Fluctuations in finite dimensional spin-glass dynamics

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1 Introduction

The out of equilibrium relaxation of glasses is well understood at the mean-field level. Fully-connected disordered spin systems, finite dimensional manifolds evolving in infinite dimensional transverse spaces with quenched random potentials, and mode-coupling-like approximations to models of interacting particles with short-ranged potentials have been successfully analyzed. The results obtained are consistent with the alternative analysis of metastable states that uses an effective free-energy density (Thouless-Anderson-Palmer approach) and the replica trick extended to access metastability and not only equilibrium properties. All these formalisms allow one to analyse global or macroscopic observables and correlation functions measured over the full system [1].

In super-cooled liquids and glasses glassy systems dynamic fluctuations are expected to be very important [2]. It has been only recently that theoretical attention has turned to their analytic description, notably in the super-cooled liquid regime [3]. In this note we shall present a very short summary of a theory of dynamic fluctuations in the *glassy* regime that is based on the assumption that time-reparametrization invariance develops asymptotically in these systems and that it is responsible for spatio-temporal fluctuations (for a recent review see [4]). We provide a rather complete list of references on this approach that should help the reader find all the details that are omitted here. In this note we focus on the application of these ideas to disordered spin models with an energy function. The proposal has also been discussed in the context of kinetically facilitated spin systems and models of particles in interaction.

2 Mean-field disordered models

The p-spin spherical disordered model mimics glassiness in the so-called fragile glasses [1]. It is defined by its energy function:

$$E_J(\boldsymbol{s}) = \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} + z \left(\sum_i s_i^2 - N \right) \;,$$

where z is a Lagrange multiplier enforcing the spherical constraint on the 'vector' spin $\boldsymbol{s} = (s_1, \ldots, s_N)$. Each component can take any real value. The $J_{i_1 \ldots i_p}$ are quenched i.i.d. Gaussian random variables with $[\, J_{i_1 \ldots i_p} \,] = 0$ and $[\, J^2_{i_1 \ldots i_p} \,] \propto N^{1-p}$ with the square brackets denoting an average of the disorder strength distribution function. The integer p defines the model and it characterizes different 'universality classes' depending on $p = 2$ or $p > 2$.

The system is coupled to its environment that generates stochastic dynamics for s_i. Since the spin-components are continuous variables one proposes a Langevin dynamics in the overdamped limit

$$\gamma \dot{s}_i(t) = -\frac{\delta E_J(\boldsymbol{s})}{s_i(t)} + \xi_i(t) \;,$$

with ξ a Gaussian white noise:

$$\langle\, \xi_i(t) \,\rangle = 0 \;, \qquad \qquad \langle\, \xi_i(t) \xi_j(t') \,\rangle = 2\gamma k_B T \delta_{ij} \delta(t - t') \;.$$

γ is the friction coefficient, T is the temperature of the bath and k_B is the Boltzmann constant ($k_B = 1$ henceforth). A rapid quench from high temperature is mimicked by a random initial conditions, $s_i(0)$, taken, e.g., from a Gaussian pdf. Such an initial condition is uncorrelated with the quenched randomness $J_{i_1 \ldots i_p}$.

In the $N \to \infty$ limit the causal dynamics can be described with the global correlation function

$$C(t, t_w) = N^{-1} \sum_{i=1}^{N} [\, \langle s_i(t) s_i(t_w) \rangle \,] \;,$$

the associated linear response function

$$R(t, t_w) = N^{-1} \sum_{i=1}^{N} \left. \frac{\langle\, \delta s_i(t) \,\rangle}{\delta h_i(t_w)} \right|_{h=0} \;,$$

or its integral over time $\chi(t, t_w) = \int_{t_w}^{t} dt' R(t, t')$. In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t, t_w) = 2T R(t', t) + \int dt' \; [\varSigma(t, t') C(t', t_w) + D(t, t') R(t_w, t')] \;,$$

$$(\partial_t - z_t) R(t, t_w) = \delta(t - t_w) + \int dt' \; \varSigma(t, t') R(t', t_w) \;,$$

where the self-energy and vertex are functions of C and R:

$$D(t,t_w) = \frac{p}{2} C^{p-1}(t,t_w) \; , \quad \Sigma(t,t_w) = \frac{p(p-1)}{2} C^{p-2}(t,t_w)\, R(t,t_w) \; ,$$

and the time-dependent Lagrange multiplier z_t is fixed by imposing $C(t,t) = 1$ [5, 6]. These equations can be solved numerically but also analytically in the long t_w limit [5, **?**] if one uses a few assumptions.

Below a critical temperature $T_d(p)$ the system cannot equilibrate with its environment and relaxes out of equilibrium. The correlation and linear reponse age (stationary is lost). A separation of time scales controlled by t_w develops and it is illustrated in Fig. 1 (a). The relaxation below the plateau q scales as

$$C^s(t,t_w) \approx q f_c \left(\frac{L(t)}{L(t_w)} \right) \; , \qquad \partial_t C^s(t,t_w) \ll C^s(t,t_w) \; , \tag{1}$$

with $f_c(1) = 1$ and $f_c(\infty) \to 0$, and it is very slow. One can then approximate the dynamic equations by dropping the time-derivatives and approximating the integrals. The equations for the slow correlation and linear response then become invariant under time-reparametrization. For example, taking $t - t_w \gg t_w$, using $z_t \to z_\infty$, dropping $\partial_t R$ and separating the fast contributions to the integrals the equation for the linear response becomes

$$\tilde{z}_\infty R^s(t,t_w) \sim \int_{t_w}^{t} dt' \; D'[C^s(t,t')]\, R^s(t,t')\, R^s(t',t_w) \; . \tag{2}$$

$\tilde{z}_\infty$ differs from z_∞ in that it got contributions from the integrals. Equation (2) is invariant under the transformation

$$t \to h_t \equiv h(t) \; , \qquad \begin{cases} C^s(t,t_w) \to C^s(h_t, h_{t_w}) \; , \\ R^s(t,t_w) \to \frac{dh_{t_w}}{dt_w}\, R^s(h_t, h_{t_w}) \; . \end{cases}$$

with h_t any positive-definite and monotonic function of time.

The methods described in [1, 5] allow one to compute analytically f_c and $\chi(C)$

$$\chi(t,t_w) \equiv \int_{t_w}^{t} dt' \, R(t,t') \sim \frac{1-q}{T} + \frac{1}{T_{eff}}\, C^s(t,t_w) = \chi(C)$$

at times t and t_w such that $1 < L(t)/L(t_w)$, but not the 'clock' $L(t)$. Note that the slow part of χ, $1/T_{eff} C^s(t,t_w)$ is finite since $T_{eff} < +\infty$ [8].

In finite dimensional models numerical simulations show that the global correlation function also shows a separation of time-scales stationary-aging, although the plateau is less clearly established [9]. A Renormalization-Group like argument based on the assumption that a separation of time-scales exists allows one to show the approximate asymptotic invariance of the slow part of the action S_{slow} in the Martin-Siggia-Rose generating functional of the Langevin equation under *global* time-reparametrizations [10, 11, 12, 13]:

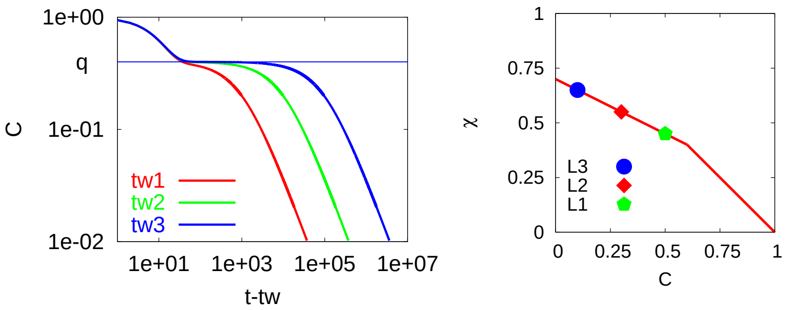

Fig. 1. (a) Sketch of the decay of the two-time correlation below $T_d(p)$ for three waiting-times $t_{w1} < t_{w2} < t_{w3}$. (b) With a solid line the parameteric construction $\chi(C)$ for t_w fixed and t running from t_w to ∞. The three points are the values obtained at a pair (t, t_w) using different functions $L_1(t) < L_2(t) < L_3(t)$.

$$
t \to h_t \equiv h(t) \; , \qquad \left\{ \begin{array}{l} C^s_r(t, t_w) \to C^s_r(h_t, h_{t_w}) \; , \\ R^s_r(t, t_w) \to \frac{dh_{t_w}}{dt_w} \, R^s_r(h_t, h_{t_w}) \; . \end{array} \right.
$$

Symmetry breaking terms become less important as $t_w \to \infty$ and $t - t_w \to \infty$.

The idea is to use this invariance to characterize the fluctuations measured on different boxes with volume V_r centered at sites $\boldsymbol{r}$ in the sample:

$$
C_r(t, t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} s_i(t) s_i(t_w) \; ,
$$

$$
\chi_r(t, t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} \int_{t_w}^{t} dt' \; \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0} \; .
$$

These are local coarse-grained two-time functions and the proposal is that they scale as

$$
C^s_r(t, t_w) \approx q_r f_c \left(\frac{L_r(t)}{L_r(t_w)} \right) \tag{3}
$$

with f_c the same scaling function as the one in the global correlation. For instance, different regions can have different $L_1 = \ln\left(\frac{t}{t_0}\right)$, $L_2 = \frac{t}{t_0}$, $L_3 = e^{\ln^a\left(\frac{t}{t_0}\right)}$ with $a > 1$, as sketched in Fig. 2 (a). The reason for this is that the time-reparametrization invariance makes it easy to modify the 'clock' from region-to-region (massless fluctuations). Instead, the scaling function f_c is hard to change (massive fluctuations).

A number of consequences of this proposal are relatively easy to put to the test numerically or even experimentally and have been summarized in [4]. One of the most stricking one is the extension of the triangular relations [14] between global correlation functions measured at times $t_1 \le t_2 \le t_3$, taken by pairs $C(t_i, t_j)$ with $i, j = 1, 2, 3$, to the local ones [15]. Indeed, the scaling (3) implies that the local correlations should be related by the *same* triangular

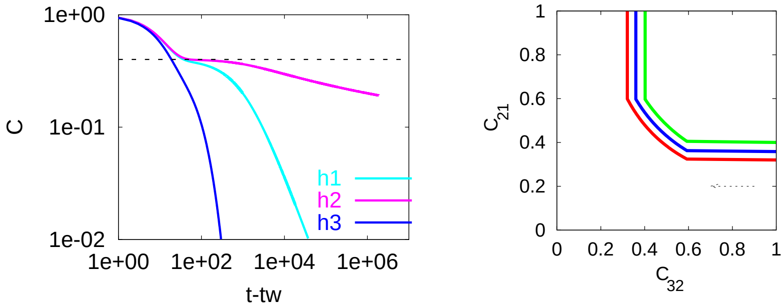

Fig. 2. (a) Decay of the local two-time correlation, at fixed t_w as a function $t - t_w$, on different slower and faster regions. (b) Sketch of the triangular relations between correlation functions measured at three times on local regions in space.

relation as the global ones. A sketch of the behaviour expected is shown in Fig. 2 (b). Each curve is traced for a region using the intermediate time as a parameter. Each region has its own different value of C^r_{13}. This fact was checked in [15] for the $3d$ Edwards-Anderson model.

In order to go further one should obtain an effective action for the local ages $L_r(t)$ – a sigma model. Of course this is a very difficult task. A possible family of models in which this action could be computed are spin-glass models with Kac-type interactions [17]. In practice, in the past we have just proposed an action $S[L_r]$ requiring it to be (i) global time-reversal invariant; (ii) local in space; (iii) positive definite [10, 11, 12, 16]. We have derived from it some predictions that we checked numerically in disordered finite dimensional spin models [10, 11, 12, 15] and kinetically constrained models [16].

The analysis of simple coarsening models [the $O(N)$ model in the large N limit] [18] suggests that dynamic fluctuations in simple coarsening systems might be different from the ones in glassy problems. At least for this mean-field model there is no invariance under generic time-reparametrization as the one discussed above and this result seems to be strongly related to the fact that the effective temperature [8], as defined from the deviations from the equilibrium fluctuation dissipation theorem in the out of equilibrium relaxation, is infinite in this case. This suggestion should be confirmed by further calculations and numerical studies in other domain growth problems and critical dynamics. One should be very careful, though, and analyse consequences of the time-reparametrization invariance scenario that are not simply due to the existence of a growing length scale (see the analysis and discussion of the dynamics of the $3d$ random field Ising model in [19]). Two candidates are the local triangular relations and the local fluctuation-dissipation relation. It would be interesting to extend the numerical study of Lennard-Jones mixtures [20] to the analysis of these local properties.

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