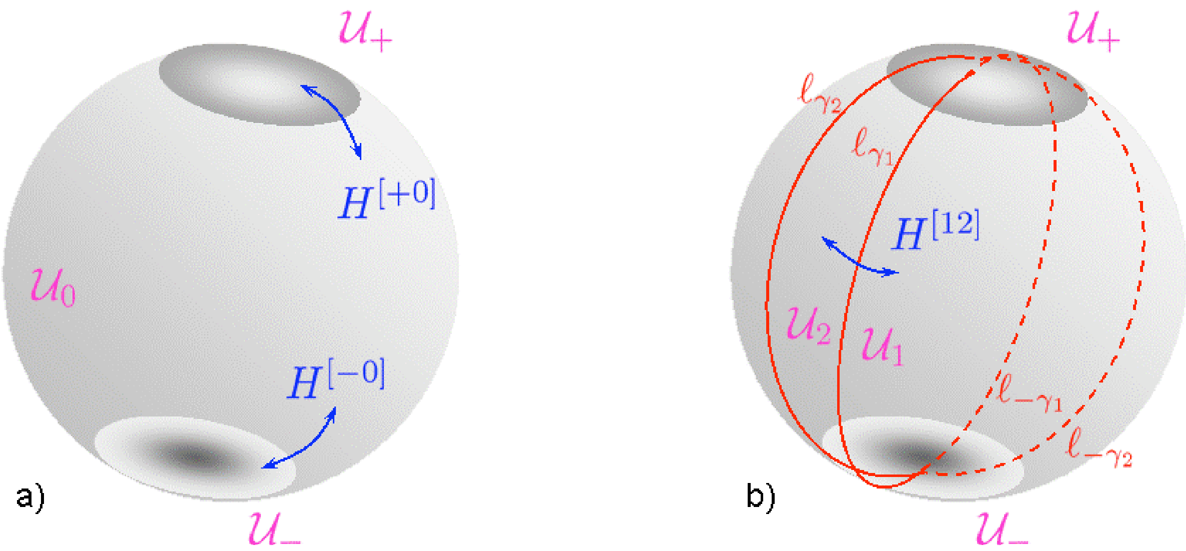

Figure 5: Melon-like structure of the twistor fiber over a fixed point in moduli space

4.7 Towards NS5-brane / Kaluza-Klein monopole corrections

On the type IIB side, the NS5-brane instanton corrections can be obtained by S-duality from the D5-brane corrections. Starting with the generating function (4.65) for p^0 D5-branes, p^a D3-branes, q_a D1-branes, q_0 D-instantons, we apply an S-duality transformation

$$\delta = \begin{pmatrix} a & b \\ -k/p^0 & p/p^0 \end{pmatrix} \in SL(2,\mathbb{Z})\,, \tag{4.68}$$

where $(p,k) \neq (0,0)$ are two integers with greatest common divisor (gcd) p^0, and the integers (a,b), ambiguous up to the addition of $(k/p^0, -p/p^0)$, are chosen such that $ap + bk = p^0$. Using (4.56), one arrives at

$$H_{k,p,\hat{\gamma}} = \frac{\sigma(\gamma)}{4\pi^2}\,\tilde{\Omega}(\gamma)\,\mathbf{E}\left(-\frac{k}{2}\,S_\alpha + \frac{p^0\,(k\hat{q}_a(\xi^a - n^a) + p^0\hat{q}_0)}{k^2(\xi^0 - n^0)} + a\,\frac{p^0 q_0'}{k} - c_{2,a}p^a\varepsilon(\delta) \right), \tag{4.69}$$

where $n^0 \equiv p/k$, $n^a \equiv p^a/k$, valued in $\mathbb{Z}/k$ and

$$S_\alpha \equiv \tilde{\alpha} + (\xi^\Lambda - 2n^\Lambda)\tilde{\xi}'_\Lambda + 2\frac{N(\xi^a - n^a)}{\xi^0 - n^0}\,. \tag{4.70}$$

The function $H_{k,p,\hat{\gamma}}$ describes the discontinuity of the Darboux coordinates across the image of the BPS ray ℓ_γ under (4.57). The set of functions $H_{k,p,\hat{\gamma}}$ provides (at least formally) a holomorphic section of $H^1(\mathcal{Z},\mathcal{O}(2))$ which is by construction invariant under S-duality. It also turns out that the set of functions is invariant under the Heisenberg and monodromy actions (4.55), (4.54), subject to two caveats: i) this invariance seems to require that the index $\bar{\Omega}(\gamma, z^a)$ be invariant under the spectral flow action (4.38), which is not true in general and ii) the set is only invariant up to some constant phases, which depend on the quadratic refinements. In spite of our inability to resolve these important issues, we boldly forge ahead.

Instanton corrections to the metric on $\mathcal{M}$ can in principle be obtained by performing the contour integrals (4.51). For simplicity, we shall treat the set of functions $H_{k,p,\hat{\gamma}}$ as a holomorphic section of $H^1(\mathcal{Z}, \mathcal{O}(-2))$, which by the Penrose transform formula

$$\Psi = \sum_j \int_{C_j} \frac{\mathrm{d}t}{t}\, e^{\Phi^{[j]}(t)}\, H^{[ij]}\left(\xi^\Lambda_{[i]}(t), \tilde{\xi}^{[j]}_\Lambda(t), \alpha^{[j]}(t)\right) \ , \tag{4.71}$$

produces a scalar valued "harmonic" function on $\mathcal{M}$ (here, "harmonic" means that Ψ is annihilated by a certain set of second-order partial differential operators determined by the quaternion-Kähler structure on $\mathcal{M}$).

In the weak coupling limit, the contour integral (4.71) is dominated by saddle points with classical action

$$\begin{aligned} S_{k,p,\hat{\gamma}} = 4\pi\, |W_{k,p,\hat{\gamma}}| + 2\pi\mathrm{i}\Big[&-(pc_0 + k\psi + p^a c_a) + \frac{p - k\tau_1}{k|p - k\tau|^2}\left(N(\tilde{p}^a) - p^0 \hat{q}_a \tilde{p}^a + (p^0)^2 \hat{q}_0\right) \\ &+ \frac{b^a}{2k}\left(\frac{1}{3}\,\kappa_{abc}(pb^b - kc^b)(3\tilde{p}^c + pb^c - kc^c) + \kappa_{abc}\tilde{p}^b\tilde{p}^c - 2p^0\hat{q}_a\right) - \frac{a}{k}\, q'_0 p^0 + c_{2,a}p^a\epsilon(\delta)\Big]. \end{aligned} \tag{4.72}$$

where $\hat{\gamma}$ is the "reduced charge vector" $(p^a, \hat{q}_a, \hat{q}_0)$ and $\tilde{p}^a \equiv p^a + kc^a - pb^a$, and the "generalized central charge" is

$$W_{k,p,\hat{\gamma}} \equiv \delta \cdot W_\gamma = \frac{\tau_2}{2|p - k\tau|^2}\left[N\left(\tilde{p}^a - i|p - k\tau|t^a\right) - p^0\hat{q}_a\left(\tilde{p}^a - \mathrm{i}|p - k\tau|t^a\right) + (p^0)^2\hat{q}_0\right], \tag{4.73}$$

This is the expected action for (p, k)5-branes, related by S-duality to the usual D5- brane instanton action

$$S_\gamma = 4\pi|W_\gamma| + 2\pi\mathrm{i}(q'_\Lambda\zeta^\Lambda - p^\Lambda\tilde{\zeta}'_\Lambda), \tag{4.74}$$

where

$$W_\gamma = \frac{\tau_2}{2}\left(\frac{N(p^a - p^0 z^a)}{(p^0)^2} - \frac{\hat{q}_a(p^a - p^0 z^a)}{p^0} + \hat{q}_0\right). \tag{4.75}$$

In the weak coupling limit $\tau_2 \to \infty$, the the (p, k)5-brane reduces to

$$S_{k,p,\hat{\gamma}} = 2\pi|k|V\tau_2^2 + \pi\mathrm{i}k\left(\sigma + \zeta^\Lambda\tilde{\zeta}'_\Lambda - 2n^\Lambda\tilde{\zeta}'_\Lambda - \bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Lambda - n^\Lambda)(\zeta^\Sigma - n^\Sigma)\right) - 2\pi\mathrm{i}m_\Lambda z^\Lambda \ . \tag{4.76}$$

where $n^0 = p/k, n^a = p^a/k, m_a = p^0\hat{q}_a/k, m_0 = ap^0q'_0/k - c_{2,a}p^a\epsilon(\delta)$. For $m_\Lambda = 0$, one recovers the chiral 5-brane action (4.24). At zero coupling $g_{(4)} = 0$ the sum over m_Λ decouples and produces a metric dependent normalization factor. Setting $k = 1$ for simplicity and using the DT/GW relation [77, 78]

$$e^{F_{\rm hol}(z,\lambda)} = \lambda^{-\frac{\chi(\hat{\mathcal{X}})}{24}\epsilon_{\rm GW}}\left[M(e^{-\lambda})\right]^{(\frac{1}{2} - \epsilon_{\rm DT})\chi(\hat{\mathcal{X}})}\, e^{F_{\rm pol}} \sum_{Q_a, J}(-1)^{2J} N_{DT}(Q_a, 2J)\, e^{-2\lambda J + 2\pi\mathrm{i}Q_a z^a} \ , \tag{4.77}$$

where $M(q) = \prod(1 - q^n)^{-n}$ is the Mac-Mahon function and $Z_{\rm DT}$ is the partition function of (ordinary, rank 1) Donaldson-Thomas invariants

$$Z_{\rm DT} \equiv \sum_{Q_a, J}(-1)^{2J}\, N_{DT}(Q_a, 2J)\, e^{-2\lambda J + 2\pi\mathrm{i}Q_a z^a}. \tag{4.78}$$

we arrive at

$$\Psi \sim \tau_2\, e^{f_1-\mathcal{K}}\, (\bar{z}^\Lambda\, \mathrm{Im}\, \tau_{\Lambda\Sigma}\bar{z}^\Sigma)^{-1/2} \sum_{n\in\Gamma_m+\theta} t_s^{-1-\frac{\chi(\hat{\mathcal{X}})}{24}}\, e^{-2\pi \mathrm{i} n^\Lambda \phi_\Lambda - \pi \mathrm{i} k\left(\sigma+\zeta^\Lambda\tilde{\zeta}'_\Lambda - 2n^\Lambda\tilde{\zeta}'_\Lambda - \bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Lambda-n^\Lambda)(\zeta^\Sigma-n^\Sigma)\right)}\,, \tag{4.79}$$

Thus, we find that NS5-brane instanton corrections in the weak coupling limit are indeed given by a Gaussian theta series, with an unexpected flux-dependent insertion and a normalization factor proportional to the one-loop topological amplitude e^{f_1}. Since e^{f_1} transforms as a section of $\mathcal{L}^{1-\frac{\chi(\mathcal{X})}{24}}$ under monodromies, this is in broad agreement with the topology of the NS-axion circle bundle σ. While this result is satisfying, it is clear that our present understanding of NS5/KKM instanton corrections is still very sketchy. The complete answer presumably requires a generalization of the KS wall-crossing formula which involves products of contact transformations rather than just symplectomorphism. Unfortunately or fortunately, this generalization does not seem to be available on the mathematical (black) market for now, and it is a great challenge to try and uncover it.

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