Automorphy in hypermultiplet moduli spaces

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Number Theory and Physics at the Crossroads
Banff, 12/05/2011

Introduction I

- Studying the vector multiplet moduli space in Calabi-Yau compactifications of type II string theories has led to classical mirror symmetry: $\mathcal{SK}_K(X) = \mathcal{SK}_c(Y)$ if (X, Y) is a mirror pair. Worldsheet instantons are governed by Gromov-Witten theory, in turn related to variations of Hodge structures.

  Candelas de la Ossa Green Parkes; Strominger Yau Zaslow; ...

- Studying the D-brane spectrum in these models has further led to homological mirror symmetry: BPS states are represented by (objects in the derived category of) coherent sheaves, respectively SLAGS, and counted by (generalized) Donaldson-Thomas invariants.

  Douglas Moore; Kontsevich; ...

Introduction II

- The hypermultiplet moduli space $\mathcal{QK}_c(X)$ in type IIA/X, respectively $\mathcal{QK}_K(\mathcal{Y})$ in type IIB/$\mathcal{Y}$ contains information about Gromov-Witten and Donaldson-Thomas invariants, and more.
- The metric on either space is constrained by SUSY to be quaternion-Kähler. It receives corrections from worldsheet, D-brane as well as NS5-brane instantons.
- D-brane instanton corrections are dictated by wall-crossing, and similar to that found by [Gaiotto Moore Neitzke] for instanton corrections in $\mathcal{N} = 2$ Seiberg-Witten type theories on $\mathbb{R}^3 \times S^1$. NS5-instantons are largely mysterious.
- In fact, the same space $\mathcal{QK}_c(X)$, respectively $\mathcal{QK}_K(\mathcal{Y})$ also arises as the vector multiplet moduli space in type IIB/$X \times S^1$, respectively type IIA/$\mathcal{Y} \times S^1$, upon renaming D5-D3-D1-D(-1) $\rightarrow$ D6-D4-D2-D0, D2 $\rightarrow$ D3, NS5$\rightarrow$ KKM, etc.

Introduction III

- Quantum mirror symmetry requires $\mathcal{QK}_c(X) = \mathcal{QK}_K(Y)$, a far-reaching generalization of classical and homological mirror symmetry.
- Moreover, the metric should have a large group of discrete isometries, including an action of $SL(2, \mathbb{Z})$ which mixes worldsheet, D-brane and five-brane instantons.
- In this talk, I will review recent progress towards understanding the HM moduli space, with emphasis on its automorphic properties. This is based on joint work with Alexandrov, Persson, Saueressig and Vandoren.

Outline

1. Classical and homological mirror symmetry
2. The perturbative hypermultiplet moduli space
3. Mirror symmetry and S-duality
4. Twistorial description of the semi-flat metric
5. The non-perturbative hypermultiplet moduli space

Outline

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Set-up I

- Consider type IIA string theory on $\mathbb{R}^{3,1} \times X$, where X is a Calabi-Yau three-fold. The low energy physics is described by $\mathcal{N} = 2, D = 4$ (ungauged) supergravity, with $n_V = h^{1,1}(X)$ vector multiplets and $n_H = h^{2,1}(X) + 1$ hypermultiplets.
- What this means is that the massless scalar fields parametrize maps

$$\mathbb{R}^{3,1} \xrightarrow{z^a, q^\Lambda} \mathcal{SK}_K(X) \times \mathcal{QK}_c(X)$$

1. $\mathcal{SK}_K(X)$ is a projective special Kähler (PSK) Riemannian manifold of complex dimension n_V, parametrizing the complexified Kähler structure on X;
2. $\mathcal{QK}_c(X)$ is a quaternion-Kähler (QK) manifold $\mathcal{M}_H$, of quaternionic dimension n_H, parametrizing the complex structure of X, as well as the string coupling, RR-axions and NS-axion.

Unlike Dave's talk, these are in general not symmetric spaces !

Vector multiplet moduli space I

- The VM moduli space $\mathcal{SK}_K(X)$ is the complexified Kähler cone of X, parametrized by the Kähler moduli

$$z^a = \int_{\gamma^a} B + \mathrm{i}\, J = b^a + \mathrm{i}\, t^a$$

where $\gamma^{a=1\dots n_V}$ is a basis of $H_2(X, \mathbb{Z})$.

- Since the string coupling is a hypermultiplet, the metric on $\mathcal{SK}_K(X)$ can be computed in classical string theory (tree-level).
- The projective special Kähler metric is encoded in the prepotential $F(X^\Lambda)$, a holomorphic function of projective coordinates X^Λ, $\Lambda = 0 \dots n_V$, homogeneous of degree two. Its third derivative $F_{\Lambda\Sigma\Xi}$ encodes the Yukawa couplings in the SUGRA action.

Vector multiplet moduli space II

- In the large volume limit $t_a \gg 1$, F is determined by the triple intersection product κ_{abc} in $H_4(X)$, the second Chern class $c_{2,a} = \int_{\gamma_a} c_2(X)$ and the genus-zero Gromov-Witten invariants $N_{0,q}$, governing worldsheet instantons:

$$F = -\frac{N(X^a)}{X^0} + \frac{1}{2} A_{ab} X^a X^b + \frac{1}{24} c_{2,a} X^a X^0 - \frac{(X^0)^2}{(2\pi i)^3} \sum_{q \in H_2^+(X)} N_{0,q}\, \mathbf{e}(q)$$

where $z^a = \frac{X^a}{X^0}$, $N(X) = \frac{1}{6}\kappa_{abc} X^a X^b X^c$, $\mathbf{e}(q) = e^{2\pi i q_a z^a}$.

- The Gopakumar-Vafa (GV) invariants $n_{0,q} \in \mathbb{Z}$, defined via

$$\sum_q N_{0,q}\, \mathbf{e}(q) = \sum_{q, d \geq 1} n_{0,q} \frac{\mathbf{e}(dq)}{d^3}$$

count the number of rational curves in homology class q.

Classical mirror symmetry I

- The Gromov-Witten invariants $N_{0,q}$ are most conveniently computed using (classical) mirror symmetry. Recall that for any (non-rigid) CY threefold X, there exists a mirror Calabi-Yau Y, such that $h_{1,1}(X) = h_{2,1}(Y)$, $h_{2,1}(X) = h_{1,1}(Y)$; if X is fibered by T^3, Y is fibered by T-dual/Mukai-transformed T^3).

Candelas et al; Strominger Yau Zaslow

- Mirror symmetry requires that $\mathcal{SK}_K(X) = \mathcal{SK}_C(Y)$. The prepotential $F(X^\Lambda)$ for $\mathcal{SK}_C(Y)$ is computable from the period integrals of the (3,0) form Ω on Y:

$$X^\Lambda = \int_{\gamma^\Lambda} \Omega \,, \qquad F_\Lambda = \int_{\gamma_\Lambda} \Omega = \partial F / \partial X^\Lambda \,,$$

where $\gamma^\Lambda, \gamma_\Lambda$ is a symplectic basis of $H_3(Y, \mathbb{Z})$, adapted to the point of maximal unipotent monodromy.

BPS spectrum and homological mirror symmetry I

- Mirror symmetry requires not only $\mathcal{M}_V^{IIA}(X) = \mathcal{M}_V^{IIB}(Y)$, but also that the full type IIA/X and type IIB/Y string theories be equivalent. In particular, the spectrum of BPS states should match.
- BPS states in type IIA/X are obtained by wrapping $D0, D2, D4, D6$ branes on complex submanifolds of X. More generally, they are described by objects in the bounded derived category of coherent sheaves $DCoh(X)$.
- BPS states in type IIB/Y are obtained by wrapping $D3$-branes on special Lagrangian cycles (SLAGs) of Y. More generally, they are realized as objects in the Fukaya category $Fuk(Y)$.
- Homological mirror symmetry states that $DCoh(X)$ and $Fuk(Y)$ are isomorphic as triangulated categories.

Douglas, Kontsevich

BPS spectrum and homological mirror symmetry II

- Both categories are graded by the charge vector $\gamma \in H_{\rm even}(X, \mathbb{Z})$ in type IIA, or $\gamma \in H_3(Y, \mathbb{Z})$ in type IIB.
- A choice of $z^a \in \mathcal{SK}_K(X)$ (resp. $\mathcal{SK}_c(Y)$) determines a stability condition on $DCoh(X)$ (resp. $Fuk(Y)$). The number of stable objects (counted with sign) with charge γ defines the generalized Donaldson-Thomas invariant $\Omega(\gamma, z^a)$.
- $\Omega(\gamma, z^a)$ is a locally constant function on $\mathcal{SK}$. It can jump on certain codimension one walls in $\mathcal{SK}$, known as lines of marginal stability (LMS), according to certain (recently established) wall-crossing formulae.

Bridgeland; Joyce Son; Kontsevich Soibelman...

- Mirror symmetry also requires that the generalized DT invariants agree.

Outline

Perturbative hypermultiplet moduli space I

- At zero string coupling, $\mathcal{QK}_{K/C}(X)$ contains as much information as $\mathcal{SK}_{K/C}(X)$. However, at finite g_s it receives non-perturbative corrections from D-brane and NS5-brane instantons. Thus, it combines Gromov-Witten theory with Donaldson-Thomas theory, and presumably new math/physics related to NS5-branes.
- $\mathcal{M}_H$ is a quaternion-Kähler space of real dimension $4(h_{1,2}(X)+1)$. Despite the name, $\mathcal{M}_V$ is not Kähler, and carries no (globally defined) complex structure.
- In type IIA/X, $\mathcal{QK}_C(X)$ parametrizes the complex structure of X, the RR-axion $C \in H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$, the NS-axion $\sigma \in S^1$ and the string coupling constant $g_s \equiv 1/R$.

Perturbative hypermultiplet moduli space II

- At tree level, i.e. in the strict weak coupling limit $R = \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$\mathrm{d}s^2_{\mathcal{QK}_c(X)} = \frac{4}{R^2}\,\mathrm{d}R^2 + \left(4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2}\right) + \frac{1}{16R^4}D\sigma^2 \,.$$

$$\mathrm{d}s^2_T = -\frac{1}{2}(\mathrm{d}\tilde\zeta_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde\zeta_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda$$

$$\mathcal{N}_{\Lambda\Lambda'} = \bar\tau_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\tau\cdot X]_\Lambda[\mathrm{Im}\tau\cdot X]_{\Lambda'}}{X^\Sigma\,\mathrm{Im}\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,, \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$$

Cecotti Girardello Ferrara; Ferrara Sabharwal

Perturbative hypermultiplet moduli space III

- The c-map construction associates a $4n+4$-dimensional QK metric to any $2n$-dimensional projective SK manifold. For example
  1. n-dim complex ball $\longrightarrow$ $n+1$-dim quaternionic ball (F quadratic)
  2. Poincaré upper half -plane $\longrightarrow$ $G_2/SO(4)$ ($F = -(X^1)^3/(X^0)$)
- The term in bracket is the metric on the Weil intermediate Jacobian $T \to \mathcal{J}_c(X) \to \mathcal{SK}_c(X)$.
- Upon modding out by translations $\sigma \equiv \sigma + 2$, the coordinate σ lives on a circle bundle with first Chern class $\mathrm{d}D\sigma/2 = \mathrm{d}\zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda = \omega_T$, where ω_T is the Kähler class on the torus T.

Perturbative hypermultiplet moduli space IV

- The one-loop correction deforms the metric on $\mathcal{M}$ into

$$ds^2_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)}\,\mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{2\,c}{R^4}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16R^4(R^2+2c)} D\sigma^2\,.$$

where $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle + 8c\mathcal{A}_K\,,\quad c = -\frac{\chi(\mathcal{X})}{192\pi}$

Antoniadis Minasian Theisen Vanhove; Gunther Hermann Louis;

Robles-Llana Saueressig Vandoren

- In particular, the one-loop deformation changes the curvature of the connection on $U(1)_\sigma$ to $\mathrm{d}D\sigma/2 = \omega_T + \frac{\chi}{24}\omega_c$. This can be traced to the effect of the anomalous couplings in 10D.

Perturbative hypermultiplet moduli space V

- The one-loop corrected metric, also known as the semi-flat metric, is believed to be exact to all orders in $g_s = 1/R$. It admits a continuous group of isometries

$$T_{H,\kappa} : (C, \sigma) \mapsto (C + H, \sigma + 2\kappa + \langle C, H \rangle)$$

with $H \in H^3(X, \mathbb{R})$, $\kappa \in \mathbb{R}$, satisfying the Heisenberg group law

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1, H_2 \rangle} \ .$$

- Instanton corrections from Euclidean D2-branes wrapping SLAGs and from Euclidean NS5-branes wrapping X are expected to break this to a discrete subgroup.
- To determine which discrete subgroup, let us discuss the qualitative form of such instanton corrections.

Perturbative hypermultiplet moduli space VI

- Euclidean D2-branes wrapping a special Lagrangian submanifold in integer homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(X,\mathbb{Z})$ induce corrections roughly of the form

$$\delta \mathrm{d}s^2|_{D2} \sim \sigma(\gamma)\,\bar{\Omega}(\gamma, z^a)\,\exp\left(-8\pi\frac{|Z_\gamma|}{g_{(4)}} - 2\pi\mathrm{i}\langle\gamma, C\rangle\right)\,.$$

  Here $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge, $\bar{\Omega}(\gamma, z^a)$ is the (generalized) DT invariant and $\sigma(\gamma)$ is a ‘quadratic refinement’ (more on this below).

- Since γ is an integer homology class, D2-instanton corrections preserve discrete translational isometries $T_{H,\kappa}$ with $H \in H^3(X,\mathbb{Z})$, $\kappa \in \mathbb{R}$. Thus, C lives in the intermediate Jacobian torus $H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$.

Perturbative hypermultiplet moduli space VII

- The quadratic refinement is essential for consistency with wall-crossing. It is a map $\sigma : H_3(X,\mathbb{Z}) \to U(1)$ such that

$$\sigma(\gamma+\gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\,\sigma(\gamma)\,\sigma(\gamma')\,.$$

- It depends on a choice of characteristics Θ,

$$\sigma(\gamma) = e^{-\mathrm{i}\pi p^\Lambda q_\Lambda + 2\pi\mathrm{i}\langle\gamma,\Theta\rangle}\;,\quad \gamma=(p^\Lambda,q_\Lambda)\;,\quad \Theta=(\theta^\Lambda,\phi_\Lambda)$$

- The characteristics Θ may be reabsorbed in C, at the cost of spoiling the transformation properties of C under monodromies. The total space of $T \to \mathcal{J}_c(X) \to \mathcal{SK}_c(X)$ is a twisted torus bundle.

Perturbative hypermultiplet moduli space VIII

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim \exp\left(-4\pi \frac{|k|}{g_{(4)}^2} - \mathrm{i}k\pi\sigma\right) \mathcal{Z}^{(k)}(z^a, C) ,$$

  where $\mathcal{Z}^{(k)}$ is the (twisted) partition function of the world-volume theory on a stack of k five-branes.

- Recall that the type IIA NS5-brane supports a self-dual 3-form flux, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a non-trivial line bundle $\mathcal{L}_{\mathrm{NS5}}$ over the space of metrics and C fields.

Witten; Henningson Nilsson Salomonson; Belov Moore; . . .

Perturbative hypermultiplet moduli space IX

- This means that $\mathcal{Z}^{(1)}(\mathcal{N}, C)$ satisfies the twisted periodicity condition

$$\mathcal{Z}^{(k)}(\mathcal{N}, C+H) = [\sigma(H)]^k \, e^{\mathrm{i}\pi k \langle H, C\rangle} \mathcal{Z}^{(k)}(\mathcal{N}, C)$$

  where $\sigma(H)$ is again a quadratic refinement of the intersection product on $H^3(X,\mathbb{Z})$. In fact, $\mathcal{Z}^{(k)}(\mathcal{N}, C)$ is, up to a $\mathcal{N}$-dependent normalization factor, a Siegel theta series of degree n_H, level k.
- For the coupling $e^{-\mathrm{i}\pi\sigma}\mathcal{Z}^{(1)}$ to be invariant under large gauge transformations and monodromies, $e^{\mathrm{i}\pi\sigma}$ must transform in the same way as $\mathcal{Z}^{(1)}$.

Perturbative hypermultiplet moduli space X

- Thus, the discrete subgroup of the Heisenberg group preserved by instanton corrections is

$$T_{H,\kappa} : (C, \sigma) \mapsto \left(C + H, \sigma + 2\kappa + \langle C, H \rangle - n^\Lambda m_\Lambda + 2\langle H, \Theta \rangle \right)$$

where $H \equiv (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{b_3}, \kappa \in \mathbb{Z}$. The additional shift of σ ensures the closure of large gauge transformations,

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2, \kappa_1+\kappa_2+\frac{1}{2}\langle H_1, H_2 \rangle + \frac{1}{2\pi i} \log \frac{\sigma(H_1+H_2)}{\sigma(H_1)\sigma(H_2)}}$$

- The action of monodromies on C, σ is tricky, but should follow by demanding invariance of the D-instanton and NS5-instantons.

Outline

Mirror symmetry I

- Mirror symmetry requires that $\mathcal{QK}_c(X)$ also describes the HM moduli space of type IIB string theory on the mirror $Y = \hat{X}$, in other words $\mathcal{QK}_c(X) = \mathcal{QK}_K(Y)$.
- In the weak coupling limit, this is a consequence of classical mirror symmetry $\mathcal{SK}_c(X) = \mathcal{SK}_K(Y)$. D2-brane instanton corrections to $\mathcal{QK}_c(X)$ are now interpreted as D5-D3-D1-D(-1). Mathematically, DT invariants of SLAGs on X become DT invariants of coherent sheaves on Y.
- One advantage of the mirror description is that S-duality exchanges (D1,F1) and (D5,NS5) instantons. Thus, one may hope to compute NS5-instantons by S-duality from D5-instantons !

Mirror symmetry II

- In the large complex structure limit of X (large volume limit of Y), and in the limit $\tau_2 \equiv 1/g_s \to \infty$, S-duality can be made manifest by changing coordinates from 'type IIA' variables to 'type IIB'

$$\zeta^0 = \tau_1 \,, \qquad \zeta^a = -(c^a - \tau_1 b^a) \,,$$

$$\tilde{\zeta}_a = c_a + \frac{1}{2} \kappa_{abc} \, b^b (c^c - \tau_1 b^c) \,, \quad \tilde{\zeta}_0 = \; c_0 - \frac{1}{6} \kappa_{abc} \, b^a b^b (c^c - \tau_1 b^c) \,,$$

$$\sigma = -2(\psi + \frac{1}{2} \tau_1 c_0) + c_a (c^a - \tau_1 b^a) - \frac{1}{6} \kappa_{abc} \, b^a c^b (c^c - \tau_1 b^c) \,.$$

One can then check that $SL(2,\mathbb{R})$ acts isometrically via

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \,, \qquad j^a \mapsto j^a |c\tau + d| \,, \qquad c_a \mapsto c_a \,,$$

$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix} \,, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}$$

Gunther Herrmann Louis; Berkooz BP; APSV

Mirror symmetry III

- The above change of variable can be viewed as the generalized 'mirror map' between $\mathcal{QK}_c(X)$ and $\mathcal{QK}_K(Y)$. In general however, it will receive all sorts of instanton corrections...
- Assuming that $SL(2,\mathbb{Z})$ indeed acts on the full metric, we can define the mirror map as the change of variable such that S-duality acts in the above canonical fashion.
- In fact, consistency with D3-brane instantons requires to modify the action of S-duality and let $c_a \mapsto c_a - c_{2,a}\,\varepsilon(\delta)$ where $\epsilon(\delta)$ is the multiplier system of the Dedekind eta function,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)/\eta(\tau) = e^{2\pi i\epsilon(\delta)}(c\tau+d)^{1/2}\,.$$

- Combining S-duality with monodromy invariance and Heisenberg yields a potentially large automorphic action on $\mathcal{QK}$.

Automorphy

Symmetries

- Monodromies

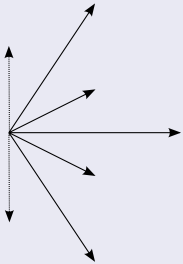
Root diagram (2D projection)

Automorphy

Symmetries

- Monodromies
- Large gauge transf.

Root diagram (2D projection)

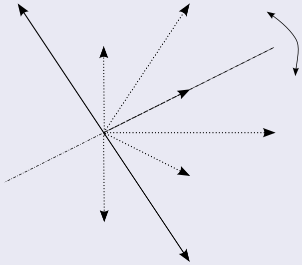


Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality

Root diagram (2D projection)

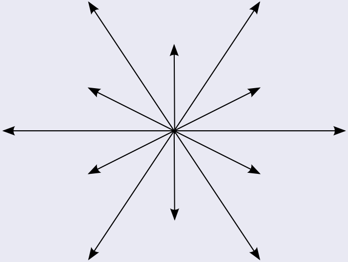


Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.

Root diagram (2D projection)



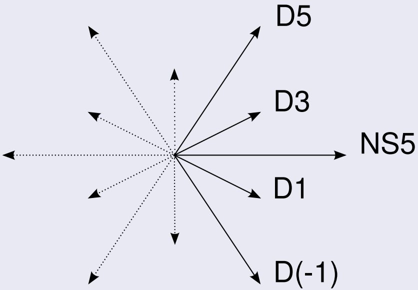
Alexeevsky; Gunaydin Koepsell Nicolai;Gunaydin Neitzke Pavlyk BP

Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 2-step nilpotent

Root diagram (2D projection)

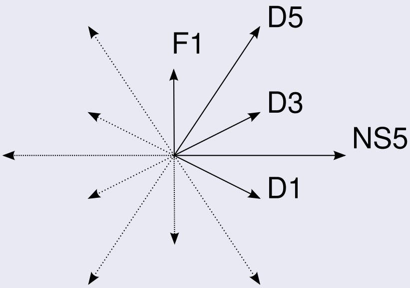


Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 2-step nilpotent
- 3-step nilpotent

Root diagram (2D projection)

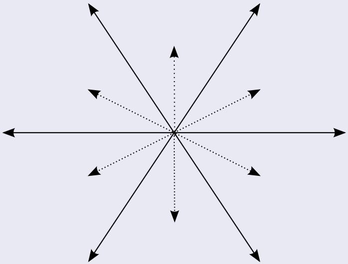


Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 2-step nilpotent
- 3-step nilpotent
- Long roots: SL(3)

Root diagram (2D projection)



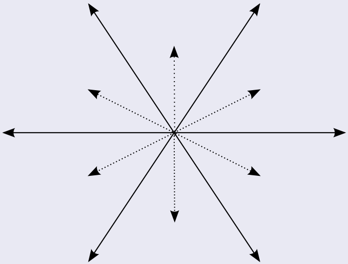
Persson BP

Automorphy

Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 2-step nilpotent
- 3-step nilpotent
- Long roots: SL(3)
- Rigid case: SU(2,1)

Root diagram (2D projection)



Bao Kleinschmidt Nilsson Persson BP

Outline

1. Classical and homological mirror symmetry
2. The perturbative hypermultiplet moduli space
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4. Twistorial description of the semi-flat metric
5. The non-perturbative hypermultiplet moduli space

Twistor techniques for QK spaces I

- QK manifolds $\mathcal{M}$ are conveniently described via their twistor space $\mathbb{C}P \to \mathcal{Z} \to \mathcal{M}$.
- $\mathcal{Z}$ admits a canonical complex structure, in fact a complex contact structure.
- Choosing a stereographic coordinate t on $\mathbb{C}P$, the contact structure is the kernel of the local (1,0)-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

where $p_3, p_\pm$ are the $SU(2)$ components of the Levi-Civita connection on $\mathcal{M}$. Dt is well-defined modulo rescalings.

Lebrun, Salamon

Twistor techniques for QK spaces II

- As in symplectic geometry, there always exist Darboux coordinates $(\Xi, \alpha) = (\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ such that

$$Dt \propto \mathrm{d}\alpha + \langle \Xi, \mathrm{d}\Xi \rangle = \mathrm{d}\alpha + \tilde{\xi}_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \ .$$

- The contact structure is encoded in complex contact transformations between local Darboux coordinate systems on their common domain $U_i \cap U_j$.
- By the moment map construction, any continuous isometry of $\mathcal{M}$ can be lifted to a holomorphic action on $\mathcal{Z}$.

Salamon; Galicki Salamon;Alexandrov BP Saueressig Vandoren

Twistor description of the perturbative metric I

- For the one-loop corrected HM metric, the following Darboux coordinates do the job, away from the north and south poles $t = 0, \infty$:

$$\begin{aligned} \Xi_{\rm sf} &= C + 2\sqrt{R^2+c}\, e^{\mathcal{K}/2} \left[t^{-1}\Omega - t\,\bar{\Omega} \right] , \\ \alpha_{\rm sf} &= \sigma + 2\sqrt{R^2+c}\, e^{\mathcal{K}/2} \left[t^{-1}\langle \Omega, C\rangle - t\,\langle \bar{\Omega}, C\rangle \right] - 8\mathrm{i}c \log t \end{aligned}$$

while the contact potential is given by $e^{\Phi} = R^2\, e^{-\mathcal{K}} + 2c$.

Neitzke BP Vandoren; Alexandrov

- Large gauge transformations $T_{H,\kappa}$ acts holomorphically on $\mathcal{Z}$ by

$$(\Xi, \alpha) \mapsto (\Xi + H, \alpha + 2\kappa + \langle \Xi, H\rangle - n^{\Lambda} m_{\Lambda} + 2\langle H, \Theta\rangle)$$

Twistor description of the perturbative metric II

- Similarly, in the zero coupling limit $R \to \infty$, S-duality acts holomorphically on $\mathcal{Z}$ via [here $\alpha' = \alpha + \xi^\Lambda \tilde\xi_\Lambda$]

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d},$$

$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{c}{2(c\xi^0 + d)} \kappa_{abc}\xi^b\xi^c - c_{2,a}\,\varepsilon(\delta),$$

$$\begin{pmatrix} \tilde\xi_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde\xi_0 \\ \alpha' \end{pmatrix} + \frac{1}{6}\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0 + d) \\ -[c^2(a\xi^0 + b) + 2c]/(c\xi^0 + d)^2 \end{pmatrix}.$$

- This reduces to the previous action on $\mathcal{QK}$, together with a $U(1)$ rotation on $\mathbb{C}P$,

$$z \mapsto \frac{c\bar\tau + d}{|c\tau + d|} z \quad \text{where} \quad z = \frac{t + \mathrm{i}}{t - \mathrm{i}}$$

Twistor description of the perturbative metric III

- To demystify these transformations, note that ξ^0, ξ^a transform like the modular parameter τ and elliptic coordinate z of usual Jacobi forms.
- The transformation rule of $\tilde{\xi}_a$ can be rephrased as the fact that $p^a \tilde{\xi}_a$ transforms like the (log of the) automorphy factor of a multi-variate Jacobi form of index $m_{ab} = \frac{1}{2}\kappa_{abc} p^c$.
- Put differently, if $\phi(\xi^0, \xi^a)$ satisfies the functional equation

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z^a}{c\tau+d}\right) = (v_\eta)^{c_{2,a}p^a}\, \mathbf{e}\left(\frac{c m_{ab} z^a z^b}{c\tau+d}\right) \phi(\tau, z^a)$$

then $\mathbf{e}\left(p^a \tilde{\xi}_a\right) \phi(\xi^0, \xi^a)$ is an invariant function on $H \ltimes N_2$, where N_2 is a Heisenberg algebra.

Twistor description of the perturbative metric IV

- The transformation of $(\tilde{\xi}_0, \alpha)$ is more exotic. It implies that if the family of functions $\phi_{p,q}(\xi^0, \xi^a)$ indexed by coprime (p, q) satisfies

$$\phi_{ap+bq,cp+dq}\left(\frac{a\tau+b}{c\tau+d}, \frac{z^a}{c\tau+d}\right) =$$

$$\mathbf{e}\left(\frac{\kappa_{abc}z^a z^b z^c}{6(c\tau+d)^2}\,\left(pc^2(c\tau+d) - q[c^2(a\tau+b)+2c]\right)\right)\,\phi_{p,q}(\tau, z^a)$$

  then $\sum'_{p,q} \mathbf{e}\left(\alpha p + \tilde{\xi}^0 q\right)\,\phi_{p,q}(\xi^0, \xi^a)$ is an invariant function on $H \ltimes N_3$ where N_3 is the 3-step nilpotent introduced before.
- Such automorphy factors are classified by $H^1(SL(2,\mathbb{Z}), A)$ where A is the sheaf of sections of rank 2 vector bundles on $H \times \mathbb{C}$...

Outline

D-instantons in twistor space I

- As in $N=2$ theories on $\mathbb{R}^3 \times S^1$, D-instanton corrections are largely dictated by wall crossing. In the vicinity of a fixed $z^a \in \mathcal{SK}$, the Darboux coordinates $\xi^\Lambda, \tilde\xi_\Lambda, \alpha$ are discontinuous across the BPS rays $\ell(\gamma) = \{t : Z(\gamma, z^a)/t \in \mathrm{i}\mathbb{R}^-\}$
- The discontinuity is given by a complex contact transformation U_γ, most conveniently formulated in terms of its action on the holomorphic Fourier modes $\mathcal{X}_\gamma = \mathbf{e}\left(-2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde\xi_\Lambda)\right)$

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'}(1-\sigma(\gamma)\mathcal{X}_\gamma)^{\langle\gamma,\gamma'\rangle\,\Omega(\gamma)}\ , \quad \alpha \mapsto \alpha - \frac{1}{2\pi^2}\Omega(\gamma)\, L[\sigma(\gamma)\mathcal{X}_\gamma]$$

where $L(z) = \sum_{n=1}^\infty \frac{z^n}{n^2} + \frac{1}{2}\log z\ \log(1-z)$ is Rogers' dilogarithm.

Gaiotto Moore Neitzke; Alexandrov BP Saueressig Vandoren

D-instantons in twistor space II

- The consistency of this prescription across walls of marginal stability is guaranteed by the KS wall-crossing formula

$$\prod^{\curvearrowleft}_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}} U_\gamma = \prod^{\curvearrowright}_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}} U_\gamma\ ,$$

Gaiotto Neitzke Moore; Kontsevich Soibelman

- The choice of branch cuts for $L(z)$ is subtle, and should follow from a careful analysis of the classical limit of the motivic KS formula.

Alexandrov, Persson, BP, in progress

- The resulting QK metric is regular across the walls. Physically, the one-instanton approximation is discontinuous, but the discontinuity is cancelled by multi-instanton corrections.

D1-F1-D(-1) instantons I

- In the large volume limit, the contributions of D1-D(-1) instantons dominate over D3-D5-NS5 instantons, and should produce a toric QK metric invariant under S-duality.
- To see how this works, recall that for coherent sheaves supported on curves, the DT invariant $\Omega(0, 0, q_a, q_0; z^a)$ is independent of q_0 and coincides with the Gopakumar invariant $n^{(0)}_{q_a}$. The corresponding symplectomorphism $\prod_{q_0} U_{q_0}$ is generated by

$$H = n^{(0)}_{q_a} \sum_{q_0=0}^{\infty} \mathrm{Li}_2 \left(e^{2\pi i (q_0 \xi^0 + q_a \xi^a)} \right)$$

which is recognized as the elliptic dilogarithm.

D1-F1-D(-1) instantons II

- By Poisson resummation on q^0, this can be converted into $H = n^{(0)}_{q_a} \sum'_{m,n} G_{m,n}$ where

$$G_{m,n}(\xi) = \begin{cases} \frac{e^{-2\pi i m q_a \xi^a}}{m^2(m\xi^0+n)}, & m \neq 0, \\ (\xi^0)^2 \, \frac{e^{2\pi i n q_a \xi^a/\xi^0}}{n^3} \,, & m = 0 \end{cases}$$

  where $G_{0,n}$ comes from Gromov-Witten instantons.

- S-duality is then ensured by the fact that

$$G_{m,n} \mapsto \frac{G_{m',n'}}{c\xi^0 + d} + \text{reg.}\,, \qquad \begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} m' \\ n' \end{pmatrix}$$

  up to a term which is regular at the zeros of $m'\xi^0 + n'$.

D3-instantons I

- Similarly, D3-D1-D(-1) instantons wrapped on a fixed divisor $\mathcal{D}$ should produce a QK metric invariant under S-duality: in one-instanton approximation, they are described formally by

$$H = \sum_{q_a, q_0} \bar{\Omega}(0, p^a, q_a, q_0; z^a) \, \mathbf{e}\left(p^a \tilde{\xi}_a - q_a \xi^a - q_0 \xi^0 \right) ,$$

where $\bar{\Omega}(\gamma, z^a) \equiv \sum_{d|\gamma} d^{-2} \Omega(\gamma/d, z^a)$ are the rational DT invariants.

D3-instantons II

- In this approximation, the DT invariants $\bar{\Omega}(0, p^a, q_a, q_0; z^a)$ are invariant under monodromy around the large volume point

$$p^a \mapsto p^a, \quad q_a \mapsto q_a - \kappa_{ab}\epsilon^b, \quad q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

The DT invariants are then functions of $\hat{q}_0 = q_0 - \frac{1}{2}\kappa^{ab} q_a q_b$, where κ^{ab} is the inverse of the matrix $\kappa_{ab} = \kappa_{abc} p^c$, and of the residue class $\mu_a \in \Lambda^*/\Lambda + \frac{1}{2}\kappa_{abc} p^b p^c$ of q_a:

$$\bar{\Omega}(0, p^a, q_a, q_0) \equiv \bar{\Omega}_{p^a,\mu_a}(\hat{q}_0)\,.$$

D3-instantons III

- From the study of the D4-D2-D0 black hole partition function, $\bar{\Omega}_{p^a,\mu_a}$ are known to be Fourier coefficients of a vector-valued holomorphic modular form of weight $(-\frac{b_2}{2}-1,0)$ and multiplier system $\mathbf{e}(c_{2a}p^a\varepsilon(\delta))$.

$$h_{p^a,\mu_a}(\xi^0) = \sum_{\hat{q}_0 \le \chi(\mathcal{D})/24} \bar{\Omega}_{p^a,\mu_a}(\hat{q}_0)\,\mathbf{e}\left(-\hat{q}_0\xi^0\right)$$

Denef Moore; Cheng de Boer Dijkgraaf Manschot Verlinde, ...

D3-instantons IV

- Formally, H may be rewritten as a multi-variate Jacobi theta series

$$H = \mathbf{e}\left(p^a \tilde{\xi}_a\right) \sum_{\mu_a \in \Lambda^*/\Lambda + \frac{1}{2}\kappa_{abc} p^b p^c} h_{p^a,\mu_a}(\xi^0)\, \Theta_{p^a,\mu_a}(\xi^0, \xi^a)$$

where Θ_{p^a,μ_a} is a theta series of index κ_{ab}

$$\Theta_{p^a,\mu_a}(\xi^0, \xi^a) = \sum_{r^a \in \mathbb{Z}^{b_2} + \kappa^{ab}\mu_b} \mathbf{e}\left(-\frac{1}{2} r^a \kappa_{ab} p^b - \kappa_{ab} r^a \xi^b - \frac{1}{2} r^a \kappa_{ab} r^b \xi^0\right)$$

Alexandrov BP Manschot Persson

- This is formal however, since κ_{ab} has signature $(1, b_2(Y) - 1)$. We expect that Θ_{p^a,μ_a} should be replaced by an indefinite theta series ła Goettsche-Zagier-Zwegers, with the sum over r^a restricted to some positive cone. Then H will be formally modular of weight $(-1, 0)$, as required for a section of $H^1(\mathcal{Z}, \mathcal{O}(2))$.

NS5-D5-instantons I

- Finally, the fully corrected metric including all NS5, D5, D3, D1, D(-1) instantons should be invariant under $SL(2,\mathbb{Z})$. Thus, NS5-instantons may be obtained from the (in principle known) D5-D3-D1-D(-1) instantons.
- Still in the one-instanton approximation, assuming that the DT invariants are invariant under monodromy

$$p^0 \mapsto p^0\,,\quad p^a \mapsto p^a + \epsilon^a p^0\,,\quad q_a \mapsto q_a - \kappa_{abc} p^b \epsilon^c - \frac{1}{2}\, p^0 \kappa_{abc} \epsilon^b \epsilon^c\,,$$
$$q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}\, \kappa_{abc} p^a \epsilon^b \epsilon^c + \frac{1}{6}\, p^0 \kappa_{abc} \epsilon^a \epsilon^b \epsilon^c\,,$$

we find that corrections from k NS5-branes are given by a non-Gaussian theta series

$$H^{(k)}_{\rm NS5} = \sum_{\substack{\mu\in(\Gamma_m/|k|)/\Gamma_m \\ n\in\Gamma_m+\mu+\theta}} \Psi^{(k,\mu)}\left(\xi^\Lambda - n^\Lambda\right) \mathbf{e}\left(k n^\Lambda (\tilde\xi_\Lambda - \phi_\Lambda) - \frac{k}{2}\,(\alpha + \xi^\Lambda \tilde\xi_\Lambda) \right)$$

NS5-D5-instantons II

where $\Psi(\xi^\Lambda)$ is a wave-function

$$\Psi^{(k,\mu)}(\xi^\Lambda) = \sum_{q_\Lambda} \bar{\Omega}(\gamma)\, \mathbf{e}\left(-\frac{k\, N(\xi^a)}{\xi^0} + \frac{p^0\, \left(k \hat{q}_a \xi^a + p^0 \hat{q}_0\right)}{k^2 \xi^0} \right)$$

Here $\gamma = (p^0, k\mu^a, q_a, q_0)$ where $p^0 = \gcd(k, k\mu^0)$,

$$\hat{q}_a = q_a + \frac{k^2}{2p^0} \kappa_{abc} \mu^b \mu^c \,, \qquad \hat{q}_0 = q_0 + \frac{k}{p^0} \mu^a q_a + \frac{k^3}{3(p^0)^2} \kappa_{abc} \mu^a \mu^b \mu^c \,,$$

- For $k = 1$, using the GV/DT correspondence, this is recognized as the topological string amplitude

$$H^{(1,0)}_{\rm NS5}(\xi^\Lambda) = \sum_{\hat{q}_a, \hat{q}_0} \Omega(1, 0, \hat{q}_a, \hat{q}_0)\, (-1)^{\hat{q}_0}\, \mathbf{e}\left(-\frac{N(\xi^a)}{\xi^0} + \frac{\hat{q}_a \xi^a + \hat{q}_0}{\xi^0} \right) .$$

Outlook I

- The hypermultiplet moduli space in type II Calabi-Yau compactifications brings together many subjects: Gromov-Witten theory, Donaldson-Thomas invariants, mirror symmetry, quaternionic geometry...
- The leading D-instanton corrections are to a large extent dictated by wall-crossing. The instanton series is expected to be divergent, due to the exponential growth of DT invariants.
- NS5-brane instanton corrections are still unclear. They should be related by S-duality to D5-instantons, hence DT invariants, but consistency with wall-crossing is far from obvious.

Outlook II

- One of the difficulties in implementing S-duality (or higher automorphic properties) is that discrete symmetries must be realized on transition functions. E.g. at the one-instanton level, we need a theory of modular forms in $H^1(\mathcal{Z},\mathcal{O}(2))$.
- Indefinite theta series and quaternionic discrete series representations are natural candidates, as they naturally live in H^1...