Progress on hypermultiplet moduli spaces

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ZMP Colloquium, Hamburg
28/01/2010

Introduction I

- Understanding the vector multiplet moduli space $\mathcal{M}_V$ in Calabi-Yau compactifications of string theory has led to one of the success stories of the math/physics interface: classical mirror symmetry, Gromov-Witten invariants, ...

Candelas de la Ossa Green Parkes; Strominger Yau Zaslow; ...

- Incorporating open strings and D-branes in the picture has considerably expanded the range of math/physics interactions: calibrations, Donaldson-Thomas invariants, K-theory, homological mirror symmetry, ...

Douglas Moore; Kontsevich; ...

Introduction II

- The hypermultiplet moduli space $\mathcal{M}_H$ in CY compactifications is yet poorly understood, but must combine all these notions with quaternion-Kähler geometry and (hopefully) new mathematics: generalized and motivic Donaldson-Thomas invariants, automorphy, …
- $\mathcal{M}_H$ should be the correct framework for a quantum version of mirror symmetry, with far-reaching mathematical implications.
- Physically, $\mathcal{M}_H$ is (probably) the correct framework for precision counting of BPS black holes in $N = 2$ supergravity (or BPS solitons in $N = 2$, Seiberg Witten gauge theories). In addition it involves the physics of NS5-branes in an essential way.

Introduction III

- Today, I will review recent progress towards understanding $\mathcal{M}_H$, based in part on my own work with Alexandrov, Saueressig and Vandoren.

APSV 2008-09

- In view of time, I will not cover related progress in understanding the moduli space of $D = 4$, $\mathcal{N} = 2$ super Yang-Mills theories reduced on a circle, though some of the ideas that we shall use have first appeared in this context.

Seiberg Witten 1996; Gaiotto Moore Neitzke 2008-09

Outline

1. Classical and homological mirror symmetry
2. The perturbative hypermultiplet moduli space
3. Twistor methods for quaternion-Kähler spaces
4. The non-perturbative hypermultiplet moduli space

Outline

Set-up I

- Consider type IIA string theory compactified on a Calabi-Yau three-fold X. The low energy physics is described by $\mathcal{N}=2, D=4$ (ungauged) supergravity, with $n_V = h^{1,1}(X)$ vector multiplets and $n_H = h^{2,1}(X)+1$ hypermultiplets.
- A vector multiplet (VM) consists of one complex-valued field t^a and one 1-form A^a_μ (hence its name), plus fermionic fields. A hypermultiplet (HM) consists of one quaternion-valued field q^Λ, plus fermions.
- The massless scalar fields $(t^a(x^\mu), q^\Lambda(x^\mu))$ provide a map from $D=4$ Minkowski space time into a Riemannian manifold $\mathcal{M}$, known as the moduli space. $\mathcal{M}=\mathcal{M}_V \times \mathcal{M}_H$ splits into the product of a projective special Kähler (PSK) manifold $\mathcal{M}_V$, of real dimension $2n_V$, and a quaternion-Kähler (QK) manifold $\mathcal{M}_H$, of real dimension $4n_H$.

Set-up II

- $\mathcal{M}_V \equiv \mathcal{SK}_K(X)$ parametrizes the complexified Kähler structure of X, while $\mathcal{M}_H \equiv \mathcal{QK}_C(X)$ parametrizes the complex structure of X, schematically

$$t^a = \int_{\gamma^a} B + \mathrm{i}\, J = b^a + \mathrm{i}\, j^a \,, \qquad q^\Lambda = \int_{\gamma^\Lambda} \Omega + j\, \mathcal{R}$$

where (J, Ω) are the Kähler and (3,0) form, $(B, \mathcal{R})$ are the NS 2-form and RR multiform, γ^a a basis of $H_2(X, \mathbb{Z})$ and γ^Λ a basis of $H_3(X, \mathbb{Z})$. *Ω is normalized such that $\int_{\gamma^0} \Omega = \sigma + iV/g_s^2 l_s^6$, where V is the volume of X and g_s the string coupling.*

- The goal is to compute the Riemannian metric on $\mathcal{M}$ from data about X. String theory provides an asymptotic expansion in powers of the string coupling constant g_s, which a particular coordinate on $\mathcal{M}_H$. The main difficulty is in understanding non-perturbative effects of order e^{-1/g_s} or smaller.

$\mathcal{M}_V$ and classical mirror symmetry I

- The VM moduli space $\mathcal{M}_V$ is very well understood. By definition, its metric is independent of g_s, so can be computed in classical string theory. Still, it depends on the symplectic structure of X in a very non-trivial way.
- Since $\mathcal{M}_V$ is a projective special Kähler manifold, its geometry is encoded in the prepotential $F(X^\Lambda)$, a holomorphic function of projective coordinates X^Λ, $t^a = X^a/X^0$, homogeneous of degree two. Its third derivative $F_{\Lambda\Sigma\Xi}$ encodes the Yukawa couplings in the SUGRA action.

$\mathcal{M}_V$ and classical mirror symmetry II

- In the limit $V \gg l_s^6$, F is determined by the intersection product $C_{abc} = \int_X J_a J_b J_c$ in $H_4(X)$ and the Euler number χ. In addition, there are exponentially suppressed corrections (worldsheet instantons); here $\mathrm{e}^q = e^{2\pi i q_a X^a / X^0}$:

$$F = -C_{abc} \frac{X^a X^b X^c}{6X^0} + \chi\zeta(3) \frac{(X^0)^2}{2(2\pi i)^3} - \frac{(X^0)^2}{(2\pi i)^3} \sum_{q \in H_2^+(X)} N_{0,q} \, \mathrm{e}^q$$

- $N_{0,q}$ are rational numbers known as the genus 0 Gromov-Witten invariants. Defining $n_{0,q}$ via the multi-covering formula

$$\sum_q N_{0,q} \, \mathrm{e}^q = \sum_{q, d \geq 1} n_{0,q} \frac{\mathrm{e}^{dq}}{d^3}$$

The integers $n_{0,q}$ count the number of rational curves in homology class q. *They can be used to define the quantum cohomology ring of X.*

$\mathcal{M}_V$ and classical mirror symmetry III

- The Gromov-Witten invariants $N_{0,q}$ are most conveniently computed using (classical) mirror symmetry. Recall that for any (non-rigid) CY threefold X, there exists a mirror Calabi-Yau Y, such that $h_{1,1}(X) = h_{2,1}(Y)$, $h_{2,1}(X) = h_{1,1}(Y)$; if X is fibered by T^3, Y is fibered by T-dual/Mukai-transformed T^3).

Candelas et al; Strominger Yau Zaslow

- Mirror symmetry requires that $\mathcal{M}_V^{IIA}(X) = \mathcal{M}_V^{IIB}(Y)$, so $\mathcal{SK}_K(X) = \mathcal{SK}_C(Y)$. The prepotential $F(X^\Lambda)$ follows from period integrals of the (3,0) form Ω on Y:

$$X^\Lambda = \int_{\gamma^\Lambda} \Omega \,, \qquad F_\Lambda = \int_{\gamma_\Lambda} \Omega = \partial_\Lambda F \,,$$

where $\gamma^\Lambda, \gamma_\Lambda$ is a symplectic basis of $H_3(Y, \mathbb{Z})$, adapted to the point of maximal unipotent monodromy.

BPS spectrum and homological mirror symmetry I

- Mirror symmetry requires not only $\mathcal{M}_V^{IIA}(X) = \mathcal{M}_V^{IIB}(Y)$, but also that the full type IIA/X and type IIB/Y string theories be equivalent. In particular, the spectrum of BPS states should match.
- BPS states in type IIA/X are obtained by wrapping $D0, D2, D4, D6$ branes on complex submanifolds of X. More generally, they are realized as coherent sheaves on X; even more accurately, as elements in the derived category of coherent sheaves $DCoh(X)$.

Douglas

- BPS states in type IIB/Y are obtained by wrapping $D3$-branes on special Lagrangian cycles (SLAGs) of Y. More precisely, they are realized as elements in the Fukaya category $Fuk(Y)$.

BPS spectrum and homological mirror symmetry II

- These algebras are graded by the charge vector $\gamma \in H_{\rm even}(X,\mathbb{Z})$ in type IIA, or $\gamma \in H_3(Y,\mathbb{Z})$ in type IIB (more accurately, $\gamma \in K(X)$)

Minasian, Moore, ...

- Each of these derived categories are endowed with a stability condition, determined by a choice of point in $\mathcal{M}_V$, which allows to decide which D-brane configurations are stable.
- The number of such configurations (counted with sign) defines the generalized Donaldson-Thomas invariant $\Omega(\gamma, t)$. It is a locally constant function on $\mathcal{M}_V$. It can jump on certain codimension one walls in $\mathcal{M}_V$, known as lines of marginal stability (LMS), according to certain (recently established) wall-crossing formulae.

Bridgeland; Joyce Son; Kontsevich Soibelman...

BPS spectrum and homological mirror symmetry III

- For D-brane charge $\gamma = [X] \oplus 0 \oplus q_a \gamma^a \oplus 2J[pt] \in H_{\rm even}(X)$ in type IIA, and t in a suitable domain, $\Omega(\gamma, t) = N_{DT}(q, 2J)$. In general, it provides a non-Abelian generalization of the Donaldson-Thomas invariants.
- The homological mirror symmetry conjecture states that $DCoh(X, t) = Fuk(Y, t)$ as an isomorphism of triangulated categories with a stability condition. In particular, the generalized Donaldson-Thomas invariants must agree.

Kontsevich

Microscopics and Macroscopics of BPS states I

- Physically, $\Omega(\gamma, t)$ arises as the Witten index of some SUSY quantum mechanical system (quiver gauge theory) describing the microscopic dynamics of open strings in a D-brane background,

$$\Omega(\gamma, t) = \mathrm{Tr}_{\mathcal{H}_{BPS(\gamma,t)}}(-1)^F$$

- BPS states have an alternative macroscopic description as certain BPS solutions in $N = 2$ supergravity. Some of them are spherically symmetric black hole solutions of Reissner-Nordström type. Others are molecule-like bound states of several BPS black holes, whose relative distances depend on the value of the VM moduli at spatial infinity.

- The Bekenstein-Hawking formula gives a powerful prediction for the growth of the BPS degeneracies [A=total horizon area]

$$\Omega(\gamma) \sim e^{\frac{1}{4}A(\gamma)} \qquad \text{as} \qquad |\gamma| \to \infty,$$

- The LMS is characterized by one of the relative distances going to infinity, i.e the bound state decaying into multi-particle states. This gives a macroscopic prediction for the wall-crossing formula obeyed by $\Omega(\gamma, t)$

Denef Moore

Topological strings I

- In addition to the metric on $\mathcal{M}_V$, there are also higher-derivative corrections to the effective action, of the type $A_h(t) R^2 T^{2h-2}$, where R is the Riemann tensor of (weakly perturbed) $\mathbb{R}^{3,1}$ and T is the curvature of the graviphoton one-form.
- A_h is known to receive exponentially suppressed (worldsheet instanton) corrections from genus h curves in X. The generating function $F_{\rm top}(t,\lambda) = \sum_{h=0}^{\infty} \lambda^{2h-2} A_h(t)$ ($A_0 \equiv F(X), X^0 = \mathrm{i}\lambda$) is computed by the topological A-model on X,

$$F_{\rm top} = F_{\rm polar} + \sum_{h,q} N_{h,q}\, \mathrm{e}^{q} \lambda^{2h-2} \ , \quad F_{\rm polar} = -C_{abc} \frac{t^a t^b t^c}{6\lambda^2} - c_{2a} t^a$$

where $c_{2a} = \int_X J_a c_2(TX)$ and $N_{g,q}$ are the genus h GW invariants. $\Psi_{\rm top} = e^{F_{\rm top}}$ is the topological string amplitude.

Bershadsky Cecotti Ooguri Vafa; Antoniadis Gava Narain Taylor

Topological strings II

- The Gromov-Witten invariants $N_{g,q}$ are in turn related to the Gopakumar-Vafa invariants $n_{h,q} \in \mathbb{Z}$ by the multicovering formula

$$\sum_{h\geq 0,q} N_{h,q}\, \mathrm{e}^{q} \lambda^{2h-2} = \sum_{h\geq 0,q,d\geq 1} \frac{1}{d} n_{h,q}\, \mathrm{e}^{dq} \left(2 \sin \frac{d\lambda}{2}\right)^{2h-2}$$

- The GW/GV/DT invariants are related via

$$e^{F_{\rm top}-F_{\rm polar}} = [M(e^{-\lambda})]^{-\chi/2} \sum_{q_a,J} (-1)^{2J}\, N_{DT}(q,2J)\, e^{-2\lambda J} \mathrm{e}^{q}$$

where $M(e^{-\lambda}) = \prod (1-e^{-n\lambda})^{-n}$ is the Mac-Mahon function.

Maulik Nekrasov Okounkov Pandharipande

- There is also a much debated conjecture relating $\Omega(\gamma, t)$ and the topological string amplitude $\Psi_{\rm top}$...

Ooguri Strominger Vafa

Outline

Perturbative hypermultiplet moduli space I

- At weak coupling g_s, the story with $\mathcal{M}_H$ runs very similar to $\mathcal{M}_V$. However, unlike $\mathcal{M}_V$, $\mathcal{M}_H$ receives non-perturbative corrections from D-brane and NS5-brane instantons. Thus, it combines Gromov-Witten theory, generalized Donaldson-Thomas theory and presumably new math/physics related to NS5-branes.
- $\mathcal{M}_H$ is a quaternion-Kähler space of real dimension $4(h_{1,2}(X)+1)$. Despite the name, $\mathcal{M}_V$ is not Kähler, and carries no (globally defined) complex structure.
- In type IIA/X, $\mathcal{M}_H \equiv \mathcal{QK}_C(X)$ parametrizes the complex structure of X, together with the string coupling constant g_s, the RR 3-form $\mathcal{R} \in \mathrm{Jac}(X) \equiv H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$ and the NS-axion $\sigma \in S^1$.

Perturbative hypermultiplet moduli space II

- In the limit $g_s \to 0$, the QK metric on $\mathcal{M}_H$ looks like

$$\mathrm{d}s^2_{\mathcal{M}_H} = \left(\frac{\mathrm{d}g_s}{g_s}\right)^2 + \mathrm{d}s^2_{\mathcal{M}_C} + g_s^2\,\mathrm{d}s^2_{\mathrm{Jac}} + g_s^4\,(\mathrm{d}\sigma + \mathcal{A})^2$$

  where $\mathrm{d}s^2_{\mathcal{M}_C}$ is the PSK metric on the moduli space of complex structures, the same as the VM moduli space in type IIB, and $\mathcal{A}$ is a connection on the circle bundle S^1_σ, with 1st Chern class $\mathrm{d}\mathcal{A} \propto \omega_{\mathrm{Jac}}$. This is known as the "c-map" or "semi-flat" metric.

  Ceccoti Ferrara Girardello; Ferrara Sabharwal

- The effect of the one-loop correction in string theory is (roughly) to shift $g_s^2 \to g_s^2 + \chi$ and $\omega_{\mathrm{Jac}} \to \omega_{\mathrm{Jac}} + \chi\,\omega_{\mathcal{M}_C}$. As a result, the metric has a curvature singularity at $g_s^2 \sim \chi$.

  Antoniadis Minasian Theisen Vanhove; Günther Herrmann Louis, ...

Perturbative hypermultiplet moduli space III

- No perturbative corrections to $\mathrm{d}s_{\mathcal{M}_H}$ are expected beyond one-loop, since they would ruin the quantization of $c_1(\mathcal{A})$.
- Instanton corrections from Euclidean D2-branes wrapping SLAGs are expected to break the translational isometries along $\mathrm{Jac}(X)$ at order e^{-1/g_s}.
- Instanton corrections from Euclidean NS5-branes wrapping X are expected to break the translational isometry along S^1_σ at order e^{-1/g_s^2}.
- The challenge is to find the exact quantum corrected QK metric on $\mathcal{M}_H$. Unfortunately, type II string perturbation theory does not tell us immediately how...

Vectors meet hypers... I

- The occurrence of $\mathcal{M}_V^{IIB}$ in the limit $g_s \to 0$ of $\mathcal{M}_H^{IIB}$ is not coincidental. Consider type IIB string theory compactified on the same CY X, further reduced on a circle of radius R to $D = 3$.
- In $D = 3$, all 1-forms are Hodge dual to 0-forms, $\mathrm{d}A_1 = *\mathrm{d}A_0$. The moduli space is now a product of two quaternion-Kähler manifolds $\mathcal{M}_H^{IIB} \times \mathcal{M}_V^{IIB'}$ of real dimensions $4h_{12}(X) + 4$ and $4h_{11}(X) + 4$. The first is just the HM moduli space in $D = 4$.
- The second factor describes the VM moduli t^a in $D = 4$, the radius R, the holonomy and hodge duals $\mathcal{R} \in \mathrm{Jac}(X)$ of the 1-forms, and the Hodge dual σ of the Kaluza-Klein connection g_{i4}. At large radius R, the QK metric looks like

$$\mathrm{d}s^2_{\mathcal{M}_V^{IIB'}} = \left(\frac{\mathrm{d}R}{R}\right)^2 + \mathrm{d}s^2_{\mathcal{M}_C} + \frac{1}{R^2}\,\mathrm{d}s^2_{\mathrm{Jac}} + \frac{1}{R^4}\left(\mathrm{d}\sigma + \mathcal{A}\right)^2$$

Vectors meet hypers... II

- This is looks the same as $\mathcal{M}_H^{IIA}$! In fact, T-duality along the circle identifies $\mathcal{M}_V^{IIB'} = \mathcal{M}_H^{IIB}$, $g_s^A = 1/R_B$. Euclidean D2-branes on $H_3(X)$ are mapped to Euclidean D3-branes on $H_4(X \times S_1)$, i.e. black holes in $D = 4$! NS5-branes on X are mapped to Taub-NUT instantons (or Kaluza-Klein monopoles) on $X \times S_1$.
- Similarly $\mathcal{M}_V^{IIA'} = \mathcal{M}_H^{IIB}$. In addition, mirror symmetry identifies $M_H^{IIA}(X)$ with $M_H^{IIB}(Y)$. To sum up, for a given CY threefold X, type II string theory associates two QK manifolds $\mathcal{QK}_K(X)$ and $\mathcal{QK}_C(X)$, such that the moduli spaces in $D = 3$ are given by

	$IIA/X \times S^1$	$IIB/X \times S^1$
$\mathcal{M}'_V \times \mathcal{M}_H$	$\mathcal{QK}_K(X) \times \mathcal{QK}_C(X)$	$\mathcal{QK}_C(X) \times \mathcal{QK}_K(X)$

Constraints on the exact HM moduli space I

- In the weak coupling limit $g_s \to 0$ (or $R \to \infty$), the QK metric must reduce to the semi-flat metric;
- D-instanton effects should be weighted by the generalized DT invariants $\Omega(\gamma, t)$ (up to multi-covering effects);
- The metric should be smooth and complete; in particular, continuous across LMS, and regular at $g_s^2 \sim \chi$;
- Under mirror symmetry, $\mathcal{M}^K(X) = \mathcal{M}^C(Y)$;

Constraints on the exact HM moduli space II

- $\mathcal{M}_K(X)$ should have an isometric action of $SL(2, \mathbb{Z})$, inherited from the 10D S-duality symmetry of type $IIB/X \times S^1$, or equivalently from the global diffeomorphisms of T^2 in M-theory on $X \times T^2$.
- There are also reasons to expect an isometric action of $SL(3, \mathbb{Z})$, coming from 4D S-duality or Ehlers symmetry, or of a Picard subgroup $SU(2, 1, \mathbb{Z}[\sqrt{-d}])$ for certain rigid CY three-folds with complex multiplication.

BP Persson; Bao Kleinschmidt Nilsson Persson BP

- When X admits a K3-fibration with a global section, one could in principle use heterotic-type II duality to compute the metric on $\mathcal{M}(X)$ using $(0, 4)$ SCFT techniques. This is promising, but little has been accomplished so far.

A word on the rigid limit I

- In the limit where X becomes singular, it is sometimes possible to decouple gravity and describe the low energy physics in terms of an ordinary field theory with $N = 2$ “rigid” supersymmetries.
- This is in particular so when X develops an A_{N-1} singularity, fibered over a Riemann surface Σ. The D2-branes wrapped on vanishing cycles lead to massless gauge bosons, described by $SU(N)$ $N = 2$ Super-Yang-Mills.
- The SK metric on the Coulomb branch (where the gauge group is broken to $U(1)^{N-1}$) is described again by a prepotential $F(X)$ (no longer homogeneous), which can be computed from period integrals on the Seiberg-Witten curve Σ.

A word on the rigid limit II

- The BPS spectrum exhibit similar chamber dependence and lines of marginal stability as in the SUGRA case.

Bilal Ferrari, . . .

- Upon reduction on a circle, the VM moduli space is enhanced to a hyperkähler manifold. When $R \to \infty$, it reduces to the 'rigid c-map' of the Coulomb branch. In addition there are $O(e^{-R})$ exponential corrections from BPS monopoles winding around the circle. The wall-crossing formula ensures that the HK metric is smooth across the LMS. The HK metric and BPS spectrum can be computed using integrable model techniques (Hitchin system).

Gaiotto Moore Neitzke

- In contrast to SUGRA, there are no $O(e^{-R^2})$ corrections, and the instanton sum converges.

Outline

QK geometry and contact geometry I

- Recall that a Riemannian manifold of real dimension $4n$ is quaternion-Kähler if its holonomy group is (exactly) $Sp(n) \times Sp(1)$. $\mathcal{M}$ is then Einstein. SUGRA requires negative scalar curvature. Let $\vec{p}$ be the $Sp(1)$ part of the Levi-Civita connection, $\mathrm{d}\vec{p} + \vec{p} \wedge \vec{p} = \frac{\nu}{2}\vec{\omega}$ the quaternionic 2-forms.
- $\mathcal{M}$ does not admit a (global) complex structure. Instead, it is more convenient to study its twistor space $\mathcal{Z}$. This is a complex contact manifold of real dimension $4n + 2$, endowed with a (non-holomorphic) projection $\pi : \mathcal{Z} \to \mathcal{M}$ with CP^1 fibers, and a real structure acting as the antipodal map on CP^1.

Salamon; Lebrun

QK geometry and contact geometry II

- Explicitly, the complex contact structure on $\mathcal{M}$ is given by the kernel of the $(1,0)$-form Dz (which transforms homogeneously under $Sp(1) = SU(2)$ frame rotations)

$$Dz = \mathrm{d}z + p_+ - \mathrm{i}p_3 z + p_- z^2$$

  Moreover, $\mathcal{M}$ carries a Kahler-Einstein metric

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dz|^2}{(1+z\bar{z})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}}$$

- Locally, there exists a “contact potential" $\Phi(x^\mu, z)$ and Darboux complex coordinates $\alpha, \xi, \tilde{\xi}$ such that

$$\mathcal{X} = 2\, e^{\Phi}\, \frac{Dz}{z} = \mathrm{d}\alpha + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda$$

  Φ provides a Kähler potential K on $\mathcal{Z}$ via $e^K = (1+z\bar{z})e^{Re(\Phi)}/|z|$.

QK geometry and contact geometry III

- The complex contact structure can be specified globally by providing contactomorphisms on the overlap of two Darboux coordinate patches. Those are conveniently specified by a Hamilton function $S^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]})$:

$$\xi^\Lambda_{[j]} = f_{ij}^{-2}\, \partial_{\tilde\xi^{[j]}_\Lambda} S^{[ij]}\,, \qquad \tilde\xi^{[i]}_\Lambda = \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\,,$$
$$\alpha^{[i]} = S^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} S^{[ij]}\,, \qquad e^{\Phi_{[i]}} = f_{ij}^2\, e^{\Phi_{[j]}}\,,$$

  where $f_{ij}^2 \equiv \partial_{\alpha^{[j]}} S^{[ij]} = \mathcal{X}^{[i]}/\mathcal{X}^{[j]}$.

- $S^{[ij]}$ are subject to consistency conditions $S^{[ijk]}$, gauge equivalence under local contact transformations $S^{[i]}$, and reality constraints.

QK geometry and contact geometry IV

- For generic choices of $S^{[ij]}$, the moduli space of solutions of the above gluing conditions, regular in each patch, is finite dimensional, and equal to $\mathcal{M}$ itself.
- On each patch U_i, $u_m^{[i]} = (\xi^\Lambda_{[i]}, \tilde\xi^{[i]}_\Lambda, \alpha^{[i]})$ admit a Taylor expansion in z around ζ_i, whose coefficients are functions on $\mathcal{M}$. The functions $u_m^{[i]}(z, x^\mu)$ parametrize the "twistor line" over $x^\mu \in \mathcal{M}$.
- The metric on $\mathcal{M}$ can be obtained by expanding $\mathcal{X}^{[i]}$ and $\mathrm{d}u_m^{[i]}$ around z_i, extracting the $SU(2)$ connection $\vec{p}$ and a basis of $(1,0)$ forms on $\mathcal{M}$ in almost complex structure $J(z_i)$, and using $\mathrm{d}\vec{p} + \frac{1}{2}\,\vec{p} \times \vec{p} = \frac{\nu}{2}\,\vec{\omega}$.
- Deformations of $\mathcal{M}$ correspond to deformations of $S^{[ij]}$, so are parametrized by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

Lebrun, Salamon

QK geometry and contact geometry V

- Any (infinitesimal) isometry κ of $\mathcal{M}$ lifts to a holomorphic isometry $\kappa_{\mathcal{Z}}$ of $\mathcal{Z}$. The moment map construction provides an element of $H^0(\mathcal{Z}, \mathcal{O}(2))$, given locally by holomorphic functions

$$\mu_\kappa = \kappa_{\mathcal{Z}} \cdot \mathcal{X} = e^{\Phi} \left(\mu_+ z^{-1} - \mathrm{i}\mu_3 + \mu_- z \right) .$$

Galicki

The moment map of the Lie bracket $[\kappa_1, \kappa_2]$ is the contact-Poisson bracket $\{\mu_{\kappa_1}, \mu_{\kappa_2}\}_{PB}$. The zeros of μ canonically associate a (local) complex structure J_κ to κ.

- Toric QK manifolds are those which admit $d+1$ commuting isometries. In this case, one can choose $\mu_{[i]}$ as the position coordinates. The transition functions must then take the form

$$S^{[ij]} = \alpha^{[j]} + \xi^\Lambda_{[i]} \tilde{\xi}^{[j]}_\Lambda - H^{[ij]} ,$$

where $H^{[ij]}$ depends on $\xi^\Lambda_{[i]}$ only.

QK geometry and contact geometry VI

- More generally, one can consider "nearly toric QK", where $H^{[ij]}$ is a general function but its derivatives wrt to $\tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]}$ are taken to be infinitesimal. For one unbroken isometry κ, $\partial_{\alpha^{[j]}} H^{[ij]} = 0$.
- The twistor lines can then be obtained by Penrose-type integrals, e.g. (in case with one isometry, no "anomalous dimensions")

$$\xi^\Lambda_{[i]} = \zeta^\Lambda + \frac{Y^\Lambda}{z} - z\bar{Y}^\Lambda - \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}z'}{2\pi \mathrm{i} z'} \frac{z'+z}{z'-z} \partial_{\tilde{\xi}^{[j]}_\Lambda} H^{[+j]}(z')$$

$$e^{\Phi_{[i]}} = \frac{1}{4}\sum_j \oint_{C_j} \frac{\mathrm{d}z'}{2\pi \mathrm{i} z'} \left(z'^{-1} Y^\Lambda - z' \bar{Y}^\Lambda \right) \partial_{\xi^\Lambda_{[j]}} H^{[+j]}(\xi(z'), \tilde{\xi}(z'))$$

The locus $z = 0$ defines the canonical complex structure J_κ.

Outline

The perturbative hypermultiplet moduli space I

- Let us now return to the HM moduli space $\mathcal{M}_H$ in type IIA compactified on X. For simplicity, assume $\chi(X) = 0$. In string perturbation theory, $\mathcal{M}_H^{\rm pert} \sim {\rm c-map}(\mathcal{M}_V^{IIB})$.
- The twistor space is governed by the Hamilton functions

$$H^{[0+]}_{\rm pert} = \frac{i}{2} F(\xi^\Lambda)\,, \quad H^{[0-]}_{\rm tree} = \frac{i}{2} \bar F(\xi^\Lambda)$$

Roček Vafa Vandoren

- As a result, the twistor lines are given [upon defining $\tilde\xi_\Lambda \equiv -2{\rm i}\tilde\xi^{[0]}_\Lambda, \alpha \equiv 4{\rm i}\alpha^{[0]} + 2{\rm i}\tilde\xi^{[0]}_\Lambda \xi^\Lambda$, $W(z) \equiv F_\Lambda \zeta^\Lambda - X^\Lambda \tilde\zeta_\Lambda$] by

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \left(z^{-1} X^\Lambda - z\,\bar X^\Lambda\right)/g_s^2\,, \\ \tilde\xi_\Lambda &= \tilde\zeta_\Lambda + \left(z^{-1} F_\Lambda - z\,\bar F_\Lambda\right)/g_s^2\,, \\ \alpha &= \sigma + \left(z^{-1} W - z\,\bar W\right)/g_s^2\,, \end{aligned}$$

Neitzke BP Vandoren; Alexandrov; APSV

Generalized Mirror Map I

- Using mirror symmetry, the perturbative contact potential may be written in terms of the GW invariants of Y [here $\tau_2 = 1/g_s$],

$$e^{\Phi} = \frac{\tau_2^2}{2} V + \frac{\tau_2^2}{4(2\pi)^3} \sum_{q_a \gamma^a \in H_2^+(Y)} n_{0,q_a} \mathrm{Re}\left[\mathrm{Li}_3\left(\mathrm{e}^{q}\right) + 2\pi q_a t^a \, \mathrm{Li}_2\left(\mathrm{e}^{q}\right)\right]$$

  while the RR multiform $\zeta^\Lambda, \tilde\zeta_\Lambda$ and NS-axion σ are related to type IIB variables $\tau_1, c^a, c_a, c_0, \psi$ by the "generalized mirror map"

$$\zeta^0 = \tau_1 \,, \qquad \zeta^a = -(c^a - \tau_1 b^a) \,,$$

$$\tilde\zeta_a = c_a + \frac{1}{2} \kappa_{abc} \, b^b (c^c - \tau_1 b^c) \,, \quad \tilde\zeta_0 = \, c_0 - \frac{1}{6} \kappa_{abc} \, b^a b^b (c^c - \tau_1 b^c) \,,$$

$$\sigma = -2(\psi + \frac{1}{2} \tau_1 c_0) + c_a (c^a - \tau_1 b^a) - \frac{1}{6} \kappa_{abc} \, b^a c^b (c^c - \tau_1 b^c) \,.$$

Gunther Herrmann Louis; Berkooz BP; APSV

S-duality and symplectic covariance I

- In the weak coupling, large IIB volume limit, $\mathcal{M}_H$ admits an isometric action of $SL(2,\mathbb{R})$

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \qquad j^a \mapsto j^a|c\tau+d|\,, \qquad c_a \mapsto c_a\,,$$
$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}\,, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}$$

- This can be lifted to a holomorphic action on Z,

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,, \qquad \dots$$

Berkovits Siegel; Robles-Llana Roček Saueressig Theis Vandoren; APSV

S-duality and symplectic covariance II

- The contact potential $e^\Phi = \frac{\tau_2^2}{2} V(j^a)$, though not invariant, transforms so that $K_{\mathcal{Z}}$ undergoes a Kähler transformation,

$$e^\Phi \mapsto \frac{e^\Phi}{|c\tau + d|}, \quad K_{\mathcal{Z}} \mapsto K_{\mathcal{Z}} - \log(|c\xi^0 + d|), \quad \mathcal{X}^{[i]} \to \frac{\mathcal{X}^{[i]}}{c\xi^0 + d}$$

- The worldsheet instanton corrections break $SL(2,\mathbb{R})$ continuous S-duality. A discrete subgroup $SL(2,\mathbb{Z})$ can be restored by summing over images:

$$\mathrm{Li}_k(e^{2\pi \mathrm{i} q_a z^a}) \to {\sum_{m,n}}' \frac{\tau_2^{k/2}}{|m\tau + n|^k} e^{-S_{m,n,q}},$$

where $S_{m,n,q} = 2\pi q_a |m\tau + n|\, t^a - 2\pi \mathrm{i} q_a (mc^a + nb^a)$ is the action of a (m,n)-string wrapped on $q_a \gamma^a$.

Robles-Llana Roček Saueressig Theis Vandoren

S-duality and symplectic covariance III

- After Poisson resummation on $n \to q_0$, we get a sum over D(-1)-D1 bound states, $e^{\Phi} = \cdots +$

$$\frac{\tau_2}{8\pi^2} \sum_{\substack{q_0 \in \mathbb{Z} \\ q_a \gamma^a \in H_2^+(Y)}} n^{(0)}_{q_a} \sum_{m=1}^{\infty} \frac{|q_\Lambda X^\Lambda|}{m} \cos\left(2\pi m\, q_\Lambda \zeta^\Lambda\right) K_1\left(2\pi m |q_\Lambda X^\Lambda| \tau_2\right)$$

Robles-Llana Saueressig Theis Vandoren

- Going back to type IIA variables, these are interpreted as Euclidean $D2$ wrapped on SLAG in a Lagrangian subspace of $H_3(X, \mathbb{Z})$ (A-cycles only). These effects correct the mirror map into

$$\tilde{\zeta}_a = \tilde{\zeta}_a^{(0)} + \frac{1}{8\pi^2} \sum_{q_a} n_{0,q} \sum_{n \in \mathbb{Z}, m \neq 0} \frac{m\tau_1 + n}{m|m\tau + n|^2} e^{-S_{m,n,q}}, \ldots$$

Alexandrov Saueressig

S-duality and symplectic covariance IV

- In the "one instanton" approximation, the contributions of B-cycles can be restored by symplectic invariance:

$$e^{\Phi} = \cdots + \frac{\tau_2}{8\pi^2} \sum_{\gamma} n_{\gamma} \sum_{m=1}^{\infty} \frac{|W_{\gamma}|}{m} \cos\left(2\pi m \Theta_{\gamma}\right) K_1\left(2\pi m |W_{\gamma}|\right)$$

$$W_{\gamma} \equiv \frac{1}{2}\tau_2 \left(q_{\Lambda} X^{\Lambda} - p^{\Lambda} F_{\Lambda} \right) , \quad \Theta_{\gamma} \equiv q_{\Lambda} \zeta^{\Lambda} - p^{\Lambda} \tilde{\zeta}_{\Lambda}$$

- At this point, n_{γ} just parametrize the allowed deformations. However, their behavior under wall-crossing and general expectations from T-duality suggest that $n_{\gamma} = \Omega(\gamma, t)$, the generalized DT invariants.

The hypermultiplet twistor space I

- The contact structure on the twistor space can be obtained by inserting an elementary symplectomorphism generated by

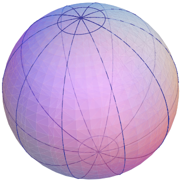
$$S_\gamma^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi_\Lambda^{[j]}, \alpha^{[j]}) = \alpha^{[j]} + \xi^\Lambda_{[i]}\, \tilde\xi_\Lambda^{[j]} + \frac{\mathrm{i}}{2(2\pi)^2}\, n_\gamma \,\mathrm{Li}_2\left(\mathcal{X}_\gamma\right) .$$

Gaiotto Moore Neitzke

across the "BPS ray" $\ell(\gamma)$,

$$\ell(\gamma) = \{ z :\ \pm W_\gamma / z \in \mathrm{i}\mathbb{R}^- \} ,$$

$$\mathcal{X}_\gamma = e^{-2\pi\mathrm{i}(q_\Lambda \xi^\Lambda_{[i]} + 2\mathrm{i} p^\Lambda \tilde\xi_\Lambda^{[j]})}$$



- As $t \in \mathcal{M}_V$ is varied, the BPS rays may cross, and the invariants n_γ should transform so as to leave the contact structure intact.

The hypermultiplet twistor space II

- BPS rays $\ell(\gamma_1)$ and $\ell(\gamma_2)$ cross at lines of marginal stability. The wall crossing formula

$$\prod_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}}^{\curvearrowleft} U_\gamma^{n^-(\gamma)} = \prod_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}}^{\curvearrowright} U_\gamma^{n^+(\gamma)} ,$$

ensures that the consistency of the twistor space across the LMS.

Gaiotto Neitzke Moore; Kontsevich Soibelman

- The metric is regular across the LMS. Physically, single instanton contributions on one side of the wall get replaced by multi-instanton configurations on the other side.

Black holes, Taub-NUT instantons and NS5-branes I

- If indeed $n_{\gamma,t} = \Omega(\gamma,t) \sim e^{\frac{1}{4}A(\gamma)}$, the instanton series is divergent, and must be treated as an asymptotic series. Its accuracy can be estimated by Borel type techniques. Schematically,

$$\sum_{Q} e^{Q^2 - Q/g_s} \sim e^{-1/g_s^2}$$

Thus NS5-brane or KK-monopoles are expected to play a crucial role in regulating the black hole sum.

BP Vandoren

Black holes, Taub-NUT instantons and NS5-branes II

- In contrast to D-instantons, NS5-brane instantons should induce genuine contact transformations, with $S^{[ij]} \propto e^{\mathrm{i}k\alpha^{[ij]}} F_k(\xi, \tilde{\xi})$.
- For gauge invariance, F_k must be a holomorphic section of the Theta line bundle over $\mathrm{Jac}(X)$. This seems to fit with known facts about the NS5-brane partition function, and about the topological string amplitude !

Witten; Freed Moore Belov; Dijkgraaf Verlinde Vonk, . . .

- One may in principle determine the NS5 instantons by $SL(2,\mathbb{Z})$ duality from the D5-instantons. Automorphy under $SL(3,\mathbb{Z})$ provides a short cut.
- There are indications that the motivic DT invariants and the quantum dilogarithm should play an important role in this story, although it is unclear yet how.

Kontsevich Soibelman; Dimofte Gukov, . . .

Conclusion I

- Determining the exact HM metric is hard, but (hopefully) not impossible. Twistor methods are essential, but can still be improved (completeness, discrete symmetries...)
- In some highly symmetric cases (e.g. Enriques or Borcea-Voisin CY), one may hope that automorphy will fix the hypermultiplet metric exactly, giving access to new CY invariants. Heterotic/type II duality may also be a very powerful approach.
- The metric on $\mathcal{M}_H$ offers a very convenient packaging of the degeneracies of 4D BPS black holes. Divergences of the BH partition function should be resolved by NS5 or TN-instantons.
- It seems that higher derivative $\tilde{F}_g$-type corrections to the hypers should be governed by a one-parameter generalization of the topological string amplitude, which mixes A and B-model data. The way to non-perturbative topological string theory ?