QK/HK correspondence, Wall-crossing and the Rogers dilogarithm

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AGMP 2011, Mulhouse
25/10/2011

based on work with Alexandrov, Persson [1110.0466], Saueressig, Vandoren [0812.4219]

Introduction I

- HyperKähler spaces appear in many mathematics and physics problems:
  - moduli space of Yang-Mills instantons on M_4, magnetic monopoles on M_3, Higgs bundles on M_2, Nahm's equations on M_1, ...
  - self-dual solutions of 4D Einstein gravity, ALE spaces
  - complexified coadjoint orbits,
  - Coulomb branch of Seiberg-Witten gauge theories on $\mathbb{R}^3 \times S^1$
  - ...
- Quaternion-Kähler spaces are less common:
  - Conformally self-dual solutions of 4D Einstein gravity + cosmological constant
  - Swann quotients of HK cones (e.g. complexified coadjoint orbits)
  - Hypermultiplet moduli spaces in $\mathcal{N} = 2$ supersymmetric string vacua in $D = 3, 4, 5, 6$
  - Vector multiplet moduli spaces in $\mathcal{N} = 2$ supersymmetric string vacua in $D = 3$

Introduction II

- In this talk, I will describe a general duality between
  - $4n$-dimensional QK spaces with one quaternionic circle action
  - $4n$-dimensional HK spaces with one rotational isometric action
- I will explain that the Coulomb branch of SW gauge theories on $\mathbb{R}^3 \times S^1$ admits a canonical hyperholomorphic connection, built out of integer data (the BPS spectrum) and involving the Rogers dilogarithm function. The existence of this connection depends on the Kontsevich-Soibelman wall-crossing formula.
- Via the QK/HK correspondence, this provides the D-instanton corrected quaternion-Kähler metric on the VM moduli space of $\mathcal{N} = 2$ string vacua in $D = 3$, or equivalently on the HM moduli space of $\mathcal{N} = 2$ string vacua in any dimension.

Introduction III

- This construction recovers the QK metric found in earlier work by Alexandrov, Saueressig, Vandoren and myself (2008), but resolves some conceptual issues.
- The QK/HK correspondence was discovered independently by Haydys (2007) and is being further investigated by Hitchin, Swann, Marcia (2011). The hyperholomorphic line bundle on the Coulomb branch of SW theories was constructed independently by Neitzke (2011).
- Similar constructions have appeared in the work of Fock and Goncharov (2010) on cluster varieties.

Outline

Outline

Basics of the QK/HK correspondence I

- Let $\mathcal{M}$ be a quaternion-Kähler (QK) manifold of real dimension $4n$, i.e. with restricted holonomy $SU(2) \times USp(2n)$. $\mathcal{M}$ admits a triplet of covariantly constant 2-forms $\vec{\omega}$, proportional to the curvature of the $SU(2)$ Levi-Civita connection:

$$\mathrm{d}\vec{p} + \frac{1}{2}\vec{p} \wedge \vec{p} = \frac{\nu}{2}\vec{\omega} \,, \qquad \nu = \frac{R}{4n(n+2)}$$

- The total space of the $\mathbb{R}^4/\mathbb{Z}_2$ bundle with connection $\vec{p}$ over $\mathcal{M}$ carries a natural hyperkähler metric with a homothetic Killing vector and a $SU(2)_R$ isometric action rotating the complex structures: the Swann bundle or HK cone $\mathcal{S}$.
- If $\nu > 0$, $\mathcal{S}$ has positive definite signature, while if $\nu < 0$, $\mathcal{S}$ has Lorentzian signature $(4, 4n)$.

Basics of the QK/HK correspondence II

- Suppose $\mathcal{M}$ admits a quaternionic circle action, i.e. a Killing vector X which exponentiates to a $U(1)_A$ action such that $\mathscr{L}_X(\vec{\omega}\wedge\vec{\omega})=0$. The $U(1)_A$ action on $\mathcal{M}$ lifts to a tri-holomorphic action on $\mathcal{S}$, with moment map $\vec{\mu}_X$ such that $\iota_X\vec{\omega}_{\mathcal{S}}=\mathrm{d}\vec{\mu}_X$.
- The HK quotient at level $\vec{m}\neq 0$

$$\mathcal{M}'=\left(\mathcal{S}\cap\{\vec{\mu}_X=\vec{m}\}\right)/U(1)_A$$

produces a $4n$-dimensional HK manifold $\mathcal{M}'$ with a $U(1)_R$ isometric action which fixes $\omega'_3=\vec{m}\cdot\vec{\omega}_S$ and rotates $\omega'_\pm=(\vec{m}\cdot\wedge\vec{\omega}_S)_\pm$.

- The level set $\mathcal{P}=\mathcal{S}\cap\{\vec{\mu}_X=\vec{m}\}$ is a circle bundle over $\mathcal{M}'$, endowed with a connection λ (the restriction of the Levi-Civita connection on $\mathcal{S}$). λ is hyperholomorphic, i.e. its curvature $\mathscr{F}=\mathrm{d}\lambda$ is of type (1,1) in all complex structures.

- Conversely, given a $4n$-dimensional HK manifold $\mathcal{M}'$ equipped with a Killing vector Y which fixes ω'_3, rotates $\omega'_\pm$ and exponentiates to a $U(1)_A$ circle action. Defining

$$\mathscr{F} = 2\mathrm{i}\,\partial\bar{\partial}\,\rho - \omega'_3$$

  where $\iota_Y \omega'_3 = \mathrm{d}\rho$ and ∂ is the Dolbeault derivative in complex structure J'_3, then $\mathscr{F}$ is the curvature of a hyperholomorphic line bundle $\mathcal{P}$ on $\mathcal{M}'$. For any connection λ on $\mathcal{P}$ invariant under $U(1)_R$, one can associate a $4n$-dimensional QK manifold $\mathcal{M} \sim \mathcal{P}/U(1)_R$ with a quaternionic isometry.
- The QK metric on $\mathcal{M}$ depends not only on $\mathscr{F}$ but on λ itself. In particular, shifting λ by the exact one-form $c\mathrm{d}\theta'$, where θ' is a coordinate along the $U(1)_R$ fiber, leads to a one-parameter family of QK metrics.

QK/HK correspondence and Toda equation I

- Let us illustrate the QK/HK correspondence in four dimensions. In this case, QK and HK metrics with one (rotational) isometry are governed by solutions of the continual Toda equation

$$\partial_{z\bar{z}} T + \partial_{\rho\rho}\, \mathrm{e}^{T} = 0 \,.$$

- HK metrics with one rotational isometry can be cast in Boyer-Finley ansatz

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{1}{2}\left[\partial_\rho T \left(\mathrm{d}\rho^2 + 4\mathrm{e}^T \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{4}{\partial_\rho T}\left(\mathrm{d}\theta' + \Theta'\right)^2\right],$$

where

$$\mathrm{d}\Theta' \equiv -\mathrm{i}\, \partial_{z\bar{z}} T\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \,.$$

QK/HK correspondence and Toda equation II

- QK metrics with one isometry can be cast in Tod's ansatz

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{1}{2}\left[\frac{P}{\rho^2}\left(\mathrm{d}\rho^2 + 4\mathrm{e}^T \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{1}{P\rho^2}\left(\mathrm{d}\theta + \Theta\right)^2\right],$$

where $P \equiv 1 - \frac{1}{2}\rho\partial_\rho T$ and

$$\mathrm{d}\Theta = \mathrm{i}(\partial_z P \mathrm{d}z - \partial_{\bar{z}} P \mathrm{d}\bar{z}) \wedge \mathrm{d}\rho - 2\mathrm{i}\,\partial_\rho(P\mathrm{e}^T)\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

- Under the QK/HK correspondence, $\mathcal{M}$ is dual to $\mathcal{M}'$ equipped with the hyperholomorphic connection

$$\lambda = \frac{2P}{\partial_\rho T}\left(\mathrm{d}\theta' + \Theta'\right) + \Theta.$$

- The ambiguity $\lambda \mapsto \lambda + c\mathrm{d}\theta'$ corresponds to the freedom of translating the solution of Toda's equation under $\rho \mapsto \rho + c$.

QK/HK correspondence and Toda equation III

- For example, $\mathcal{M} = S^4$ ($\epsilon = 1$) or $\mathcal{M} = H^4$ ($\epsilon = -1$) are dual to $\mathcal{M}' = \mathbb{R}^4$ with

$$\begin{aligned} \mathrm{d}s^2_{\mathcal{M}'} =& \epsilon \left[\mathrm{d}R^2 + R^2 \left(\mathrm{d}\delta^2 + \sin^2\delta\, \mathrm{d}\beta^2 + \cos^2\delta\, \mathrm{d}\gamma^2 \right) \right] \;, \\ T =& 2\log \frac{\epsilon(4\rho - 1)}{4\cosh(z + \bar{z})} \,, \quad \rho = \frac{1}{4}(1 + \epsilon R^2), \\ \theta' =& \beta + \gamma, \quad z = \tfrac{1}{2}\left(\log\tan\delta + \mathrm{i}(\beta - \gamma) \right) \end{aligned}$$

equipped with a flat hyperholomorphic connection and $U(1)_R$ action

$$\lambda = -\frac{1}{4}\mathrm{d}(\beta + \gamma) \,, \qquad \mathscr{F} = 0 \,, \qquad Y = \partial_\beta + \partial_\gamma$$

QK/HK correspondence and Toda equation IV

- As another example, $\mathcal{M}' = \mathbb{R}^4$ in bi-polar coordinates

$$\begin{aligned} \mathrm{d}s^2_{\mathcal{M}'} =& \mathrm{d}R^2 + R^2(\mathrm{d}\theta')^2 + \mathrm{d}\tilde{R}^2 + \tilde{R}^2(\mathrm{d}\tilde{\theta}')^2, \\ T =& \log(\rho + c)\ , \quad \rho = \tfrac{1}{2}\, R^2 - c, \quad z = \tilde{R} e^{\mathrm{i}\tilde{\theta}'}/\sqrt{2} \end{aligned}$$

with a connection with constant anti-self-dual curvature

$$\lambda = \tfrac{1}{2}\left(R^2\,\mathrm{d}\theta' - \tilde{R}^2\,\mathrm{d}\tilde{\theta}'\right) + c\mathrm{d}\theta'\ , \qquad Y = \partial_{\theta'}$$

is dual to $\mathcal{M} = \mathbb{C}P^{1,1}$ for $c = 0$, or more generally to the 'one-loop deformed universal hypermultiplet metric ' if $c \neq 0$,

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{\rho + 2c}{4\rho^2(\rho + c)}\,\mathrm{d}\rho^2 + \frac{\rho + 2c}{16\rho^2}\left((\mathrm{d}\zeta^2 + 4\mathrm{d}\tilde{\zeta}^2\right) + \frac{\rho + c}{64\rho^2(\rho + 2c)} D\sigma^2$$

where $z = -\frac{1}{4}\,(\zeta - 2\mathrm{i}\tilde{\zeta}), \theta = -\frac{1}{8}\,\sigma, D\sigma = \mathrm{d}\sigma + \tilde{\zeta}\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\tilde{\zeta}$.

QK/HK correspondence in twistor space I

- Recall that the HK metric on $\mathcal{M}'$ can be conveniently encoded in a complex symplectic structure on the twistor space $\mathcal{Z}' = \mathbb{P}^1_\zeta \times \mathcal{M}'$,

$$\omega' = \omega'_+ - \mathrm{i}\zeta\omega'_3 + \zeta^2\omega'_- , \qquad \omega'_\pm = -\frac{1}{2}(\omega'_1 \mp \mathrm{i}\omega'_2)$$

- Locally, there exists complex Darboux coordinates $(\eta^\Lambda_{[i]}, \mu^{[i]}_\Lambda)$ such that

$$\omega' = \mathrm{d}\eta^\Lambda_{[i]} \wedge \mathrm{d}\mu^{[i]}_\Lambda .$$

- On the overlap of 2 patches, the Darboux coordinates are related by a complex symplectomorphism determined by a generating function $H^{[ij]}_{\mathrm{HK}}(\eta_{[i]}, \mu^{[j]}, \zeta)$. For HK manifolds with a $U(1)_R$ isometry, $H^{[ij]}_{\mathrm{HK}}$ is independent of ζ.

QK/HK correspondence in twistor space II

- These gluing conditions can be rewritten as a set of integral equations

$$\eta^\Lambda_{[i]}(\zeta) = x^\Lambda + \zeta^{-1} v^\Lambda - \zeta \bar{v}^\Lambda - \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \partial_{\mu^{[j]}_\Lambda} H^{[ij]}_{\mathrm{HK}}(\zeta')$$

$$\mu^{[i]}_\Lambda(\zeta) = \varrho_\Lambda + \tfrac{1}{2} \textstyle\sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \partial_{\eta^\Lambda_{[i]}} H^{[ij]}_{\mathrm{HK}}(\zeta')$$

  where $v^\Lambda, x^\Lambda, \varrho_\Lambda$ are coordinates on $\mathcal{M}'$.
- The HK metric can be obtained by plugging the solution of these equations into $\omega' = \mathrm{d}\eta^\Lambda_{[i]} \wedge \mathrm{d}\mu^{[i]}_\Lambda$, and Taylor expanding around any point on $\mathbb{P}^1$.

QK/HK correspondence in twistor space III

- Similarly, the QK metric on $\mathcal{M}$ can be encoded in a complex contact structure on the twistor space $\mathcal{Z}$, a non-trivial $\mathbb{P}^1_t$ bundle over $\mathcal{M}$, the kernel of

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2 ,$$

- Locally, there exists Darboux coordinates $(\xi^\Lambda_{[i]}, \tilde\xi^{[i]}_\Lambda, \alpha^{[i]})$ such that

$$\mathcal{X}^{[i]} \equiv 4\, e^{\Phi^{[i]}}\, \frac{Dt}{\mathrm{i}t} = \mathrm{d}\alpha^{[i]} + \tilde\xi^{[i]}_\Lambda \mathrm{d}\xi^\Lambda_{[i]} \,.$$

  where the 'contact potential' e^Φ is holomorphic along the fibers.

- On the overlap of 2 patches, the Darboux coordinates are related by a complex contactomorphism determined by a generating function $H^{[ij]}_{\mathrm{QK}}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]})$. For QK manifolds with a $U(1)_A$ isometry, $H^{[ij]}_{\mathrm{QK}}$ can be chosen to be independent of $\alpha^{[j]}$.

QK/HK correspondence in twistor space IV

- These gluing conditions can be rewritten as a set of integral equations

$$\xi^\Lambda_{[i]}(t) = A^\Lambda + t^{-1}\,Y^\Lambda - t\,\bar Y^\Lambda - \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\,\partial_{\tilde\xi^{[j]}_\Lambda} H^{[ij]}_{\mathrm{QK}},$$

$$\tilde\xi_\Lambda^{[i]}(t) = B_\Lambda + \tfrac{1}{2}\sum_j \oint_{C_j} \tfrac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\tfrac{t'+t}{t'-t}\,\partial_{\xi^\Lambda_{[j]}} H^{[ij]}_{\mathrm{QK}},$$

$$\alpha^{[i]}(t) = \theta + \tfrac{1}{2}\sum_j \oint_{C_j} \tfrac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\tfrac{t'+t}{t'-t}\left(H^{[ij]}_{\mathrm{QK}} - \xi^\Lambda_{[i]}\partial_{\xi^\Lambda_{[i]}} H^{[ij]}_{\mathrm{QK}}\right) + 4\mathrm{i}c\log t.$$

  where $A^\Lambda, B_\Lambda, \theta$ and $Y^\Lambda/U(1)$ serve as coordinates on $\mathcal{M}$.

- The QK metric can be obtained by plugging the solution of these equations into the complex contact form, and Taylor expanding around any point on $\mathbb{P}^1$.

QK/HK correspondence in twistor space V

- For a dual pair of QK/HK spaces, the Darboux coordinates are identified up to a rotation on $\mathbb{P}^1$,

$$\eta^{\Lambda}_{[i]}(\zeta) = \xi^{\Lambda}_{[i]}(t)\,, \qquad \mu^{[i]}_{\Lambda}(\zeta) = \tilde\xi^{[i]}_{\Lambda}(t)\,, \qquad t = \zeta\, e^{-\mathrm{i}\theta'}\ .$$

  in particular $\mathcal{Z}$ and $\mathcal{Z}'$ are described by the same transition functions

$$H^{[ij]}_{\rm HK}(\Xi) = H^{[ij]}_{\rm QK}(\Xi)\,.$$

- In addition, the contact Darboux coordinate $\alpha^{[i]}$ on $\mathcal{Z}$ yields a holomorphic section

$$\Upsilon^{[i]}(\zeta) = e^{-2\mathrm{i}\pi\alpha^{[i]}(t)}$$

  of a complex line bundle $\mathscr{L}_{\mathcal{Z}'}$, trivial along the twistor lines.

QK/HK correspondence in twistor space VI

- By the usual twistor correspondence, this determines a hyperholomorphic connection

$$\lambda = \frac{1}{4}\left(\bar\partial^{(\zeta)}\alpha^{[i]} + \partial^{(\zeta)}\bar\alpha^{[i]}\right),$$

where $\partial^{(\zeta)}$ is the Dolbeault derivative in complex structure $J(\zeta)$. In particular, the log-norm of the section Υ gives a Kähler potential,

$$\mathscr{F} = \frac{\mathrm{i}}{2}\,\partial\bar\partial^{(\zeta)}\mathrm{Im}\alpha^{[i]} \,.$$

- One advantage of this point of view is that the twistor space $\mathcal{Z}'$ is trivially fibered over $\mathbb{P}^1$, while $\mathcal{Z}$ is a non-trivial fibration by $\mathbb{P}^1$'s. This simplification is key for understanding wall-crossing in QK moduli spaces.

Outline

Seiberg-Witten theories in 4 dimensions I

- Consider a $\mathcal{N}=2$ gauge theory in $\mathbb{R}^4$ with a rank r gauge group G, broken to $U(1)^r$ on the Coulomb branch. The massless scalar fields take values in a rigid special Kähler manifold $\mathcal{B}$, with Kähler potential

$$K_{\mathcal{B}} = \mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) \ , \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma} F$$

  where $F_\Lambda = \partial_{X^\Lambda}$ is the derivative of the prepotential F.
- F can be obtained by realizing $\mathcal{B}$ as the parameter space of a suitable family of genus r Riemann surfaces Σ_u, and computing period integrals

$$X^\Lambda = \int_{A^\Lambda} \lambda \ , \qquad F_\Lambda = \int_{B_\Lambda} \lambda \ .$$

  where λ is a suitable meromorphic one-form and (A^Λ, B_Λ) a symplectic basis of $H_1(\Sigma_u, \mathbb{Z})$.

Reduction on a circle I

- Upon compactifying the $\mathcal{N}=2$ gauge theory on a circle $S^1(R)$, the moduli space $\mathcal{B}$ is enlarged to a HK space $\mathcal{M}'$ which includes the holonomies $C=(\zeta^\Lambda,\tilde\zeta_\Lambda)$ of the gauge fields $\mathcal{A}^\Lambda$ and their magnetic duals $\tilde{\mathcal{A}}_\Lambda$ along the circle.
- Large gauge transformations imply that C lives in the rank $2r$ torus $\mathcal{T}=H^1(\Sigma_u,\mathbb{R})/H^1(\Sigma_u,\mathbb{Z})$. Topologically, $\mathcal{M}'$ is the Jacobian of the family of Riemann surfaces Σ_u.
- As $R\to\infty$, the HK metric takes the semi-flat form

$$\mathrm{d}s^2_{\mathcal{M}'}=R\,\mathrm{d}s^2_{\mathcal{B}}+\frac{1}{R}\left(\mathrm{d}\tilde\zeta_\Lambda-\tau_{\Lambda\Sigma}\zeta^\Sigma\right)[\mathrm{Im}\tau]^{\Lambda\Lambda'}\left(\mathrm{d}\tilde\zeta_{\Lambda'}-\tau_{\Lambda'\Sigma'}\zeta^{\Sigma'}\right).$$

  This HK metric is known as the rigid c-map of the rigid special Kähler metric on $\mathcal{B}$.

Reduction on a circle II

- The complex Darboux coordinates on the twistor space $\mathcal{Z}'$ of the semi-flat torus bundle $\mathcal{M}'$ can be chosen as the complex moment maps of the torus action,

$$\xi^\Lambda = \zeta^\Lambda + \frac{R}{2}\left(\zeta\,\bar{X}^\Lambda - \zeta^{-1}\,X^\Lambda\right) \equiv \xi^\Lambda|_{\rm sf}$$
$$\tilde{\xi}_\Lambda = \tilde{\zeta}_\Lambda + \frac{R}{2}\left(\zeta\,\bar{F}_\Lambda - \zeta^{-1}\,F_\Lambda\right) \equiv \tilde{\xi}_\Lambda|_{\rm sf}$$

Neitzke BP, unpublished

- The coordinates $\xi^\Lambda, \tilde{\xi}_\Lambda$ are multi-valued on the torus, they may be traded for the holomorphic Fourier modes

$$\mathcal{X}_\gamma \equiv \exp\left[2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde{\xi}_\Lambda)\right]\ , \qquad \gamma\in\Gamma = H^1(\Sigma_u,\mathbb{Z})$$

The instanton corrected metric I

- At finite R, we expect instanton corrections to the semi-flat metric, from Euclidean dyons wrapping around S^1. These corrections will break continous translational invariance along the torus fiber.
- In the one-instanton approximation, these corrections are expected to be of order $\Omega(\gamma, u)\, e^{-R|Z(\gamma,u)|+2\pi\mathrm{i}\langle\gamma,C\rangle}$, where Z is the central charge and Ω is the BPS index,

$$Z(\gamma; u) = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda \,, \qquad \Omega(\gamma, u) = -\frac{1}{2}\mathrm{Tr}(-1)^{2J_3}\, J_3^2$$

The instanton corrected metric II

- $\Omega(\gamma, u)$ is a locally constant function of u, but it is discontinuous on certain codimension-one walls of marginal stability in $\mathcal{B}$, where the central charges of two charge vectors γ_1, γ_2 align,

$$W(\gamma_1, \gamma_2) = \{u \ / \ \arg(Z(\gamma_1; u) = \arg(Z(\gamma_2; u)\}$$

  due to the decay of bound states of n dyons with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, with fixed total charge $\gamma = \sum \alpha_i$. Yet, after including multi-instanton corrections, the exact metric should be smooth.

The KS wall-crossing formula I

- In the context of Donaldson-Thomas invariants on a CY three-fold, Kontsevich and Soibelman have shown that the jump of $\Omega_{\rm DT}(\gamma, u)$ satisfies the following property. Consider the Lie algebra spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the algebra of (twisted) Hamiltonian vector fields on the symplectic torus $\Gamma^* \otimes \mathbb{R}$,

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \, e_{\gamma_1+\gamma_2}$$

For a given charge vector γ and VM moduli u^a, consider the operator $U_\gamma(u^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; u^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2} \right)$$

The KS wall-crossing formula II

- The KS wall-crossing formula states that the product

$$A(u^a) = \prod_{\substack{\gamma = M\gamma_1 + N\gamma_2, \\ M\geq 0, N\geq 0}} U_\gamma(u^a) \ ,$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As u^a crosses W, $\Omega(\gamma; u^a)$ jumps and the order of the factors is reversed, such that A stays constant. Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0, \\ M/N\downarrow}} U^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0, \\ M/N\uparrow}} U^-_{M\gamma_1+N\gamma_2} \ ,$$

This provides an algorithm to compute $\Omega^+(\gamma)$ in terms of $\Omega^-(\gamma)$.

The KS wall-crossing formula III

- The KS formula admits a motivic generalization, which allows to compute the jump of the refined BPS index (or P.S.C.)

$$\Omega(\gamma, y, u) = \mathrm{Tr}'(-y)^{2J_3} = \sum_{n\in\mathbb{Z}} (-y)^n \Omega_n(\gamma, u)$$

Dimofte Gukov; Gaiotto Moore Neitzke

The operators U_γ are replaced by

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \left[\Psi(y^n \hat{\mathcal{X}}_\gamma)\right]^{-(-1)^n \Omega_n(\gamma, z^a)}, \quad \Psi(x) = \prod_{n=0}^{\infty} (1 + (-y)^{2n+1} x)^{-1}$$

where Ψ is the quantum dilogarithm and $\hat{\mathcal{X}}_\gamma$ are generators of the quantum torus

$$\hat{\mathcal{X}}_\gamma \, \hat{\mathcal{X}}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle} \, \hat{\mathcal{X}}_{\gamma+\gamma'} \, .$$

In the limit $y \to 1$, it reduces to the 'numerical' KS formula.

The KS wall-crossing formula IV

- For example, the $SU(3)$ Argyres-Douglas theory has 2 chambers, with BPS spectrum $\{\gamma_1, \gamma_2\}$ on one side, $\{\gamma_2, \gamma_1 + \gamma_2, \gamma_1\}$ on the other side, all with $\Omega(\gamma) = 1, \sigma(\gamma) = -1$.
- The motivic KS formula is equivalent to the pentagon identity for the quantum dilogarithm,

$$\Psi(x_1)\,\Psi(x_2) = \Psi(x_2)\,\Psi(x_{12})\,\Psi(x_1)\;, \qquad x_1 x_2/y = y x_2 x_1 \equiv -x_{12}$$

- In the classical limit $y \to 1$, it reduces to

$$U_{0,1}\, U_{1,1}\, U_{1,0} = U_{1,0}\,, U_{0,1}$$

where $U_{p,q}: \quad [x, y] \mapsto [(1 + x^p y^q)^q x, (1 + x^p y^q)^{-p} y]$

The KS wall-crossing formula V

- In the semi-classical limit, it implies the five-term relation for $\mathcal{L}(z) \equiv \mathrm{Li}_2(-z) + \frac{1}{2}\log(z)\log(1+z)$,

$$\mathcal{L}(x) + \mathcal{L}\left(\frac{-xy}{1-x}\right) + \mathcal{L}\left(\frac{y}{1+x+xy}\right) = \mathcal{L}(x(1+y)) + \mathcal{L}(y)$$

equivalent to the usual five-term relation for the Rogers dilogarithm $L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z)$,

$$L(x) - L\left(\frac{x(1-y)}{1-xy}\right) - L\left(\frac{y(1-x)}{1-xy}\right) + L(y) - L(xy) = 0$$

Instanton corrections in twistor space I

- The Darboux coordinates for the instanton-deformed HK metric on $\mathcal{M}'$ are given by solutions of a system of integral equations

$$\mathcal{X}_\gamma = \mathcal{X}^{\rm sf}_\gamma \, e^{\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta+\zeta'}{\zeta-\zeta'} \log\left(1-\sigma(\gamma')\, \mathcal{X}_{\gamma'}(\zeta')\right)}$$

  where $\ell(\gamma) \subset \mathbb{P}^1$ is the 'BPS ray'

$$\ell(\gamma) = \{ t \ : Z(\gamma; u^a)/t \in i\mathbb{R}^- \}$$

  where $\sigma(\gamma) = \pm 1$ is a choice of quadratic refinement
- This may be solved iteratively by plugging $\mathcal{X}_\gamma \to \mathcal{X}^{\rm sf}_\gamma$ etc, eventually leading to a multi-instanton expansion for the components of the metric.

Gaiotto Moore Neitzke

Instanton corrections in twistor space II

- In particular, across a BPS ray $\ell(\gamma)$, the Darboux coordinates jump by the symplectomorphism

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'}(1-\sigma(\gamma)\mathcal{X}_{\gamma})^{\langle\gamma,\gamma'\rangle\,\Omega(\gamma,u)}\,. \quad [*]$$

- As the moduli u^a vary in $\mathcal{B}$, BPS rays may cross each other. Upon identifying U_γ with the symplectomorphism $[*]$, the numerical KSwall-crossing formula guarantees that the twistor space is well-defined and that the HK metric is smooth.

Hyperholomorphic line bundle and Rogers dilog I

- We now restrict to superconformal field theories, such that the prepotential is a homogeneous function of degree 2. $\mathcal{M}'$ then admits a $U(1)_R$ action $X^\Lambda \to e^{\mathrm{i}\theta'} X^\Lambda$. Thus, it admits a canonical hyperholomorphic line bundle and a QK dual metric.
- Identifying the generating functions $H^{[ij]}_{\mathrm{HK}}(\eta,\mu)$ for [$*$], we find that the gluing conditions for the holomorphic section $\tilde\Upsilon \equiv \Upsilon\, e^{-\mathrm{i}\pi\xi^\Lambda\tilde\xi_\Lambda}$ of the line bundle $\mathscr{L}_{\mathcal{Z}'}$:

$$\tilde\Upsilon \mapsto \tilde\Upsilon\, \exp\left(\frac{\mathrm{i}}{2\pi}\,\Omega(\gamma,u)\, L_{\sigma(\gamma)}(\sigma(\gamma)\mathcal{X}_\gamma)\right)$$

where $L_\epsilon(z)$ is a variant of the Rogers dilogarithm,

$$L_\epsilon(z) \equiv \mathrm{Li}_2(z) + \frac{1}{2}\log(\epsilon^{-1} z)\log(1-z)\,.$$

- In particular, the coordinate $\tilde{\alpha} = -2\alpha - \xi^\Lambda \tilde{\xi}_\Lambda$ along the $\mathbb{C}^\times$ fiber jumps across the BPS ray ℓ_γ by

$$\Delta_\gamma \tilde{\alpha} = \frac{1}{2\pi^2}\,\Omega(\gamma)\,L\left(e^{-2\pi \mathrm{i} \langle \gamma, \Xi \rangle}\right),$$

- The semi-classical limit of the motivic KS wall-crossing formula guarantees the consistency of this prescription as u is varied.
- Requiring that Υ smoothly extends to the north and south pole,

$$\tilde{\alpha} = \sigma + \zeta^{-1}\mathcal{W} - \zeta\bar{\mathcal{W}} - \frac{\mathrm{i}}{8\pi^3}\sum_\gamma \int_{\ell_\gamma} \frac{\mathrm{d}\zeta'}{\zeta'}\,\frac{\zeta+\zeta'}{\zeta-\zeta'}\,\Omega(\gamma)\,L_{\sigma(\gamma)}(\sigma(\gamma)\mathcal{X}_\gamma)$$

where

$$\mathcal{W} = X^\Lambda \tilde{\zeta}_\Lambda - F_\Lambda \zeta^\Lambda + \frac{1}{8\pi^2}\sum_\gamma \Omega(\gamma)\,Z_\gamma \int_{\ell_\gamma} \frac{\mathrm{d}\zeta'}{\zeta'}\,\log\left(1-\sigma(\gamma)\mathcal{X}_\gamma\right).$$

Hyperholomorphic line bundle and Rogers dilog III

- This in turn determines a hyperholomorphic circle bundle over $\mathcal{M}'$, smooth across the wall:

$$\lambda = \frac{1}{\mathrm{i}}(\partial - \bar{\partial})K_{\mathcal{B}} + \frac{1}{8}\left(\zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda - \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda\right) + \mathrm{inst.}$$

- This construction extends to the case where there is no $U(1)_R$ symmetry on $\mathcal{M}'$, at the cost of allowing $1/\zeta^2$ and ζ^2 terms in $\tilde{\alpha}$ (Neitzke, 2011). However, for applications to the hypermultiplet moduli space in string vacua, the superconformal case suffices.

Outline

Hypermultiplet moduli space revisited I

- In type II string theory compactified on a Calabi-Yau three-fold $\mathcal{X}$, the hypermultiplet moduli space is a QK manifold $\mathcal{M}$, determined classically from a prepotential $F(X)$.
- The one-loop corrected metric on $\mathcal{M}$ turns out to be dual to the (Lorentzian) rigid c-map metric with prepotential $F(X)$ under the QK/HK correspondence. The one-loop correction corresponds to shifting $\lambda \mapsto \lambda + c\mathrm{d}\theta'$ where $c = -\chi(\mathcal{X})/(192\pi)$.
- D-instanton corrections to the QK metric preserve the quaternionic Killing vector ∂_σ. They can be obtained by applying the QK/HK correspondence to the instanton corrected HK metric, using the Donaldson-Thomas invariants of X in place of $\Omega(\gamma, u)$.

Hypermultiplet moduli space revisited II

- This recovers and clarifies the construction of the D-instanton corrected metric in Alexandrov BP Saueressig Vandoren (2008). The 'contact transformations across BPS rays' on $\mathcal{Z}$ are now interpreted as gauge transformations of the dual line bundle on $\mathcal{Z}'$, and are now proven to be consistent with wall-crossing.
- Some problems remain however: the topology of the axion circle bundle in the one-loop corrected metric is still puzzling, the D-instanton series is divergent, ...
- The outstanding problem is to understand NS5-brane corrections, which break the remaining $U(1)_A$ isometry. This should in principle follow by S-duality from the D-instanton corrections, but S-duality in twistor space is tough...