D3-instantons, Mock Theta series and Twistors

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Introduction

- In $D = 4$ string vacua with $N = 2$ supersymmetries, the moduli space splits into a product $\mathcal{M} = \mathcal{SK} \times \mathcal{QK}$ of a special Kähler manifold, parametrized by vector multiplets and a quaternion-Kähler manifold, parametrized by hypermultiplets.
- The study of $\mathcal{SK}$ and the associated spectrum of BPS states has had many applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, Donaldson-Thomas invariants, black hole precision counting, etc...
- Understanding $\mathcal{QK}$ may be even more rewarding: quantum extension of mirror symmetry, new geometric invariants, new checks of dualities, richer automorphic properties...

HM multiplet moduli space in $D = 4$

- In type IIB compactified on a CY threefold Y, the QK metric on $\widehat{\mathcal{QK}}$ near $g_4 = 0$ is obtained from $\mathcal{SK}$ by the (one-loop deformed) c-map construction.

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- In addition there are $\mathcal{O}(e^{-1/g_4})$ and $\mathcal{O}(e^{-1/g_4^2})$ corrections from D5-D3-D1-D(-1) brane instantons and NS5-branes, respectively.

Becker Becker Strominger

- D-instanton corrections are controlled by the generalized Donaldson-Thomas invariants, and essentially dictated by consistency with wall crossing. NS5-brane instantons are far less understood.

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S-duality

- S-duality of type IIB string theory requires that $\mathcal{QK}$ admits an isometric action of $SL(2,\mathbb{Z})$. Since S-duality commutes with the large volume limit, it should hold at any level in the hierarchy

$$\begin{pmatrix} \text{one}-\text{loop} \\ D(-1) \end{pmatrix} > \begin{pmatrix} F1 \\ D1 \end{pmatrix} > D3 > \begin{pmatrix} D5 \\ NS5 \end{pmatrix}$$

- It is known that F1-D1-D(-1) instantons are consistent with S-duality. This is how they were first derived, although S-duality is no longer manifest in the standard twistorial construction.

Robles-Llana Roček Saueressig Theis Vandoren; Alexandrov Saueressig

- At the other end, S-duality determines NS5-instanton contributions from D5-instantons, at least in principle.

Alexandrov Persson BP

Punch-line

- Our goal is to show that D3-instanton corrections are also invariant under S-duality, at least in the one-instanton approximation, large volume limit.
- The argument is a souped-up version of the proof of modular invariance of the D4-D2-D0 black hole partition function.

Maldacena Strominger Witten; Gaiotto Strominger Yin; Denef Moore; . . .

- The relevance of mock theta series is not surprising: for fixed D3-brane charge, the sum over D1-branes is a theta series of indefinite signature $(1, b_2 - 1)$. But it must also be holomorphic in twistor space !

Outline

Outline

HM moduli space in type IIB I

- The HM moduli space in type IIB compactified on a CY 3-fold Y is a QK manifold $\mathcal{M}$ of real dimension $4(h_{1,1}+1)$ describing
  1. the 4D dilaton g_4,
  2. the complexified Kähler moduli $z^a = b^a + \mathrm{i} t^a \in \mathcal{SK}$
  3. the RR scalars $C = (\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in T = H^{\mathrm{even}}(Y,\mathbb{R})/H^{\mathrm{even}}(Y,\mathbb{Z})$
  4. the NS axion σ dual to B-field in 4 dimensions
- At tree level, i.e. in the strict weak coupling limit $R=\infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the *c*-map metric

$$ds^2_{\mathcal{M}} = \frac{\mathrm{d}g_4^2}{g_4^2} + \mathrm{d}s^2_{\mathcal{SK}} + g_4^2\,\mathrm{d}s^2_T + g_4^4\,D\sigma^2\,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda$$

Cecotti Girardello Ferrara; Ferrara Sabharwal

- At large volume, the metric on $\mathcal{SK}$ is governed by the prepotential

$$F(X) = -\frac{1}{6}\kappa_{abc}\frac{X^a X^b X^c}{X^0} + \chi(Y)\,\frac{\zeta(3)(X^0)^2}{2(2\pi\mathrm{i})^3} + F_{\rm GW}(X)$$

where κ_{abc} is the cubic intersection form and $F_{\rm GW}$ are Gromov-Witten instanton corrections:

$$F_{\rm GW}(X) = -\frac{(X^0)^2}{(2\pi\mathrm{i})^3}\sum_{k_a\gamma^a\in H_2^+(Y)} n^{(0)}_{k_a}\,\mathrm{Li}_3\left[\mathrm{E}\left(k_a\frac{X^a}{X^0}\right)\right],$$

- We ignore the one-loop correction in this talk.

Classical S-duality I

- In the large volume, classical limit, the metric is invariant under $SL(2,\mathbb{R}) \ltimes N$, where $SL(2,R)$ acts by

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \qquad t^a \mapsto t^a |c\tau+d|\,, \qquad \tilde{c}_a \mapsto \tilde{c}_a$$

$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}\,, \qquad \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}\,,$$

$$\tau = \tau_1 + \mathrm{i}\tau_2\,, \quad \zeta^0 = \tau_1\,, \quad \zeta^a = -(c^a - \tau_1 b^a)\,,$$

$$\tilde{\zeta}_a = \tilde{c}_a + \frac{1}{2}\kappa_{abc}\, b^b (c^c - \tau_1 b^c)\,, \quad \tilde{\zeta}_0 = \tilde{c}_0 - \frac{1}{6}\kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c)\,,$$

$$\sigma = -2\psi - \tau_1 \tilde{c}_0 + \tilde{c}_a (c^a - \tau_1 b^a) - \frac{1}{6}\kappa_{abc}\, b^a c^b (c^c - \tau_1 b^c)\,.$$

and N is the 3-step nilpotent algebra N of translations along $(b^a, c^a), \tilde{c}_a, (c_0, \psi)$.

- One expects D5-D3-D1-D(-1)-instanton corrections of the form

$$\delta \mathrm{d}s^2 \sim \sum_{\gamma} \sigma(\gamma)\,\bar\Omega(\gamma; z^a)\,\exp\left(-8\pi |Z_\gamma|/g_4 - 2\pi \mathrm{i} \langle \gamma, C\rangle\right) + \dots$$

1. $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge
2. $\bar\Omega(\gamma; z^a)$ are the generalized DT invariants, roughly the number of stable coherent sheaves with charge γ
3. the dots stand for loop corrections around single instantons, multi-instantons, NS5-branes...

- The exact form is essentially dictated by consistency with QK geometry and wall crossing, and best expressed in twistor space

Gaiotto Moore Neitze; Kontsevich Soibelman; Alexandrov BP Saueressig Vandoren

Twistors in a nutshell I

- The QK metric on $\mathcal{M}$ is encoded in the complex contact structure on the twistor space $\mathcal{Z}$, a canonical $\mathbb{P}^1$ bundle over $\mathcal{M}$. The contact structure is the kernel of the (1,0)-form, well-defined modulo rescalings,

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

where $p_3, p_\pm$ are the $SU(2)$ components of the connection on $\mathcal{M}$.

Salamon; Lebrun

- Locally, there always exist Darboux coordinates $(\Xi, \tilde{\alpha})$ such that

$$Dt \propto \mathrm{d}\tilde{\alpha} + \langle \Xi, \mathrm{d}\Xi \rangle = \mathrm{d}\tilde{\alpha} + \tilde{\xi}_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda \ .$$

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Twistors in a nutshell II

- Continuous isometries are classified by $H^0(\mathcal{Z},\mathcal{O}(2))$, and always lift (by moment map construction) to a holomorphic action on $\mathcal{Z}$.
- Infinitesimal deformations of $\mathcal{M}$ are classified by $H^1(\mathcal{Z},\mathcal{O}(2))$, and correspond to deformations of the complex contact structure
- For the tree level HM metric, the following Darboux coordinates do the job, away from the north and south poles $t=0,\infty$ (here X is the symplectic vector (X^Λ, F_Λ) :

$$\begin{aligned} \Xi_{\rm sf} &= C + 2R\, e^{\mathcal{K}/2} \left[t^{-1} X - t\, \bar{X} \right] \ , \\ \tilde{\alpha}_{\rm sf} &= \sigma + 2R\, e^{\mathcal{K}/2} \left[t^{-1} \langle X, C\rangle - t\, \langle \bar{X}, C\rangle \right] \ , \end{aligned}$$

Neitzke BP Vandoren; Alexandrov

- We ignore the coordinates $\tilde{\xi}_0, \tilde{\alpha}$ in the sequel.

S-duality and D-instantons in twistor space I

- The action of S-duality on $\mathcal{M}$, combined with a suitable $U(1)$ rotation along the fiber,

$$z \mapsto \frac{c\bar\tau+d}{|c\tau+d|}\, z, \qquad z \equiv \frac{t+\mathrm{i}}{t-\mathrm{i}},$$

lifts to a holomorphic action on $\mathcal{Z}$ via (here $\alpha = -\frac{1}{2}(\tilde\alpha + \xi^\Lambda \tilde\xi_\Lambda)$)

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,,$$
$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c\,, \ldots$$

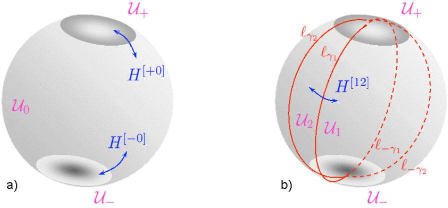
- (ξ^0, ξ^a) transform like (modular parameter, elliptic variable). $\mathrm{E}\left(p^a\tilde\xi_a\right)$ transforms like the automorphic factor of a Jacobi form.
- Note that S-duality fixes the points $t = \pm\mathrm{i}$ along the $\mathbb{P}^1$ fiber.

S-duality and D-instantons in twistor space II

- D-instanton corrections correct the Darboux coordinates into solutions of the integral equations

$$\Xi = \Xi_{\rm sf} + \sum_{\gamma} \Omega(\gamma; z^a)\, \langle \cdot, \gamma \rangle \int_{\ell_\gamma} \frac{\mathrm{d}t'}{8\pi^2\, t'}\, \frac{t+t'}{t-t'}\, \log\left[1 - \sigma(\gamma)\, \mathcal{X}_\gamma(t')\right]$$

where ℓ_γ are the BPS rays $\{t \; : \; Z(\gamma; z^a)/t \in \mathrm{i}\mathbb{R}^-\}$ and $\mathcal{X}_\gamma = \mathrm{E}\left(-\langle \Xi, \gamma \rangle\right)$ are the holomorphic Fourier modes



GMN; Alexandrov BP Saueressig Vandoren; Alexandrov

S-duality and D-instantons in twistor space III

- The consistency of this prescription across lines of marginal stability is guaranteed by the KS formula. The resulting metric, including multi-instanton corrections, is smooth across the walls.
- Similar eqs allowing to compute $\tilde{\alpha}$ and Φ once Ξ is known.
- These eqs can be solved iteratively, by first substituting $\Xi \to \Xi_{\rm sf}$ on the rhs, integrating, etc. leading to an infinite (divergent) series of multi-instanton corrections.
- Do D3-D1-D(-1) instanton corrections preserve S-duality ?

Outline

S-duality and D1-F1-D(-1) instantons I

- Continuous S-duality is broken by Gromov-Witten instantons at tree level and by the one-loop correction.
- In the large volume limit, retaining D1-F1-D(-1) instantons, S-duality mixes (one-loop,D(-1)) and (D1,F1). It was shown that $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ remains unbroken provided

$$\Omega(0,0,0,q_0) = -\chi(Y) \, , \qquad \Omega(0,0,q_a,q_0) = n^{(0)}_{q_a}$$

Robles-Llana Roček Saueressig Theis Vandoren

- The (GMN-type) 'type IIA' Darboux coordinates are not covariant, but they can be mapped by a contact transformation (and Poisson resummation on q_0) to a set of 'type IIB' Darboux coordinates which transform as above.

Alexandrov Saueressig; Alexandrov BP

S-duality and D3-D1-F1-D(-1) instantons I

- Since the same BPS invariants govern D3-D1-D(-1) instantons in IIB/$\mathcal{X}$ and D4-D2-D0 black holes in IIA/$\mathcal{X}$, S-duality is expected to follow from the modularity of the D4-D2-D0 black hole partition

$$\mathcal{Z}_{\rm BH}(\tau, y^a) = \sum_{q_a, q_0} \Omega^{\rm MSW}_{p^a, q_a, q_0} \, \mathrm{E}\left(-(q_0 + \tfrac12 q_-^2)\tau - q_+^2 \bar\tau + q_a y^a \right)$$
$$= \mathrm{Tr}'(2J_3)^2 (-1)^{2J_3} \, \mathrm{E}\left(\left(L_0 - \tfrac{c_L}{24} \right) \tau - \left(\bar L_0 - \tfrac{c_R}{24} \right) \bar\tau + q_a y^a \right)$$

  where q_+, q_- are the projections of q_a on $H^{1,1}$ and $(H^{1,1})^\perp$.
- When p^a is a very ample primitive divisor, $\mathcal{Z}_{\rm BH}$ is the modified elliptic genus of the MSW superconformal CFT, a multivariate Jacobi form of weight $(-\frac32, \frac12)$, index $\kappa_{ab} = \kappa_{abc} p^c$ and multiplier system $v_\eta^{c_{2a} p^a}$.

Maldacena Strominger Witten;

Denef Moore; Gaiotto Strominger Yin; de Boer Cheng Dijkgraaf Manschot Verlinde

S-duality and D3-D1-F1-D(-1) instantons II

- Spectral flow invariance of the SCFT implies that $\Omega^{\rm MSW}(p^a, q_a, q_0)$ depends only on p^a, $\hat{q}_0 \equiv q_0 - \frac{1}{2} q_a \kappa^{ab} q_b$ and on the residue $\mu^a \in \Lambda^*/\Lambda$ of q_a modulo Λ. Thus

$$\mathcal{Z}_{\rm BH}(\tau, y^a) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{1}{2}p} h_{p^a,\mu_a}(\tau)\, \overline{\theta_{p^a,\mu_a}(\tau, y^a, p^a)}\,,$$

where θ_{p^a,μ_a} is a signature $(1, b_2 - 1)$ Siegel-Narain theta series,

$$\theta_{p^a,\mu_a}(\tau, y^a, t^a) = \sum_{k \in \Lambda + \mu + \frac{1}{2}p} (-1)^{p \cdot k}\, \mathrm{E}\left(\frac{1}{2}(k_+)^2 \tau + \frac{1}{2}(k_-)^2 \bar{\tau} + k \cdot y\right)$$

and

$$h_{p^a,\mu_a} = \sum_{\hat{q}_0} \Omega^{\rm MSW}(p^a, \mu_a, \hat{q}_0)\, \mathrm{E}\left(-\hat{q}_0 \tau\right)$$

is a weight $(-\frac{b_2}{2} - 1, 0)$ vector-valued modular form.

S-duality and D3-D1-F1-D(-1) instantons III

- There is an important catch: the MSW degeneracies $\Omega^{\rm MSW}_{p^a,q_a,q_0}$ agree with the generalized DT invariants only at the 'large volume attractor point'

$$\Omega^{\rm MSW}(p^a, q_a, q_0) = \lim_{\lambda \to +\infty} \bar{\Omega}\,(0, p^a, q_a, q_0; b^a(\gamma) + \mathrm{i}\lambda\, t^a(\gamma))$$

- Away from this point, DT invariants get contributions from bound states of MSW micro-states. To exhibit modular invariance, we need to first express the generalized DT invariants in terms of MSW invariants, and then do the multi-instanton expansion in powers of $\Omega^{\rm MSW}_{p^a,q_a,q_0}$.
- We shall restrict to the one-instanton approximation, effectively identifying $\bar{\Omega}(0, p^a, q_a, q_0; z^a) = \Omega^{\rm MSW}(p^a, q_a, q_0)$. Moreover we work in the large volume limit, zooming around $t = \pm i$ $(z = 0, \infty)$.

S-duality and D3-D1-F1-D(-1) instantons IV

- By expanding the integral equations to first order in Ω^{MSW}, and allowing corrections of the same order to the mirror map between $\zeta^\Lambda, \tilde\zeta_\Lambda, \sigma$ and $c^a, \tilde c_a, \tilde c_0, \sigma$, one finds

$$\delta\xi^0 = 0\ , \quad \delta\xi^a = 2\pi\mathrm{i} p^a\, \mathcal{J}_p(z)\ , \quad \delta\tilde\xi_a = -D_a\, \mathcal{J}_p(z)\ , \quad \dots$$

where $S_{\mathrm{cl}} = \frac{\tau_2}{2}\,\kappa_{abc}p^a t^b t^c - \mathrm{i}\,\tilde c_a p^a$ is the classical D3-brane action,

$$\mathcal{J}_p(z) = \sum_{q_\Lambda} \int_{\ell_\gamma} \frac{\mathrm{d}z'}{(2\pi)^3\mathrm{i}(z'-z)}\, \Omega^{\mathrm{MSW}}(p^a, q_a, q_0)\mathrm{E}\left(p^a\tilde\xi_a - q_\Lambda \xi^\Lambda\right),$$

where ℓ_γ runs from $-\infty$ to $+\infty$, passing through the saddle point at $z'_\gamma = -\mathrm{i}(q+b)_+/\sqrt{p\cdot t^2}$.

- Corrections to the Darboux coordinates have a modular anomaly, best exposed by rewriting the Penrose-type integral along z as an Eichler integral

$$\mathcal{J}_{\boldsymbol{p}}(z) = \frac{\mathrm{i}\, e^{-2\pi S_{\mathrm{cl}}}}{8\pi^2} \sum_{\boldsymbol{\mu}\in\Lambda^*/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau) \int_{\bar\tau}^{-\mathrm{i}\infty} \frac{\overline{\Upsilon_{\boldsymbol{\mu}}(w,\bar\tau;\bar z)}\,\mathrm{d}\bar w}{\sqrt{\mathrm{i}(\bar w-\tau)}}$$

where, restricting to $z=0$ for simplicity,

$$\overline{\Upsilon_{\boldsymbol{\mu}}(w,\bar\tau;0)} = \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac12\boldsymbol{p}} (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\,(k+b)_+$$

$$\times\,\mathrm{E}\left(-\tfrac12(\boldsymbol{k}+\boldsymbol{b})_+^2\,\bar w-\tfrac12(\boldsymbol{k}+\boldsymbol{b})_-^2\,\tau+\boldsymbol{c}\cdot(\boldsymbol{k}+\tfrac12\boldsymbol{b})\right).$$

S-duality and D3-D1-F1-D(-1) instantons VI

- The Eichler integral of an analytic modular form $F(\tau,\bar\tau)$ of weight $(\mathfrak{h},\bar{\mathfrak{h}})$ (known as the shadow) is defined by

$$\Phi(\tau) = \int_{\bar\tau}^{-\mathrm{i}\infty} \frac{F(\tau,\bar w)\,\mathrm{d}\bar w}{[\mathrm{i}(\bar w-\tau)]^{2-\bar{\mathfrak{h}}}}$$

  It transforms with modular weight $(\mathfrak{h}+2-\bar{\mathfrak{h}},0)$, up to modular anomaly given by a period integral,

$$\Phi(\gamma\tau) = (c\tau+d)^{\bar{\mathfrak{h}}+2-\mathfrak{h}} \left(\Phi(\tau) - \int_{-d/c}^{-\mathrm{i}\infty} \frac{F(\tau,\bar w)\,\mathrm{d}\bar w}{[\mathrm{i}(\bar w-\tau)]^{2-\bar{\mathfrak{h}}}} \right) .$$

- In particular, $\mathcal{J}_{\boldsymbol{p}}(z)$ transforms with modular weight $(-1,0)$, up to modular anomaly of the form above.

- Miraculously, the modular anomalies in the Darboux coordinates can be absorbed all at once by a contact transformation generated by

$$H = \frac{1}{8\pi^2} \mathrm{E}\left(p^a \tilde{\xi}_a\right) \sum_{\mu \in \Lambda^*/\Lambda + \frac{1}{2}p} h_{p^a,\mu_a}(\xi^0)\, \Theta_{p^a,\mu_a}(\xi^0, \xi^a)$$

where Θ_{p^a,μ_a} is Zwegers' indefinite theta series, viewed as a holomorphic function in twistor space,

$$\theta_{p^a,\mu_a}(\tilde{\xi}^0, \xi^a) = \sum_{k \in \Lambda + \mu + p/2} (\mathrm{sign}[(k+b)\cdot t] - \mathrm{sign}[(k+b)\cdot t_1]) \times (-1)^{p\cdot k}\, \mathrm{E}\left(-k_a \xi^a - \frac{1}{2}\xi^0\, k_a \kappa^{ab} k_b\right)$$

Here t_1 is an arbitrary point on the boundary of the Kahler cone.

S-duality and D3-D1-F1-D(-1) instantons VIII

- The fact that h_{p^a,μ_a} transforms with multiplier system $v_\eta^{p^a c_{2,a}}$ implies that $\tilde{c}_a$ must transform with an additional shift $\tilde{c}_a \mapsto \tilde{c}_a - c_{2,a} \log v_\eta$ under S-duality.
- Amusingly, the holomorphic theta series provides the modular completion of the Eichler integral, rather than the other way around ! The latter arises as a Penrose-type integral (after Fourier transform along the fiber).
- The contact potential has no modular anomaly, and is given by the modular derivative of the MSW elliptic genus.

Outline

Conclusion I

- In the one-instanton approximation and large volume limit, D3-instanton corrections turn out to be consistent with S-duality, albeit in a very non-trivial manner.
- At finite volume, D3-instanton corrections on $\mathcal{M}$ are no longer given by Gaussian theta series (or Eichler integrals thereof), although they are still formally Gaussian on $\mathcal{Z}$.
- At two-instanton level, one expects a non-trivial interplay between modularity and wall-crossing.

Manschot; Alim Haghighat Hecht Klemm Rausch Wottschke

- It would be worth revisiting previous linear analysis of NS5 instantons. Is S-duality automatic, or does it require special properties of the D5-D3-D1-D(-1) DT invariants ?

Alexandrov Persson BP