# Hyperinstantons, black holes and wall-crossing

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LPTHE, Paris

Bad Honnef 2011
14/03/2011

*based on work with Alexandrov, Saueressig, Vandoren, Persson, ...*

# *Instantons corrections to hypermultiplet moduli spaces*, black holes and wall-crossing

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# Introduction

- In $D = 4$ string vacua with $\mathcal{N} = 2$ supersymmetries , the moduli space splits into a product $\mathcal{M} = VM_4 \times HM_4$ corresponding to vector multiplets and hypermultiplets.
- The study of $VM_4$ and of the BPS spectrum has had tremendous applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, Donaldson-Thomas invariants, black hole precision counting, etc...
- Understanding $HM_4$ may be even more rewarding: quantum extension of mirror symmetry, new geometric invariants, new checks of Het/II duality richer automorphic properties...
- By gauging isometries of $HM_4$, one may construct vacua with spontaneously broken $\mathcal{N} = 2$ SUSY, possibly relevant for phenomenology.

# Hypers = Vectors

Upon compactification on $S^1(R)$,

- the HM moduli space stays unchanged:

$$\mathrm{HM}_3 = \mathrm{HM}_4$$

- the VM moduli space is enhanced to a quaternion-Kähler manifold which includes $\mathrm{VM}_4$, the radius $R$ of the circle, the electric and magnetic holonomies of the $D = 4$ Maxwell fields, and the NUT potential $\sigma$, dual to the Kaluza-Klein gauge field in $D = 3$:

$$\mathrm{VM}_3 \approx \text{c-map}(\mathrm{VM}_4) + \text{1-loop} + \mathcal{O}(e^{-R}) + \mathcal{O}(e^{-R^2})$$

- $\mathrm{VM}_3$ and $\mathrm{HM}_3$ are two sides of the same coin, exchanged by T-duality along the circle. Let us focus on $\mathrm{VM}_3$ for now.

# Instantons = Black holes + KKM

- The $\mathcal{O}(e^{-R})$ corrections come from BPS black holes in $D = 4$, whose Euclidean wordline winds around the circle: thus $VM_3$ encodes the $D = 4$ BPS spectrum, with chemical potentials for every electric and magnetic charges, and naturally incorporates chamber dependence.

*Seiberg Witten; Shenker*

- The $\mathcal{O}(e^{-R^2})$ corrections come from Kaluza-Klein monopoles, i.e. gravitational instantons of the form $\mathrm{TN}_k \times \mathcal{X}$.
- Including these additional contributions may lead to enhanced automorphic properties, analogous to the $SL(2,\mathbb{Z}) \to Sp(2,\mathbb{Z})$ enhancement in $N = 4$ dyon counting.

*Dijkgraaf Verlinde Verlinde; Gunaydin Neitzke BP Waldron*

# SYM vs. SUGRA

- A much simpler version of this problem occurs in (Seiberg-Witten) $\mathcal{N}=2$ SYM field theories on $\mathbb{R}^3 \times S^1$. In this case $VM_3$ is a hyperkähler manifold of the form

$$\mathrm{VM}_3 \approx \text{rigid c-map}(\mathrm{VM}_4) + \mathcal{O}(e^{-R})$$

- The $\mathcal{O}(e^{-R})$ corrections similarly come from BPS dyons in $D=4$. Understanding their effect on the twistor space of $\mathrm{VM}_3$ leads to a physical derivation of the KS wall-crossing formula.

*Gaiotto Moore Neitzke, Kontsevich Soibelman*

- The extension to $\mathcal{N}=2$ string vacua is non-trivial, due (in part) to the intricacy of quaternion-Kähler geometry, the exponential growth of BPS degeneracies, and poor understanding of KK monopoles.

# Back to HM

- On the flip side of the coin, $R = 1/g_{(4)}$ encodes the string coupling. The $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections to $\mathrm{HM}_4$ now originate from Euclidean D-branes and NS5-branes, respectively.

  *Becker Becker Strominger*

- When $\mathcal{X}$ is K3-fibered, $HM_4$ can in principle be computed exactly using Het/type II duality: since the heterotic string coupling belongs to $\mathrm{VM}_4$, $HM_4$ is determined by the $(0,4)$ heterotic SCFT at tree level (still non-trivial due to non-perturbative $\alpha'$ corrections)

  *Aspinwall*

- Recent progress has instead occurred on the type II side, by combining wall-crossing, S-duality, mirror symmetry with twistor techniques.

  *Robles-Llana Rocek Saueressig Theis Vandoren*

  *Alexandrov BP Saueressig Vandoren*

# Outline

# Outline

# BPS dyons in Seiberg-Witten theories I

- Consider a $\mathcal{N}=2$ gauge theory in $\mathbb{R}^4$ with a rank $r$ gauge group $G$, broken to $U(1)^r$ on the Coulomb branch. The low energy dynamics is described by the effective action

$$S=\int \partial_{X^\Lambda}\bar\partial_{\bar X^\Sigma}K\,\partial_\mu X^\Lambda\,\partial\mu\bar X^\Sigma+\frac{1}{4\pi}\mathrm{Re}\left[\tau_{\Lambda\Sigma}\mathcal{F}^\Lambda\wedge\left(\mathcal{F}^\Sigma+\mathrm{i}\star\mathcal{F}^\Sigma\right)\right]$$

where the Kähler potential and gauge kinetic function are expressed locally in terms of a prepotential $F(X^\Lambda)$ via

$$K=\mathrm{i}(\bar X^\Lambda F_\Lambda-X^\Lambda\bar F_\Lambda)\ ,\qquad \tau_{\Lambda\Sigma}=\partial_{X^\Lambda}\partial_{X^\Sigma}F$$

- Globally, $X^\Lambda$ is not a good coordinate, rather $\Omega\equiv(X^\Lambda,F_\Lambda)$ is a holomorphic (Lagrangian) section of a flat bundle over $\mathrm{VM}_4$, with non-trivial monodromies.

- $\mathrm{VM}_4$ can be realized as the parameter space of a suitable family of genus $r$ Riemann surfaces $\Sigma_u$. The section $\Omega(u)$ is given by

$$X^\Lambda = \int_{A^\Lambda} \lambda \,, \qquad F_\Lambda = \int_{B_\Lambda} \lambda \,.$$

  where $\lambda$ is a suitable meromorphic one-form and $(A^\Lambda, B_\Lambda)$ a symplectic basis of $H_1(\Sigma_u, \mathbb{Z})$.
- The lattice of electric and magnetic charges $\gamma = (p^\Lambda, q_\Lambda)$ is $\Gamma = H^1(\Sigma_u, \mathbb{Z})$, with DSZ product

$$\langle \gamma, \gamma' \rangle = \gamma \cap \gamma' = q_\Lambda p'^\Lambda - q'_\Lambda p^\Lambda \in \mathbb{Z}$$

- States which saturate the BPS bound

$$\mathcal{M} \geq |Z(\gamma; u)| \,, \qquad Z(\gamma; u) = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda$$

  belong to short multiplets of the $\mathcal{N} = 2$ SUSY algebra.

# BPS dyons in Seiberg-Witten theories III

- Such multiplets may pair up into long multiplets but the index

$$\Omega(\gamma) = -\frac{1}{2}\mathrm{Tr}(-1)^{2J_3}\, J_3^2$$

  stays constant under this process.

- On certain codimension-one walls of marginal stability in $\mathrm{VM}_4$, where the central charges of two charge vectors $\gamma_1, \gamma_2$ align,

$$W(\gamma_1, \gamma_2) = \{u \ / \ \arg(Z(\gamma_1; u) = \arg(Z(\gamma_2; u)\}$$

  the index $\Omega(\gamma)$ for $\gamma = M\gamma_1 + N\gamma_2$ may jump, due to the decay of bound states of $n$ dyons with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, with fixed total charge $\gamma = \sum \alpha_i$.

# Wall-crossing formulae I

- Defining the rational index

$$\bar{\Omega}(\gamma) \equiv \sum\nolimits_{d|\gamma} \Omega(\gamma/d)/d^2 \,,$$

the jump $\Delta\bar{\Omega}(\gamma) = \bar{\Omega}^-(\gamma) - \bar{\Omega}^+(\gamma)$ takes the form

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod\nolimits_{i=1}^n \bar{\Omega}^+(\alpha_i) \,,$$

where the sum runs over all unordered decompositions of the total charge vector $\gamma$ into a sum of $n$ vectors $\alpha_i \in \tilde{\Gamma}$. The coefficients $g(\{\alpha_i\})$ are universal functions of $\alpha_i$.

- $|\mathrm{Aut}(\{\alpha_i\})| = \prod r_k!$ if $\{\alpha_i\}$ consists of $r_1$ charges $\beta_1$, $r_2$ charges of type $\beta_2$, etc.

# The Coulomb branch wall-crossing formula I

- Physically, $g(\{\alpha_i\})$ is the index of the quantum mechanics of $n$ distinguishable particles in $\mathbb{R}^3$ with charges $\alpha_i$, interacting by Coulomb & Lorentz forces. This can be computed by localization,

$$g(\{\alpha_i\}) = \lim_{y\to 1}\left[\frac{(-1)^{\sum_{i<j}\langle\alpha_i,\alpha_j\rangle+n-1}}{(y-1/y)^{n-1}}\sum_{\{z_i\}} s(\{z_i\})\, y^{\sum_{i<j}\alpha_{ij}\,\mathrm{sign}(z_j-z_i)}\right]$$

where $\{z_i\}$ runs over the solutions of

$$\forall i=1\dots n,\quad \sum_{j\neq i}\frac{\langle\alpha_i,\alpha_j\rangle}{|z_i-z_j|}=\sum_{j\neq i}\langle\alpha_i,\alpha_j\rangle\,,\quad \sum_{i=1}^{n} z_i=0\,,$$

and $s(\{z_i\})=(-1)^{\#\{i;z_{i+1}<z_i)\}}$. In SUGRA, $\{z_i\}$ corresponds to the locations of the centers of a collinear multi-centered BPS black hole solution.

Manschot BP Sen

# The KS wall-crossing formula I

- Mathematically, the same $g(\{\alpha_i\})$ can be extracted from the Kontsevich-Soibelman wall-crossing formula. Consider the Lie algebra spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle \, e_{\gamma_1+\gamma_2}$$

For a given charge vector $\gamma$ and VM moduli $u^a$, consider the operator $U_\gamma(u^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; u^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2}\right)$$

We shall see later that the operators $U_\gamma$ naturally arise in the construction of HM moduli spaces.

# The KS wall-crossing formula II

- The KS wall-crossing formula states that the product

$$A(u^a) = \prod_{\substack{\gamma = M\gamma_1 + N\gamma_2, \\ M\geq 0, N\geq 0}} U_\gamma(u^a) \ ,$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As $u^a$ crosses $W$, $\Omega(\gamma; u^a)$ jumps and the order of the factors is reversed, but the operator $A$ stays constant. Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0, \\ M/N\downarrow}} U^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0, \\ M/N\uparrow}} U^-_{M\gamma_1+N\gamma_2} \ ,$$

# The KS wall-crossing formula III

- To compute $\Delta\Omega(M\gamma_1 + N\gamma_2)$, it suffices to project the KS formula to the finite-dimensional algebra

$$\mathcal{A}_{M,N} = \mathcal{A}/\{ \sum_{m>M \,\text{or}\, n>N} \mathbb{R} \cdot e_{m\gamma_1+n\gamma_2} \} \, .$$

  and use the Baker-Campbell-Hausdorff formula.

- For example, the projection of the KS formula to $\mathcal{A}_{1,1}$

$$\begin{aligned} & \exp(\bar\Omega^+(\gamma_1) e_{\gamma_1}) \, \exp(\bar\Omega^+(\gamma_1+\gamma_2) e_{\gamma_1+\gamma_2}) \, \exp(\bar\Omega^+(\gamma_2) e_{\gamma_2}) \\ = & \exp(\bar\Omega^-(\gamma_2) e_{\gamma_2}) \, \exp(\bar\Omega^-(\gamma_1+\gamma_2) e_{\gamma_1+\gamma_2}) \, \exp(\bar\Omega^-(\gamma_1) e_{\gamma_1}) \end{aligned}$$

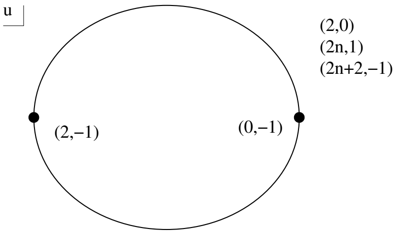
  leads to the primitive wall-crossing formula,

$$g(\alpha_1, \alpha_2) = (-1)^{|\langle \alpha_1, \alpha_2 \rangle|+1} \, |\langle \alpha_1, \alpha_2 \rangle|$$

*Denef Moore*

# The KS wall-crossing formula IV

- For $SU(2)$ Seiberg-Witten theory with no flavor, the jump of the BPS spectrum on the wall where $a/a_D \in \mathbb{R}^+$




is encoded in the identity

$$U_{2,-1} \cdot U_{0,1} = U_{0,1} \cdot U_{2,1} \cdot U_{4,1} \dots U_{2,0}^{(-2)} \dots U_{3,-1} \cdot U_{2,-1} U_{1,-1}$$

*Seiberg Witten; Bilal Ferrari; Denef*

# Reduction on a circle I

- Upon compactifying the $\mathcal{N}=2$ gauge theory on a circle $S^1(R)$, the moduli space $\mathrm{VM}_3$ is enlarged to include the holonomies $C=(\zeta^\Lambda,\tilde\zeta_\Lambda)$ of the gauge fields $\mathcal{A}^\Lambda$ and their magnetic duals $\tilde{\mathcal{A}}_\Lambda$ along the circle.
- Large gauge transformations imply that $C$ lives in the rank $2r$ torus $\mathcal{T}=H^1(\Sigma_u,\mathbb{R})/H^1(\Sigma_u,\mathbb{Z})$. Topologically, $\mathrm{VM}_3$ is the Jacobian of the family of Riemann surfaces $\Sigma_u$.
- As $R\to\infty$, the metric takes the semi-flat form

$$\mathrm{d}s^2_{\mathrm{VM}_3}=R\,\mathrm{d}s^2_{\mathrm{VM}_4}+\frac{1}{R}\left(\mathrm{d}\tilde\zeta_\Lambda-\tau_{\Lambda\Sigma}\zeta^\Sigma\right)[\mathrm{Im}\tau]^{\Lambda\Lambda'}\left(\mathrm{d}\tilde\zeta_{\Lambda'}-\tau_{\Lambda'\Sigma'}\zeta^{\Sigma'}\right)\ .$$

- At finite $R$, we expect $\mathcal{O}(e^{-|Z(\gamma,u)|+\mathrm{i}\langle C,\gamma\rangle})$ instanton corrections to the semi-flat metric, from Euclidean dyons wrapping around $S^1$.

# Reduction on a circle II

- As required by SUSY, the semi-flat metric (aka the rigid $c$-map of the (rigid special Kähler) metric on $\mathrm{VM}_4$) is hyperkähler.
- Indeed, it is Kähler in complex coordinates $(X^\Lambda, W_\Lambda = \tilde\zeta_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Lambda)$, with Kähler potential

$$K_c = \mathrm{i} R (X^\Lambda \bar F_\Lambda - \bar X^\Lambda F_\Lambda) + \frac{1}{R}(W_\Lambda + \bar W_\Lambda)[\mathrm{Im}\tau]^{\Lambda\Sigma}(W_\Sigma + \bar W_\Sigma)$$

- Raising the indices from the Kähler form $\omega^3 = \partial\bar\partial K_c$ and the complex symplectic form $\omega^1 + \mathrm{i}\omega^2 = \mathrm{d}W_\Lambda \wedge \mathrm{d}X^\Lambda$, one obtains three complex structures $J_1, J_2, J_3$ such that

$$J_i\, J_j = -\delta_{ij} + \epsilon_{ijk} J_k \ .$$

- Instanton corrections must preserve the HK property, and are most conveniently expressed in the language of twistors (or projective superspace, in physics parlance).

# Twistor techniques for HK manifolds I

- Recall that any HK manifold $\mathcal{M}$ admits a family of complex structures

$$J(t,\bar{t}) = \frac{1-t\bar{t}}{1+t\bar{t}}\, J^3 + \frac{t+\bar{t}}{1+t\bar{t}}\, J^2 + \mathrm{i}\,\frac{t-\bar{t}}{1+t\bar{t}}\, J^1$$

  parametrized by $t \in \mathbb{P}^1 = S^2$. This complex structure extends to a complex structure on the twistor space $\mathcal{Z} = \mathbb{P}^1 \times \mathcal{M}$.

- Using the hyperkähler metric on $\mathcal{Z}$, one obtains a triplet of Kähler forms $\omega^i$. The complex symplectic form

$$\omega^{[0]}(t) = \omega^+ - \mathrm{i}t\,\omega^3 + t^2\,\omega^- , \quad \omega^\pm = -\frac{1}{2}(\omega^1 \mp \mathrm{i}\omega^2)$$

  is holomorphic w.r.t. to $J(t,\bar{t})$. It is regular at $t=0$, but has a pole at $t=\infty$.

# Twistor techniques for HK manifolds II

- Since the complex two-form is defined up to overall factor, one may instead consider

$$\omega^{[\infty]}(t) \equiv t^{-2}\,\omega^{[0]}(t) = \omega^- - \mathrm{i}\omega^3/t + \omega^+/t^2 \ ,$$

$\omega$ is then real w.r.t. to the antipodal map $t \mapsto -1/\bar{t}$,

$$\omega^{[\infty]}(t) = \overline{\omega^{[0]}(-1/\bar{t})}$$

- More generally, one may introduce a covering of $\mathbb{P}^1$ by open sets $U_i$, and a complex two-form $\omega^{[i]}$, holomorphic on $U_i$, such that

$$\omega^{[i]} = f_{ij}^2\,\omega^{[j]} \quad \text{mod } dt\ , \qquad \omega^{[\bar{i}]}(t) = \overline{\omega^{[i]}(-1/\bar{t})}$$

where $f_{ij}$ are the transition functions of the $\mathcal{O}(1)$ bundle on $\mathbb{P}^1$. The knowledge of $\omega(t)$ allows to reconstruct the HK metric.

*Hitchin*

# Twistor techniques for HK manifolds III

- Locally, one can choose complex Darboux coordinates, regular in patch $U_i$, such that
$$\omega^{[i]} = \mathrm{d}\xi^{\Lambda}_{[i]} \wedge \mathrm{d}\tilde{\xi}^{[i]}_{\Lambda}$$
- They must be related by a complex symplectomorphism on the overlap of two patches $U_i \cap U_j$, and satisfy reality properties
$$\overline{\xi^{\Lambda}_{[i]}(-1/\bar{t})} = -\xi^{\Lambda}_{[\bar{i}]}(t) \,, \quad \overline{\tilde{\xi}^{[i]}_{\Lambda}(-1/\bar{t})} = -\tilde{\xi}^{[\bar{i}]}_{\Lambda}(t)$$
- Any triholomorphic isometry of $\mathcal{S}$ yields a triplet of moment maps $\vec{\mu}_{\kappa} = (v, \bar{v}, x)$, such that $\kappa \cdot \vec{\omega} = d\vec{\mu}_{\kappa}$. This yields a global $\mathcal{O}(2)$-section $\xi$ (aka $\mathcal{O}(2)$ multiplet)
$$\xi = \frac{v}{t} + x - \bar{v} t \,,$$

# Twistor techniques for HK manifolds IV

- For toric hyperkähler manifolds (i.e. $4d$-dimensional HK manifolds with $d$ commuting tri-holomorphic isometries), one may choose the corresponding moment maps $\xi^\Lambda$ as 'position' coordinates. The remaining 'momenta' $\tilde\xi_\Lambda^{[i]}$ are determined by

$$\tilde\xi_\Lambda^{[i]} - \tilde\xi_\Lambda^{[j]} = \partial_{\xi^\Lambda} H^{[ij]} \,, \qquad \xi^\Lambda = \xi^\Lambda_{[i]} = \frac{v^\Lambda}{t} + x^\Lambda - \bar v^\Lambda t$$

where $H^{[ij]}(\xi^\Lambda)$ are holomorphic functions on $U_i \cap U_j$, satisfying cocycle and reality conditions.

- The space of solutions to these gluing conditions is the HK space itself. In coordinates $v^\Lambda, \bar v^\Lambda, x^\Lambda, \rho_\Lambda$ adapted to the toric action,

$$\tilde\xi_\Lambda^{[i]}(t) = \rho_\Lambda + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}\, t'} \frac{t+t'}{t'-t} \partial_{\xi^\Lambda} H^{[0j]}(t') \,.$$

*Alexandrov BP Saueressig Vandoren*

# Twistor techniques for HK manifolds V

- The complex coordinates and Kähler potential $K$ of the HK metric are given by the Legendre transform of the 'tensor Lagrangian' associated to the 'generalized prepotential' $H = \{H^{[ij]}\}$,

$$K(v^\Lambda, \bar{v}^\Lambda, w_\Lambda, \bar{w}_\Lambda) = \langle \oint \frac{dt}{2\pi \mathrm{i} t} H(\xi^\Lambda, t) - x^\Lambda (w_\Lambda + \bar{w}_\Lambda) \rangle_{x^\Lambda}$$

  E.g. $H = \xi^2/t + m\,\xi \log \xi$ produces Taub-NUT space with mass parameter $m$.

*Hitchin Karlhede Linström Rocek*

- Perturbations of hyperkähler metrics correspond to deformations of the complex symplectomorphisms relating the Darboux coordinate systems on $U_i \cap U_j$.

# The instanton corrected metric I

- For the semi-flat metric, the complex Darboux coordinates can be chosen as the complex moment maps of the torus action,

$$\xi^\Lambda|_{\rm sf} = \zeta^\Lambda + \frac{R}{2}\left(t\,\bar X^\Lambda - t^{-1}\,X^\Lambda\right)$$
$$\tilde\xi_\Lambda|_{\rm sf} = \tilde\zeta_\Lambda + \frac{R}{2}\left(t\,\bar F_\Lambda - t^{-1}\,F_\Lambda\right)$$

*Neitzke BP, unpublished*

- In the presence of instanton corrections, the Darboux coordinates are deformed. To express them, it is convenient to introduce the holomorphic Fourier modes

$$\mathcal{X}_\gamma \equiv \exp\left[2\pi{\rm i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde\xi_\Lambda)\right]$$

# The instanton corrected metric II

- The Darboux coordinates for the instanton-deformed HK metric on $\mathrm{VM}_3$ are given by solutions of a system of integral equations

$$\mathcal{X}_\gamma = \mathcal{X}^{\rm sf}_\gamma \exp\left[-\sum_{\gamma'} \Omega(\gamma';u)\, \langle\gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{2\pi\mathrm{i}\, t'}\, \frac{t'+t}{t'-t} \log\left(1-\sigma(\gamma')\, \mathcal{X}_{\gamma'}(t')\right)\right]$$

  where $\ell(\gamma) \subset \mathbb{P}^1$ is the 'BPS ray'

$$\ell(\gamma) = \{t \ : Z(\gamma; u^a)/t \in \mathrm{i}\mathbb{R}^-\}$$

  where $\sigma(\gamma)$ is a choice of quadratic refinement (next page)
- This may be solved iteratively by plugging $\mathcal{X}_\gamma \to \mathcal{X}^{\rm sf}_\gamma$ etc, eventually leading to a multi-instanton expansion for the components of the metric.

*Gaiotto Moore Neitzke*

# The instanton corrected metric III

- $\sigma$ is a quadratic refinement of the DSZ inner product, i.e. a map $H_3(\mathcal{X},\mathbb{Z}) \to U(1)$ such that

$$\sigma(\gamma+\gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\,\sigma(\gamma)\,\sigma(\gamma')\,.$$

- Quadratic refinements are parametrized by characteristics $(\theta,\phi)\in\mathcal{T}$,

$$\sigma(\gamma) = e^{-\mathrm{i}\pi p^\Lambda q_\Lambda + 2\pi\mathrm{i}\langle\gamma,\Theta\rangle}\,, \quad \gamma=(p^\Lambda, q_\Lambda)\,, \quad \Theta=(\theta^\Lambda,\phi_\Lambda)$$

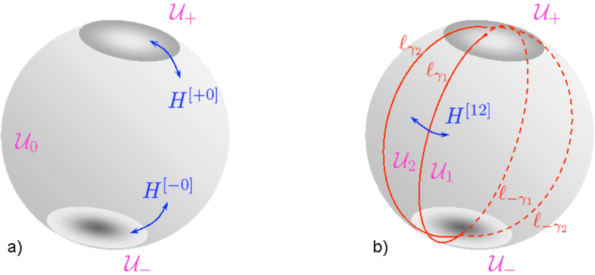
- The quadratic refinement is needed for consistency with wall-crossing. The choice of characteristics can be reabsorbed into a shift of $C=(\zeta^\Lambda,\tilde\zeta_\Lambda)$.

# The instanton corrected metric IV

- In particular, across a BPS ray $\ell(\gamma)$, the Darboux coordinates jump by the symplectomorphism

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'}(1 - \sigma(\gamma)\mathcal{X}_{\gamma})^{\langle\gamma,\gamma'\rangle\,\Omega(\gamma,u^a)} .$$

This provides a geometric realization of the operators $\mathcal{U}_{\gamma}(u^a)$ appearing in the KS wall-crossing formula !



# The instanton corrected metric V

- As the moduli $u^a$ vary in $\mathrm{VM}_4$, BPS rays may cross each other. The KS formula guarantees that the twistor space is well-defined and the HK metric is smooth across walls of marginal stability.
- Physically, one-instanton contributions may jump, but the full multi-instanton sum is continuous.
- The integral equations above are formally identical to Zamolodchikov's Y-system in studies of integrable models.

*GMN; Alexandrov Roche*

# Outline

# The perturbative metric I

- The HM moduli space in type IIA compactified on a CY 3-fold (family) $\mathcal{X}$ is a quaternion-Kähler manifold $\mathcal{M}$ of real dimension $2b_3(\mathcal{X}) = 4(h_{2,1} + 1)$.
- $\mathcal{M} \equiv \mathcal{Q}_c(\mathcal{X})$ encodes
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complex structure of the CY family $\mathcal{X}$,
  3. the periods of the RR 3-form $C$ on $\mathcal{X}$,
  4. the NS axion $\sigma$, dual to the Kalb-Ramond $B$-field in 4D
- To write down the metric explicitly, let us choose a symplectic basis $\mathcal{A}^\Lambda, \mathcal{B}_\Lambda$, $\Lambda = 0 \dots h_{2,1}$ of $H_3(\mathcal{X}, \mathbb{Z})$.

# The perturbative metric II

- The complex structure moduli space $\mathcal{M}_c(\mathcal{X})$ may be parametrized by the periods $\Omega(z^a) = (X^\Lambda, F_\Lambda) \in H_3(\mathcal{X}, \mathbb{C})$ of the (3,0) form

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega_{3,0} \, , \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega_{3,0} \, ,$$

up to holomorphic rescalings $\Omega \mapsto e^f \Omega$.

- $\mathcal{M}_c(\mathcal{X})$ is endowed with a special Kähler metric

$$\mathrm{d}s^2_{\mathcal{SK}} = \partial\bar\partial\mathcal{K} \, , \qquad \mathcal{K} = -\log[\mathrm{i}(\bar X^\Lambda F_\Lambda - X^\Lambda \bar F_\Lambda)]$$

and a $\mathbb{C}^\times$ bundle $\mathcal{L}$ with connection $\mathcal{A}_K = \frac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar a} \mathrm{d}\bar z^{\bar a})$.

- $\Omega$ transforms as $\Omega \mapsto e^f \rho(M)\, \Omega$ under a monodromy $M$ in $\mathcal{M}_c(\mathcal{X})$, where $\rho(M) \in Sp(b_3, \mathbb{Z})$.

# The perturbative metric III

- Topologically trivial harmonic C-fields on $\mathcal{X}$ may be parametrized by the real periods

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C .$$

- Large gauge transformations require that $C \equiv (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ have unit periodicities, i.e. $C$ lives in the torus

$$C \in T = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z}) .$$

- Due to monodromies, the torus $T$ is non-trivially fibered over $\mathcal{M}_c(\mathcal{X})$. The total space of this fibration is the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$.

# The perturbative metric IV

- $T$ carries a canonical symplectic form and complex structure induced by the Hodge $\star_{\mathcal{X}}$, hence a Kähler metric

$$\mathrm{d}s_T^2 = -\frac{1}{2}(\mathrm{d}\tilde{\zeta}_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde{\zeta}_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

where

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\tau\cdot X]_\Lambda[\mathrm{Im}\tau\cdot X]_{\Lambda'}}{X^\Sigma\,\mathrm{Im}\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,, \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$$

- $\mathcal{N}$ (resp. $\tau$) is the Weil (resp. Griffiths) period matrix of $\mathcal{X}$. While $\mathrm{Im}\tau$ has signature $(1, b_3 - 1)$, $\mathrm{Im}\mathcal{N}$ is negative definite.

# The tree-level metric

- At tree level, i.e. in the strict weak coupling limit $R = \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{1}{16R^4}D\sigma^2 \,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda$$

*Cecotti Girardello Ferrara; Ferrara Sabharwal*

- The *c*-map (aka semi-flat) metric admits continuous isometries

$$T_{H,\kappa} : (C,\sigma) \mapsto (C + H, \sigma + 2\kappa + \langle C, H\rangle)$$

with $H \in H^3(\mathcal{X}, \mathbb{R})$, $\kappa \in \mathbb{R}$, satisfying the Heisenberg relations

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle} \,.$$

# The one-loop corrected metric I

- The one-loop correction deforms the metric on $\mathcal{M}$ into

$$ds^2_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)}\, \mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\, \mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{2\,c}{R^4}\, e^{\mathcal{K}}\, |X^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16 R^4 (R^2+2c)} D\sigma^2 \,.$$

where $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle + 8c\mathcal{A}_K\,, \quad c = -\chi(\mathcal{X})/(192\pi)$

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- The one-loop correction to $g_{rr}$ was computed by reducing the CP-even $R^4$ coupling in 10D on $\mathcal{X}$. The correction to $D\sigma$ can be obtained with less effort by reducing CP-odd couplings in 10D.

# The one-loop corrected metric II

- The one-loop correction to $D\sigma$ has important implications for the topology of the HM moduli space, as we shall discuss later.
- The one-loop corrected metric is presumably exact to all orders in $1/R$. It will receive $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections from D-instantons and NS5-brane instantons, eventually breaking all continuous isometries.
- Note the curvature singularity at finite distance $R^2 = -2c$ when $\chi(\mathcal{X}) > 0$ ! This should hopefully be resolved by instanton corrections.

# Topology of the HM moduli space I

- At least at weak coupling, $\mathcal{M}$ is foliated by hypersurfaces $\mathcal{C}(R)$ of constant string coupling. We shall now discuss the topology of the leaves $\mathcal{C}(R)$.
- Quotienting by translations along the NS axion $\sigma$, we already saw that $\mathcal{C}(R)/\partial_\sigma$ reduces to the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$, in particular $C \in T = H^3(\mathcal{X},\mathbb{R})/H^3(\mathcal{X},\mathbb{Z})$.
- This is consistent with the fact that Euclidean D2-branes wrapping a special Lagrangian submanifold in integer homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(\mathcal{X},\mathbb{Z})$ induce corrections roughly of the form

$$\delta \mathrm{d}s^2|_{\mathrm{D2}} \sim \sigma(\gamma)\,\bar{\Omega}(\gamma, z^a)\,\exp\left(-8\pi\frac{|Z_\gamma|}{g_{(4)}} - 2\pi\mathrm{i}\langle\gamma, C\rangle\right)\,.$$

Here $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge.

# Topology of the HM moduli space II

- NS5-brane instantons will further break continuous translations along $\sigma$ to discrete shifts $\sigma \mapsto \sigma + 2$ (in our conventions). Thus $\mathcal{C}(R)$ is a circle bundle over $\mathcal{J}_c(\mathcal{X})$, with fiber parametrized by $e^{i\pi\sigma}$.
- The horizontal one-form $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C \rangle - \frac{\chi(\mathcal{X})}{24\pi}\mathcal{A}_K$ implies that the first Chern class of $\mathcal{C}$ is

$$\mathrm{d}\left(\frac{D\sigma}{2}\right) = \omega_T + \frac{\chi(\mathcal{X})}{24}\,\omega_c\,, \quad \omega_T = \mathrm{d}\tilde{\zeta}_\Lambda \wedge \mathrm{d}\zeta^\Lambda\,, \quad \omega_c = -\frac{1}{2\pi}\mathrm{d}\mathcal{A}_K$$

  where $\omega_T, \omega_c$ are the Kähler forms on $T$ and $\mathcal{M}_c(\mathcal{X})$.
- To identify the circle bundle $\mathcal{C}(R)$, let us examine NS5-brane instantons.

# Five-brane instantons I

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\rm NS5} \sim \exp\left(-4\pi \frac{|k|}{g_{(4)}^2} - \mathrm{i} k \pi \sigma\right) \mathcal{Z}^{(k)}(z^a, C) \ ,$$

  where $\mathcal{Z}^{(k)} = \mathrm{Tr}[(2J_3)^2(-1)^{2J_3}]$ is the (twisted) partition function of the world-volume theory on a stack of $k$ five-branes.
- Recall that the type IIA NS5-brane supports a self-dual 3-form flux, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a non-trivial line bundle $\mathcal{L}_{\rm NS5}$ over the space of metrics and $C$ fields.

*Witten; Henningson Nilsson Salomonson; Belov Moore; . . .*

# Five-brane instantons II

- This means that $\mathcal{Z}(\mathcal{N}, C)$ satisfies the twisted periodicity condition

$$\mathcal{Z}(\mathcal{N}, C+H) = \sigma(H)\, e^{\mathrm{i}\pi\langle H,C\rangle} \mathcal{Z}(\mathcal{N}, C)$$

  where $\sigma(H)$ is a quadratic refinement of the intersection product on $H^3(\mathcal{X},\mathbb{Z})$, with characteristics $\Theta = (\theta^\Lambda, \phi_\Lambda)$.

- $\mathcal{L}_{\rm NS5}|_{\mathcal{T}}$ admits a unique holomorphic section, given by the Siegel theta series

$$\mathcal{Z}^{(1)} = N \sum_{n^\Lambda \in \Gamma_m + \theta} e^{\mathrm{i}\pi(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + 2\pi\mathrm{i}(\tilde\zeta_\Lambda - \phi_\Lambda)n^\Lambda + \mathrm{i}\pi(\theta^\Lambda\phi_\Lambda - \zeta^\Lambda\tilde\zeta_\Lambda)} ,$$

  where $\Gamma_m$ is a Lagrangian sublattice of $\Gamma = H^3(\mathcal{X},\mathbb{Z})$.

- This agrees with the chiral five-brane partition function obtained by holomorphic factorization of the partition function of a non-chiral 3-form $H = \mathrm{d}\mathcal{B}$ on $\mathcal{X}$, with Gaussian action. The $C$-independent normalization factor $N$ is tricky.

# Topology of the NS axion I

- For the coupling $e^{-\mathrm{i}\pi\sigma}\mathcal{Z}^{(1)}$ to be invariant under large gauge transformations and monodromies, $e^{\mathrm{i}\pi\sigma}$ must parametrize the fiber of $\mathcal{L}_{\mathrm{NS5}}$. This implies that $\sigma$ picks up additional shifts under discrete translations along $\mathcal{T}$,

$$T_{H,\kappa} : (C,\sigma) \mapsto \left(C + H, \sigma + 2\kappa + \langle C, H\rangle - n^\Lambda m_\Lambda + 2\langle H, \Theta\rangle\right)$$

where $H \equiv (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{b_3}$, $\kappa \in \mathbb{Z}$. This shift is needed for the closure of large gauge transformations,

$$T_{H_1,\kappa_1} T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\frac{1}{2}\langle H_1,H_2\rangle + \frac{1}{2\pi\mathrm{i}}\log\frac{\sigma(H_1+H_2)}{\sigma(H_1)\sigma(H_2)}}$$

- The invariance of $e^{-\mathrm{i}\pi\sigma}\mathcal{Z}^{(1)}$ under monodromies is guaranteed by the anomaly inflow mechanism, but details remain to be worked out.

# Twistor techniques for QK spaces I

- QK manifolds $\mathcal{M}$ are conveniently described via their twistor space $\mathbb{P}^1 \to \mathcal{Z} \to \mathcal{M}$.
- Unlike the HK case, the fibration is non-trivial, and $\mathcal{Z}$ admits a complex contact structure rather than a complex symplectic structure.
- Choosing a stereographic coordinate $t$ on $\mathbb{P}^1$, the contact structure is the kernel of the local (1,0)-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

where $p_3, p_\pm$ are the $SU(2)$ components of the Levi-Civita connection on $\mathcal{M}$. $Dt$ is well-defined modulo rescalings.

*Lebrun, Salamon*

# Twistor techniques for QK spaces II

- As in the symplectic case, there always exist Darboux coordinates $(\Xi, \tilde{\alpha}) = (\xi^\Lambda, \tilde{\xi}_\Lambda, \tilde{\alpha})$ such that

$$Dt \propto \mathrm{d}\tilde{\alpha} + \langle \Xi, \mathrm{d}\Xi \rangle = \mathrm{d}\tilde{\alpha} + \tilde{\xi}_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \, .$$

- The contact structure is encoded in complex contact transformations between local Darboux coordinate systems on their common domain $U_i \cap U_j$.
- By the moment map construction, any continuous isometry of $\mathcal{M}$ can be lifted to a holomorphic action on $\mathcal{Z}$.

*Salamon; Galicki Salamon;Alexandrov BP Saueressig Vandoren*

# Twistor description of the perturbative metric

- For the one-loop corrected HM metric, the following Darboux coordinates do the job, away from the north and south poles $t = 0, \infty$:

$$\Xi_{\rm sf} = C + 2\sqrt{R^2 + c}\, e^{\mathcal{K}/2} \left[ t^{-1}\Omega - t\,\bar{\Omega} \right] , \quad \Phi_{\rm sf} = 2 \log R ,$$
$$\tilde{\alpha}_{\rm sf} = \sigma + 2\sqrt{R^2 + c}\, e^{\mathcal{K}/2} \left[ t^{-1} \langle \Omega, C \rangle - t\, \langle \bar{\Omega}, C \rangle \right] - 8 \mathrm{i} c \log t$$

*Neitzke BP Vandoren; Alexandrov*

- Large gauge transformations $T_{H,\kappa}$ acts holomorphically on $\mathcal{Z}$ by

$$(\Xi, \tilde{\alpha}) \mapsto (\Xi + H, \tilde{\alpha} + 2\kappa + \langle \Xi, H \rangle - n^{\Lambda} m_{\Lambda} + 2\langle H, \Theta \rangle)$$

- At fixed values of $t, z^a$, $\mathcal{Z}$ is a complexified twisted torus $\mathbb{C}^{\times} \ltimes [H^3(\mathcal{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^{\times}]$.

# D-instantons in twistor space I

- As in Seiberg-Witten theories, D-instanton corrections are essentially dictated by wall crossing. Across a BPS ray $\ell_\gamma$, the Darboux coordinates $\xi^\Lambda, \tilde\xi_\Lambda, \tilde\alpha$ jump by a complex contact transformation

$$\mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'}(1-\sigma(\gamma)\mathcal{X}_\gamma)^{\langle\gamma,\gamma'\rangle\,\Omega(\gamma)}\ , \quad \tilde\alpha \mapsto \tilde\alpha - \frac{1}{2\pi^2}\Omega(\gamma)\, L[\sigma(\gamma)\mathcal{X}_\gamma]$$

  where $L(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} + \frac{1}{2}\log z\ \log(1-z)$ is Rogers' dilogarithm.
- The consistency of this prescription across lines of marginal stability is guaranteed by the KS formula. This structure can also (and was first) argued using type IIB S-duality and mirror symmetry.

# D-instantons in twistor space II

- Due to exponential growth of $\bar{\Omega}(\gamma)$, the D-instanton series

$$\sum \Omega(\gamma) e^{-\frac{|Z(\gamma)|}{g} + \mathrm{i}\langle C, \gamma\rangle}$$

is divergent, and must be treated as an asymptotic series.

- Cutting off the series at $||\gamma|| < Q$, and assuming that the ambiguity in the series is “on the order of the last term in the sum", one can optimize the cut-off $Q$ such that

$$\min_{\gamma} \left[ e^{S_{BH}(\gamma) - \frac{|Z(\gamma)|}{g}} \right] \sim e^{-1/g^2}$$

*BP Vandoren*

- This ambiguity is suggestive of NS5/KKM instanton. Note however that BPS NS5-instantons depend on the NS-axion while the D-instantons don't.

# Outline

# HM moduli space in type IIB I

- The HM moduli space in type IIB compactified on a CY 3-fold $\hat{\mathcal{X}}$ is a QK manifold $\mathcal{M} \equiv \mathcal{Q}_K(\hat{\mathcal{X}})$ of real dimension $4(h_{1,1}+1)$
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complexified Kähler moduli $z^a = b^a + \mathrm{i} t^a = X^a/X^0$
  3. the periods of $C = C^{(0)} + C^{(2)} + C^{(4)} + C^{(6)} \in H^{\mathrm{even}}(\hat{\mathcal{X}}, \mathbb{R})$
  4. the NS axion $\sigma$
- Near the infinite volume point, $\mathcal{M}_K(\hat{\mathcal{X}})$ is governed by

$$F(X) = -\frac{N(X^a)}{X^0} + \chi(\hat{\mathcal{X}}) \frac{\zeta(3)(X^0)^2}{2(2\pi\mathrm{i})^3} + F_{\mathrm{GW}}(X)$$

where $N(X^a) \equiv \frac{1}{6}\kappa_{abc} X^a X^b X^c$, $\kappa_{abc}$ is the cubic intersection form, and $F_{\mathrm{GW}}$ are Gromov-Witten instanton corrections:

$$F_{\mathrm{GW}}(X) = -\frac{(X^0)^2}{(2\pi\mathrm{i})^3} \sum_{k_a\gamma^a \in H_2^+(\hat{\mathcal{X}})} n^{(0)}_{k_a} \,\mathrm{Li}_3\left[\mathrm{E}^{k_a \frac{X^a}{X^0}}\right],$$

# HM moduli space in type IIB II

- D-instantons are now Euclidean D5-D3-D1-D(-1), described mathematically by coherent sheaves $E$ on $\mathcal{X}$. Their charges are related to the Chern classes via the Mukai map

$$q'_\Lambda X^\Lambda - p^\Lambda F_\Lambda = \int_{\hat{\mathcal{X}}} e^{-(B+\mathrm{i}J)}\, \mathrm{ch}(E)\, \sqrt{\mathrm{Td}(\hat{\mathcal{X}})}$$

  where $q_\Lambda = q'_\Lambda - A_{\Lambda\Sigma} p^\Sigma$, integer for suitable $A$.
- Quantum mirror symmetry implies $\mathcal{Q}_c(\mathcal{X}) = \mathcal{Q}_K(\hat{\mathcal{X}})$. At the perturbative (resp. D-instanton) level, this reduces to classical (resp. homological) mirror symmetry.
- The exact HM metric should admit an isometric action of $SL(2,\mathbb{Z})$, corresponding to type IIB S-duality in 10 dimensions. This action is most easily described in the “primed" frame.

# S-duality in twistor space I

- At tree level, an element $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on $\mathcal{Z}$ via

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d}\,,$$

$$\tilde{\xi}'_a \mapsto \tilde{\xi}'_a + \frac{c}{2(c\xi^0 + d)} \kappa_{abc}\xi^b\xi^c - c_{2,a}\,\varepsilon(\delta)\,,$$

$$\begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} + \frac{1}{6}\,\kappa_{abc}\xi^a\xi^b\xi^c \begin{pmatrix} c^2/(c\xi^0 + d) \\ -[c^2(a\xi^0 + b) + 2c]/(c\xi^0 + d)^2 \end{pmatrix}.$$

where $\alpha' = (\tilde{\alpha} + \xi^\Lambda \tilde{\xi}'_\Lambda)/(4\mathrm{i})$. Here $\epsilon(\delta)$ is the multiplier system of the Dedekind eta function, required for consistency of D3-instantons,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right)/\eta(\tau) = e^{2\pi\mathrm{i}\epsilon(\delta)}(c\tau + d)^{1/2}\,.$$

# S-duality in twistor space II

- Continuous S-duality is broken by Gromov-Witten instantons at tree level and by the one-loop correction. A discrete subgroup can be preserved provided D(-1) and D1-instantons combine with GW instantons into a Kronecker-Eisenstein series:

$$\tau_2^{3/2}\,\mathrm{Li}_3(e^{2\pi \mathrm{i} q_a z^a}) \to \sum_{m,n}' \frac{\tau_2^{3/2}}{|m\tau+n|^3} e^{-S_{m,n,q}} \,,$$

where $S_{m,n,q} = 2\pi q_a |m\tau + n|\, t^a - 2\pi \mathrm{i} q_a (mc^a + nb^a)$ is the action of an $(m,n)$-string wrapped on $q_a\gamma^a$.

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- After Poisson resummation on $n \to q_0$, we recover the sum over D(-1)-D1 bound states, with $\Omega(0,0,q_a,q_0) = n^{(0)}_{q_a}$, $\Omega(0,0,0,0) = -\chi(\hat{\mathcal{X}})$. In particular, $\mathrm{Li}_3$ turns into elliptic $\mathrm{Li}_2$ !

# S-duality in twistor space III

- In the presence of D3-branes, S-duality requires that the sum over D3-D1-D(-1) instantons should be a multi-variable Jacobi form of index $m_{ab} = \frac{1}{2}\kappa_{abc}p^c$ and multiplier system $e^{-2\pi i c_{2a} p^a \epsilon(\delta)}$.

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- The trouble is that $m_{ab}$ has indefinite signature $(1, b_2(\hat{\mathcal{X}}) - 1)$, and the dimension of the space $H^0$ of such Jacobi forms vanishes. $H^1$ however is non-zero, and is probably where the D3-D1-D(-1) partition sum lives. This is presumably related to Mock modular forms, but details remain to be worked out.
- S-duality relates D5 and NS5. Starting from the known form of D5-D3-D1-D(-1) corrections, one may construct a Poincaré-type series to obtain the contributions from $k$ five branes in one-instanton approximation.

*Alexandrov Persson Pioline*

# Outline

# Conclusion

- Instanton corrections to HM moduli spaces are efficiently described using twistors: they amount to deformations of the complex symplectomorphisms (resp., complex contact transformations) between local Darboux coordinate systems.
- D-instanton corrections are essentially dictated by wall-crossing. The KS formula guarantees that the instanton-corrected metric is smooth across walls of marginal stability: the jumps in $\Omega(\gamma, t)$ cancel out of the multi-instanton series.
- NS5-instantons are still poorly understood. In principle, they are determined by S-duality, but this is subtle beyond one-instanton level. The consistency of D3-D1-D(-1) with S-duality also remains to be shown.
- What about hyperinstantons in flux compactifications ?