Classical & quantum glasses

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Slides & useful notes in front page

Cargèse, June 2011

Plan

- **First lesson**.

  Introduction. Overview of disordered systems and methods.

- **Second lesson.**

  Statics of classical and quantum disordered systems.

- **Third lesson**.

  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson**.

  Quantum dynamics. Formalism. Results for mean-field models.

Isolated systems

Dynamics of a classical isolated system.

Foundations of statistical physics.

Question : does the dynamics of a particular system reach a flat distribution over the constant energy surface in phase space ?

Ergodic theory, $\in$ mathematical physics at present.

Dynamics of a quantum isolated system :

a problem of current interest, recently boosted by cold atom experiments.

Question : after a quantum quench, i.e. a rapid variation of a parameter in the system, are at least some observables described by thermal ones ? When, how, which ? **but we shall not discuss these issues here**.

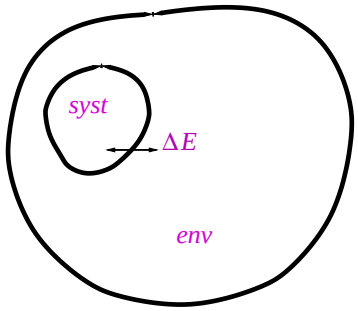
Dissipative systems

Aim

Our interest is to describe the **statics** and **dynamics** of a **classical or quantum system** coupled to a **classical or quantum environment**.

The Hamiltonian of the ensemble is

$$\boxed{H = H_{syst} + H_{env} + H_{int}}$$



The dynamics of all variables are given by Newton or Heisenberg rules, depending on the variables being classical or quantum.

The total energy is conserved, $E =$ ct but each contribution is not, in particular, $E_{syst} \neq$ ct, and we'll take $E_{syst} \ll E_{env}$.

Reduced system

Model the environment and the interaction

E.g., an esemble of harmonic oscillators and a bi-linear coupling :

$$\boxed{H_{env} + H_{int} = \sum_{\alpha=1}^{\mathcal{N}} \left[\frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2 \right] + \sum_{\alpha=1}^{\mathcal{N}} c_\alpha q_\alpha x}$$

Classically (coupled Newton equations) and **quantum mechanically** (easier in a path-integral formalism) one can integrate out the oscillator variables. Assuming the environment is coupled to the sample at the initial time, $\mathcal{T}$, and that its variables are characterized by a Gibbs-Boltzmann distribution or density function at inverse temperature β one finds a **colored Langevin equation** (classical) or a **reduced dynamic generating functional** $\mathcal{Z}_{red}$ (quantum mechanically).

(see later explicit calculations)

What we know

Collective phenomena lead to **phase transitions**.

E.g., **thermal PM - FM transitions in classical magnetic systems**.

We understand the nature of the **equilibrium phases** and the **phase transitions**. We can describe the phases with **mean-field theory** and the critical behavior with the **renormalization group**.

Quantum and thermal fluctuations conspire against the ordered phases.

We understand the **equilibrium and out of equilibrium relaxation at the critical point or within the phases**. We describe it with the **dynamic RG** at the critical point or the **dynamic scaling hypothesis** in the ordered phase.

E.g., **growth of critical structures or ordered domains**.

What we do not know

In systems with **competing interactions** there is no consensus upon :

- whether there are phase transitions,
- which is the nature of the putative ordered phases,
- which is the dynamic mechanism.

Examples are :

- systems with quenched disorder ;
- systems with geometric frustration ;
- glasses of all kinds.

Static and dynamic **mean-field theory** has been developed – both classically and quantum mechanically – and they yield new concepts and predictions.

Extensions of the **RG** have been proposed and are currently being explored.

Quenched disorder

Quenched variables are frozen during time-scales over which other variables fluctuate.

Time scales

$$\boxed{\tau_0 \ll \tau_{exp} \ll \tau_{eq}^{qd}}$$

τ_{eq}^{qd} could be the diffusion time-scale for magnetic impurities the magnetic moments of which will be the variables of a **magnetic system** ;

or the flipping time of impurities that create random fields acting on other magnetic variables.

Weak disorder (modifies the critical properties but not the phases) *vs.*

strong disorder (that modifies both).

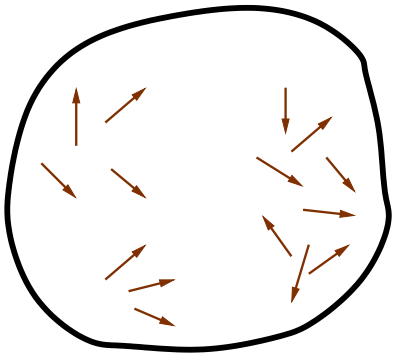
e.g. **random ferromagnets** *vs.* **spin-glasses**.

The case of spin-glasses

Magnetic impurities (spins) randomly placed in an inert host

Quenched random interactions

Magnetic impurities in a metal host



RKKY potential

$$V(r_{ij}) \propto \frac{\cos 2k_F r_{ij}}{r_{ij}^3} s_i s_j$$

very rapid oscillations about 0

and slow power law decay.

Standard lore : there is a 2nd order static phase transition at T_s

separating a paramagnetic from a spin-glass phase.

No dynamic precursor above T_s. Glassy dynamics below T_s with aging, memory effects, etc.

Measurements

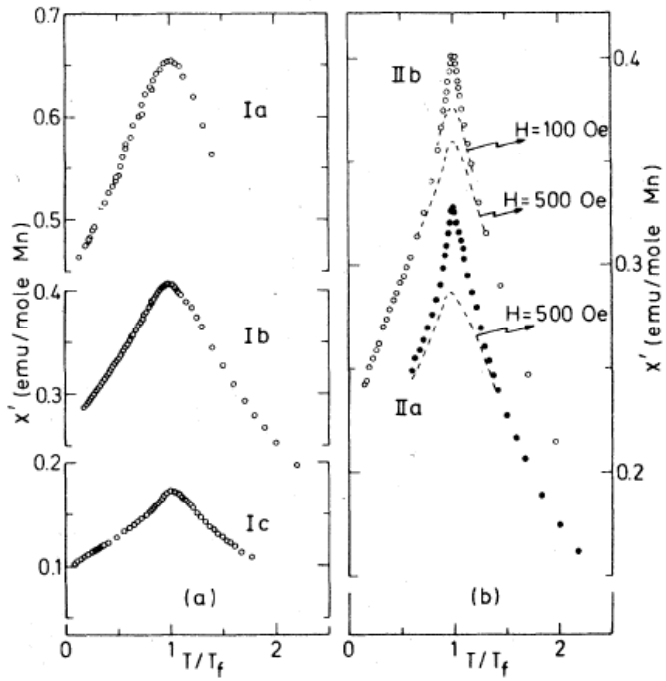
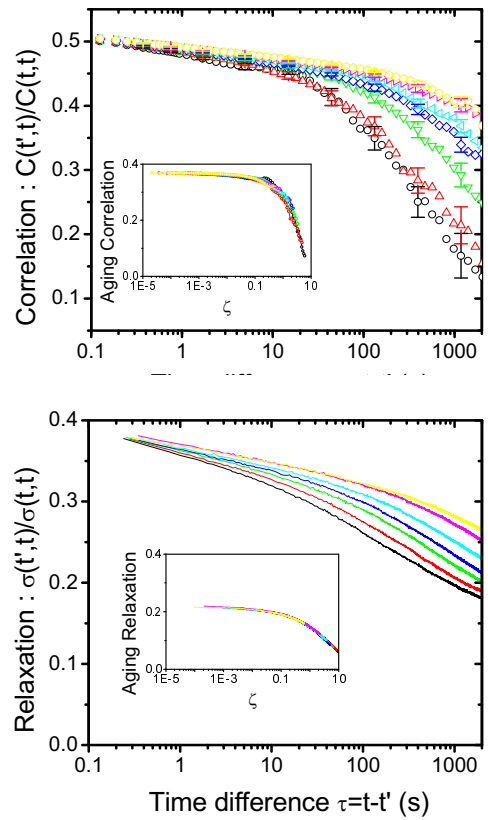


FIG. 1. Susceptibility χ' as a function of reduced temperature T/T_f for various CuMn spin-glass (powder) alloys: (a) Ia(0.23 at.% Mn) $T_f = 2.85$ K, Ib(0.46 at.% Mn) $T_f = 5.00$ K, and Ic(1.48 at.% Mn) $T_f = 12.40$ K; (b) IIa(0.57 at.% Mn) $T_f = 6.00$ K and IIb(0.70 at.% Mn) $T_f = 7.65$ K. The dashed lines represent data in an external magnetic field as labeled. The samples Ia, Ib, Ic and IIa, IIb, respectively, have been prepared and annealed using different methods, as described in text.



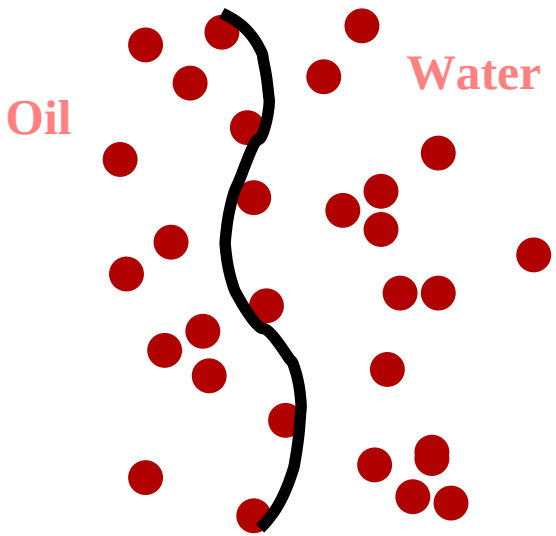
J. Mydosh et al. 81.
Transition

Hérisson & Ocio 02.
Dynamics

Pinning by impurities

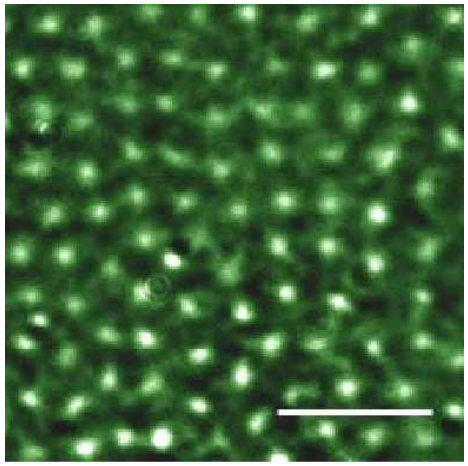
Competition between elasticity and quenched randomness

d-dimensional elastic manifold in a transverse N-dimensional quenched random potential.



Interface between two phases ;
vortex line in type-II supercond ;
stretched polymer.

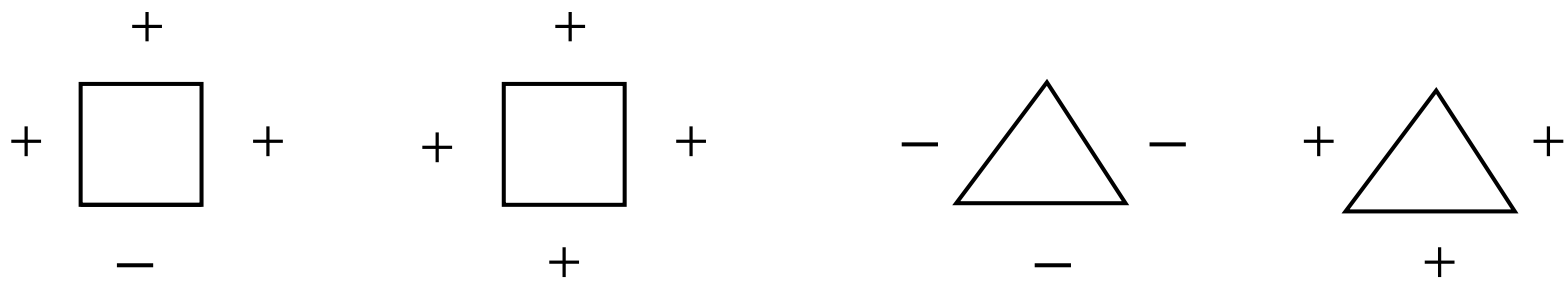
Distorted Abrikosov lattice



Goa et al. 01.

Frustration

$$H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$ **Ising model**



Disordered **Geometric**

$$E_{gs}^{frust} > E_{gs}^{FM}$$ and $$S_{gs}^{frust} > S_{gs}^{FM}$$

Frustration enhances the ground-state enegy and entropy

Disordered example

$$E_{gs}^{frust} = -2J \quad > \quad E_{gs}^{FM} = -4J$$
$$S_{gs}^{frust} = \ln 6 \quad > \quad S_{gs}^{FM} = \ln 2$$

Geometric case

$$E_{gs}^{frust} = 3J \quad > \quad E_{gs}^{FM} = -3J$$
$$S_{gs}^{frust} = \ln 3 \quad > \quad S_{gs}^{FM} = \ln 2$$

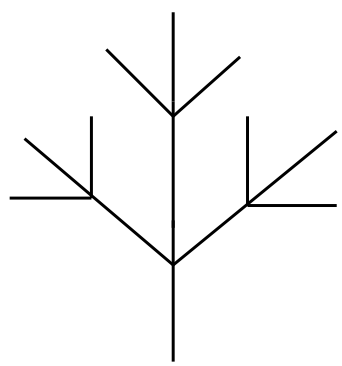
Frustration

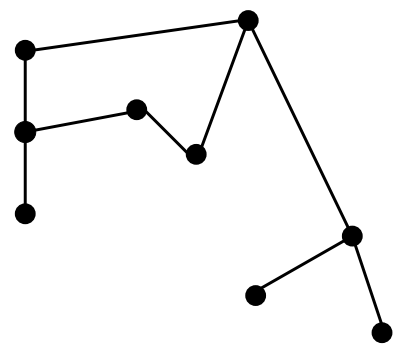
$$H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$ **Ising model**

One cannot satisfy all couplings simultaneously if $\prod_{loop} J_{ij} < 0$.

One can expect to have **metastable states** too.

Trick : Avoid frustration by defining a spin model on a tree, with no loops, or in a dilute graph, with loops of length $\ln N$.



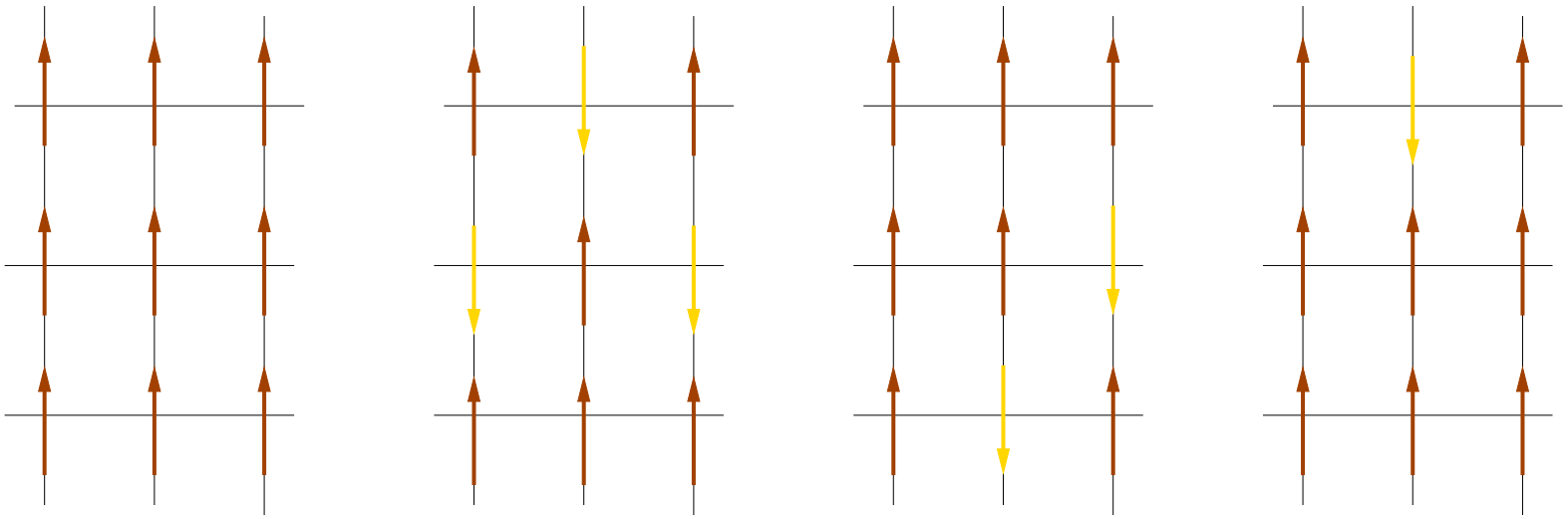


But it could be an over-simplication.

Heterogeneity

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$,

contrary to what happens in a ferromagnetic sample, for instance.



Homogeneous **Heterogeneous**

$h_i = -4J \;\; \forall\, i.$ $h_j = 2J$ $h_k = 0$ $h_l = -2J.$

Each sample is *a priori* different but,

do they all have a different thermodynamic and dynamic behavior ?

Self-averageness

The disorder-induced free-energy density distribution approaches a Gaussian with vanishing dispersion in the thermodynamic limit :

$$\boxed{\lim_{N\to\infty} f_N(\beta, J) = f_\infty(\beta) \qquad \text{independently of disorder}}$$

- Experiments : all *typical* samples behave in the same way.
- Theory : one can perform a (hard) average of disorder,

$$\boxed{-\beta f_\infty(\beta) = \lim_{N\to\infty}[\ln \mathcal{Z}_N(\beta, J)]}$$

Exercise : Prove it for the $1d$ Ising chain ; argument for finite d systems.

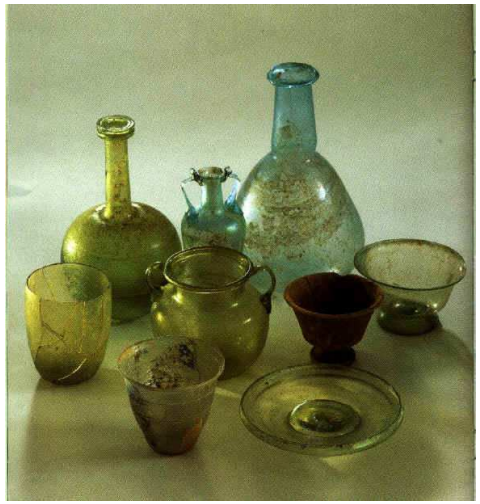
Intensive quantities are also self-averaging.

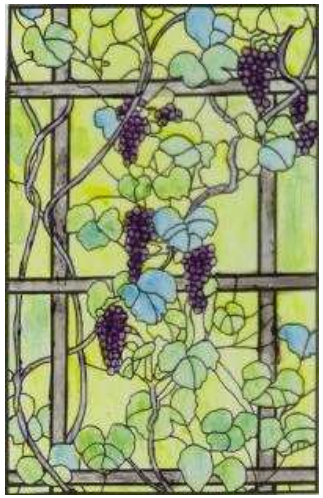
Replica theory

$$\boxed{-\beta f_\infty(\beta) = \lim_{N\to\infty} \lim_{n\to 0} \frac{\mathcal{Z}_N^n(\beta, J) - 1}{Nn}}$$

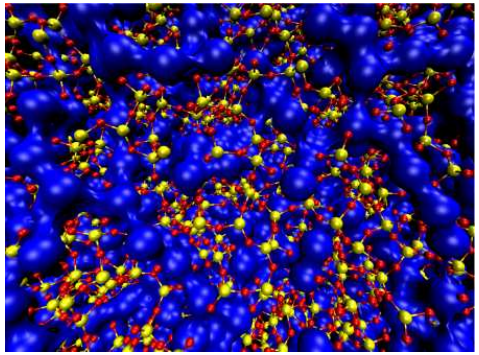
Everyday-life glasses

- 3000 BC Glass discovered in the Middle East. LUXURIOUS OBJECTS.
- 1st century BC Blowpipe discovered on the Phoenician coast. Glass manufacturing flourished in the Roman empire. EVERYDAY-LIFE USE.
- By the time of the Crusades glass manufacture had been revived in Venice. CRISTALLO
- After 1890, the engineering of glass as a material developed very fast everywhere.



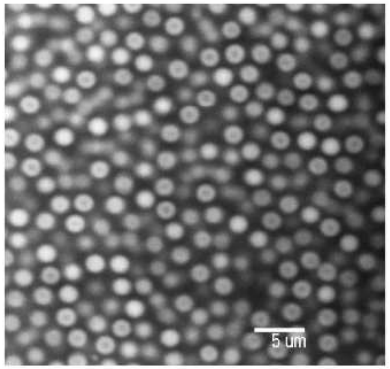


What do glasses look like ?



Simulation

Molecular (Sodium Silicate)

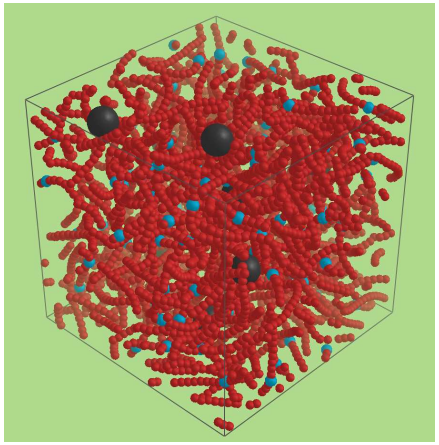


Confocal microscopy

Colloids (e.g. $d \sim 162$ nm in water)

Experiment

Granular matter



Simulation

Polymer melt

Structural glasses

Characteristics

- Selected **variables** (molecules, colloidal particles, vortices or polymers in the pictures) are coupled to their surroundings (other kinds of molecules, water, etc.) that act as **thermal baths in equilibrium**.
- There is no quenched disorder.
- The interactions each variable feels are still in competition, e.g. Lenard-Jones potential, **frustration**.
- Each variables feels a different set of forces, **time-dependent heterogeneity**.

Sometimes one talks about **self-generated disorder**.

Correlation functions

Structure and dynamics

The two-time dependent density-density correlation :

$$g(r;t,t_w) \equiv \langle\, \delta\rho(\vec{x},t)\delta\rho(\vec{y},t_w)\,\rangle \qquad \text{with} \qquad r = |\vec{x}-\vec{y}|$$

The average over different dynamical histories (simulation/experiment) $\langle \ldots \rangle$ implies

isotropy (all directions are equivalent)

invariance under translations of the reference point $\vec{x}$.

Its Fourier transform, $F(q;t,t_w) = N^{-1}\sum_{i,j=1}^{N}\langle\, e^{i\vec{q}(\vec{r}_i(t)-\vec{r}_j(t_w))}\,\rangle$.

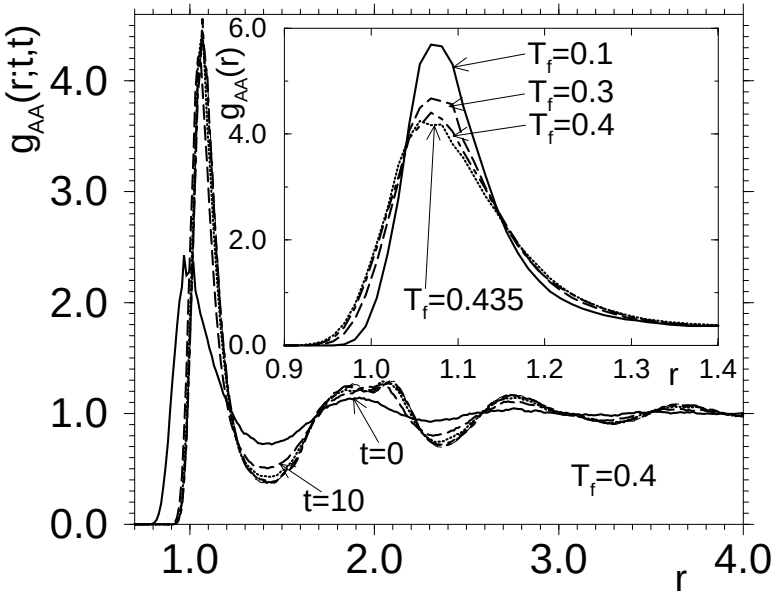
The incoherent intermediate or self correlation :

$$F_s(q;t,t_w) = N^{-1}\sum_{i=1}^{N}\langle\, e^{i\vec{q}(\vec{r}_i(t)-\vec{r}_i(t_w))}\,\rangle$$

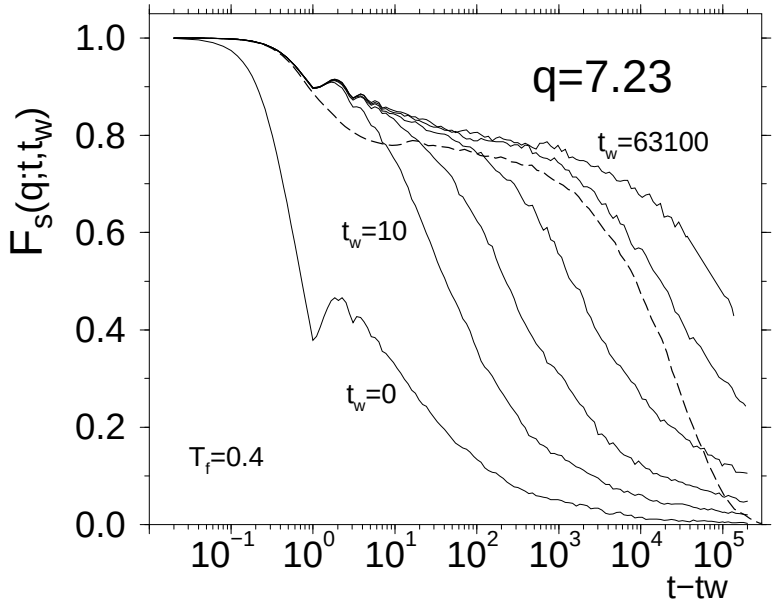
Hansen & McDonald 06.

Structural glasses

No obvious structural change



but slowing down !



LJ mixture $V_{\alpha\beta}(r) = 4\epsilon_{\alpha\beta}\left[\left(\frac{\sigma_{\alpha\beta}}{r}\right)^{12} - \left(\frac{\sigma_{\alpha\beta}}{r}\right)^{6}\right]$

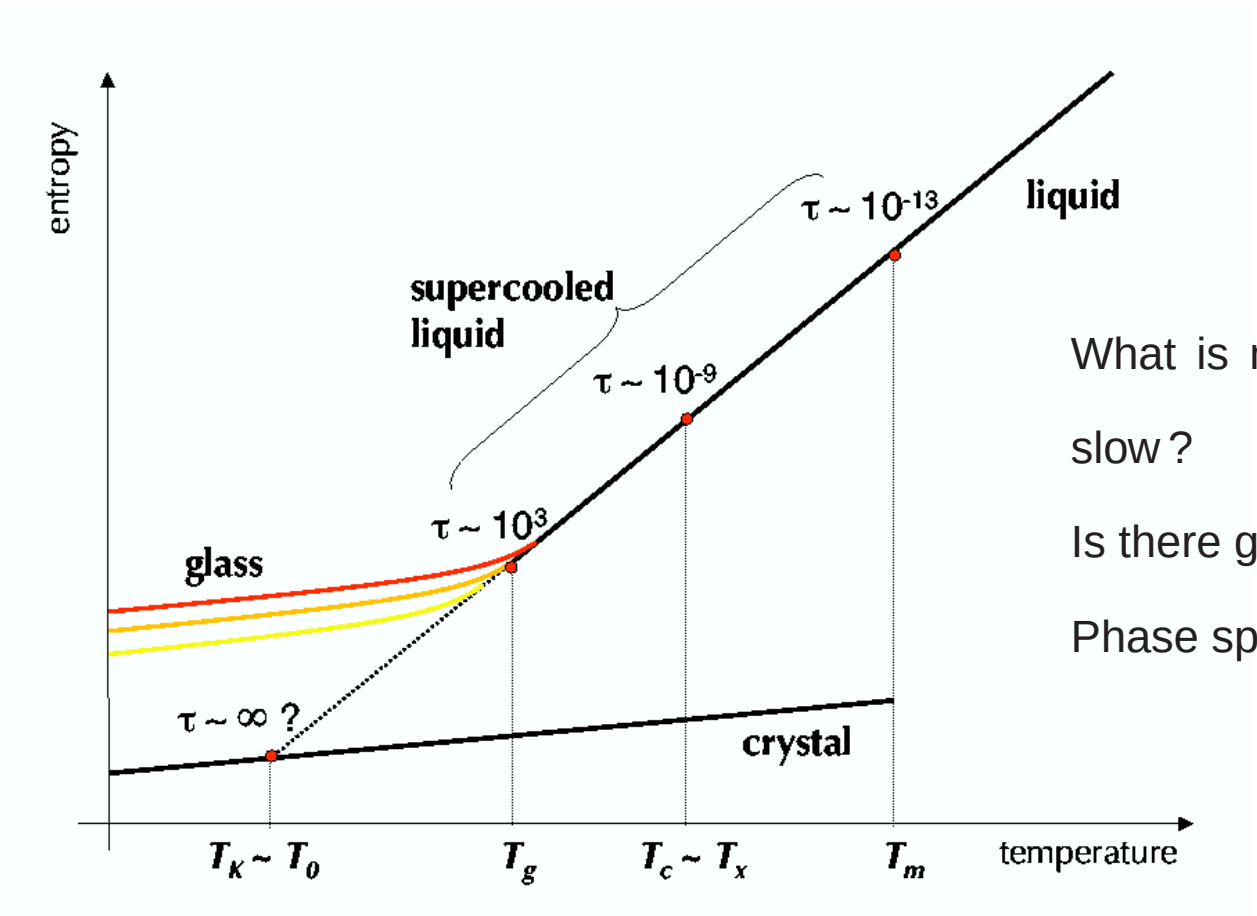
$\tau_0 \ll \tau_{exp} \ll \tau_{eq}$ that changes by > 10 orders of magnitude !

Time-scale separation & slow non-equilibrium dynamics

Molecular dynamics – J-L Barrat & Kob 99.

Time-scales

Calorimetric measurement of entropy



What is making the relaxation so slow ?

Is there growth of static order ?

Phase space picture ?

Fluctuations

Energy scales

Thermal fluctuations

– irrelevant for granular matter since $mgd \gg k_B T$; dynamics is induced by macroscopic external forces.

– important for magnets, colloidal suspensions, etc.

Quantum fluctuations

– when $\hbar\omega \gtrsim k_B T$ quantum fluctuations are important.

– one can even set $T \to 0$ and keep just quantum fluctuations.

Examples : quantum magnets, Wigner crystals, etc.

Summary

Structural glasses

Crytallization at T_m is avoided by cooling fast enough.

Liquid — Supercooled liquid — **Glass**

Exponential relax — Non-exponential relax

Equilibrium — Metastable equilibrium — **Non-equilibrium**

Separation of time-scales &

An exponential number of metastable states !

Stationary — **Aging**

Aging means that correlations and reponses depend on t *and* t_w

ac susceptibilities depend on ω *and* t_w

There might be an **equilibrium transition** to an **ideal glass** at T_s.

Methods

for classical and quantum disordered systems

Statics

TAP Thouless-Anderson-Palmer	fully-connected (complete graph)
Replica theory	Gaussian approx. to field-theories
Cavity or Peierls approx.	dilute (random graph)
Bubbles & droplet arguments functional RG[1]	finite dimensions

Dynamics

Generating functional for classical field theories (MSRJD).

Schwinger-Keldysh closed-time path-integral for quantum dissipative models (the previous is recovered in the $\hbar \to 0$ limit).

Perturbation theory, renormalization group techniques, self-consistent approx.

[1] See P. Le Doussal's seminar.

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End of 1st talk

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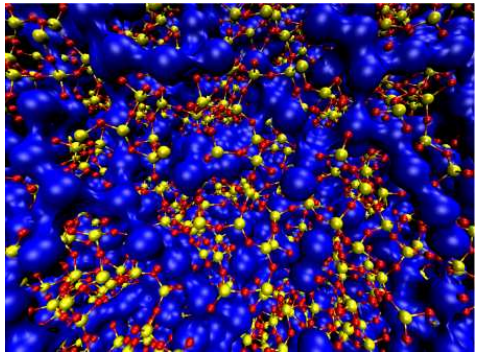

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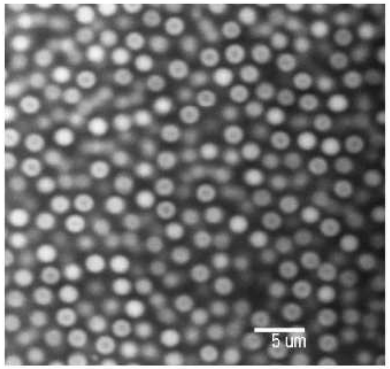

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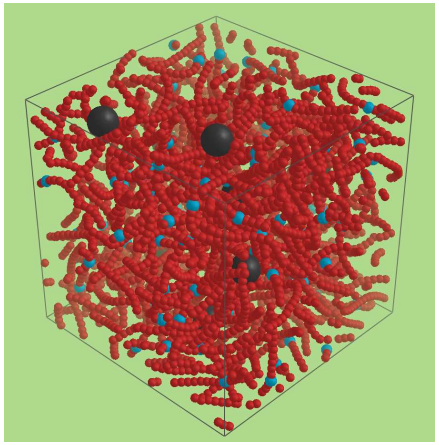

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Hansen & McDonald 06.

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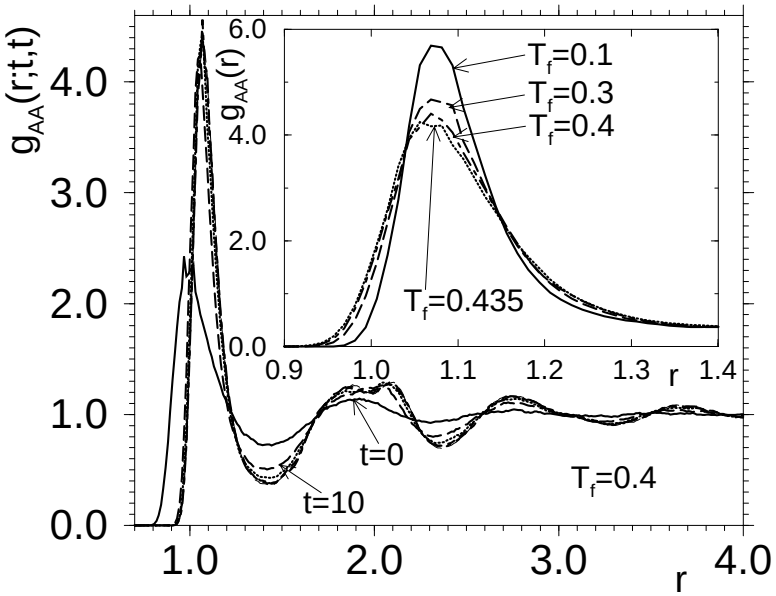

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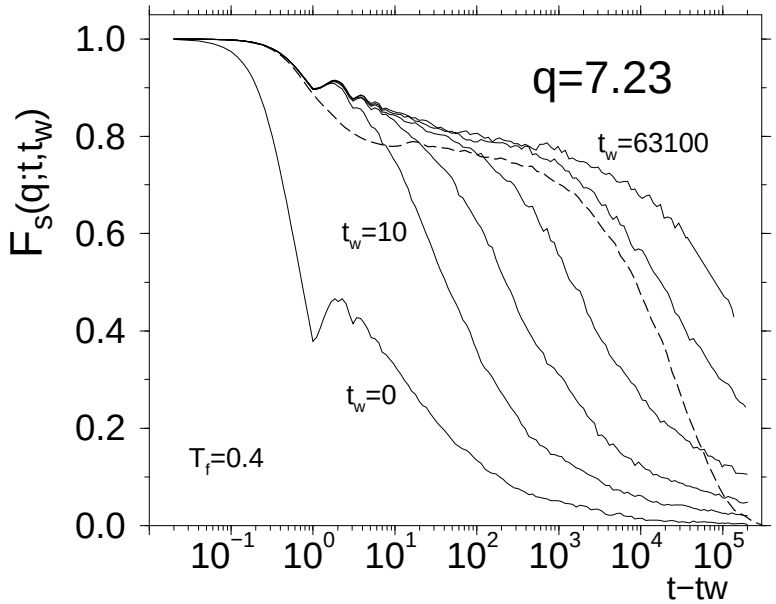

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$$\tau_0 \ll \tau_{exp} \ll \tau_{eq}$$

Time-scale separation & slow non-equilibrium dynamics

Molecular dynamics – J-L Barrat & Kob 99.

Why are they all ‘glasses’ ?

What do they have in common ?

– No obvious spatial order : disorder (differently from crystals).

– Many metastable states

Rugged landscape

– Slow non-equilibrium relaxation

$$\boxed{\tau_0 \ll \tau_{exp} \ll \tau_{eq}}$$

Time-scale separation

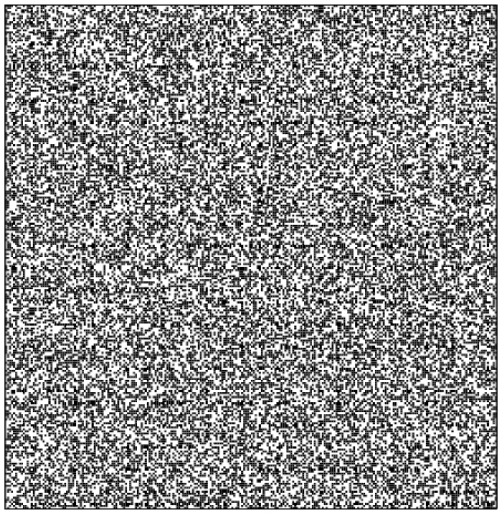
– Hard to make them flow under external forces.

Pinning, creep, slow non-linear rheology.

PM-FM transition

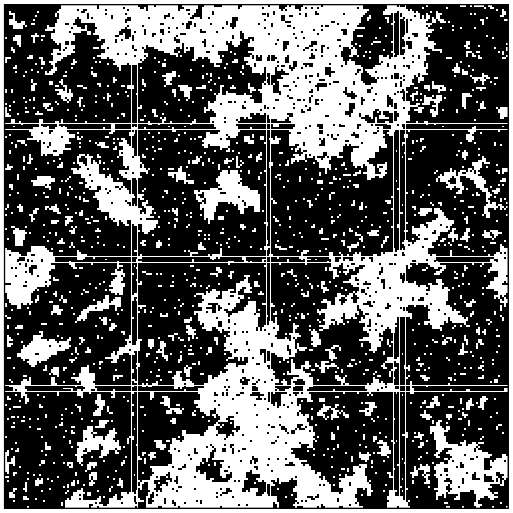
e.g., up & down spins in a $2d$ Ising model (IM)

$\langle \phi \rangle = 0$



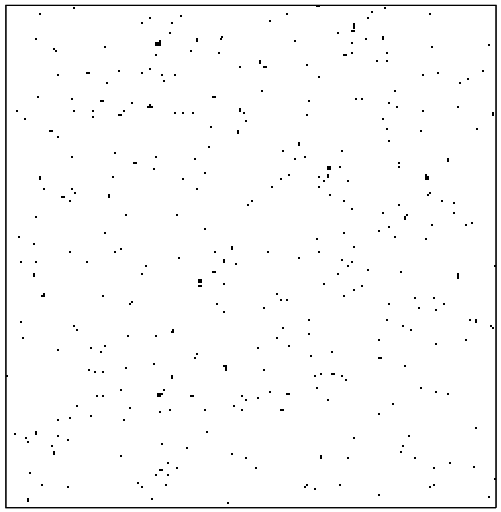
$T \to \infty$

$\langle \phi \rangle = 0$



$T = T_c$

$\langle \phi \rangle \neq 0$



$T < T_c$

In a canonical setting the control parameter is T/J.

Mean-field theory for PM-FM

Fully connected Ising model

Normalize J by the size of the system N to have $\mathcal{O}(1)$ local fields

$$\boxed{H = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - h\sum_i s_i}$$

The partition function reads $\mathcal{Z}_N = \int_{-1}^{1} du\, e^{-\beta N \mathbf{f}(u)}$ with $Nu = \sum_i s_i$

$$\boxed{\mathbf{f}(u) = -\frac{J}{2}u^2 - hu + T\left[\frac{1+u}{2}\ln\frac{1+u}{2} + \frac{1-u}{2}\ln\frac{1-u}{2}\right]}$$

Energy terms and entropic contribution stemming from $\mathcal{N}(\{s_i\})$ yielding the same u value.

Use the **saddle-point**, $\lim_{N\to\infty} f_N(\beta J, \beta h) = \mathbf{f}(u_{sp})$, with

$$\boxed{u_{sp} = \tanh\left(\beta J u_{sp} + \beta h\right) = \langle u \rangle}$$

Exercise : do these calculations, see notes.

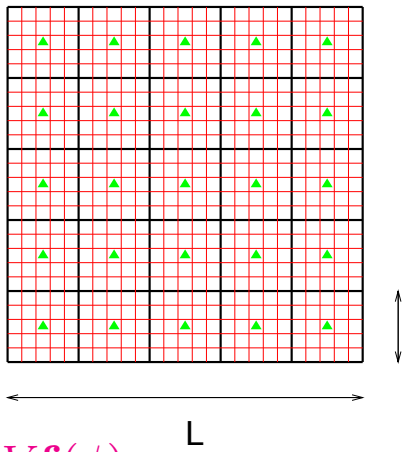
Ginzburg-Landau for PM-FM

Continuous scalar statistical field theory

Coarse-grain the spin

$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$.

Set $h = 0$.



The partition function is $\mathcal{Z}_V = \int \mathcal{D}\phi \, e^{-\beta V \mathbf{f}(\phi)}$ with V the volume and

$$\boxed{\mathbf{f}(\phi) = \int d^d r \, \left\{ \frac{1}{2} [\nabla \phi(\vec{r})]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{\lambda}{4} \phi^4(\vec{r}) \right\}}$$

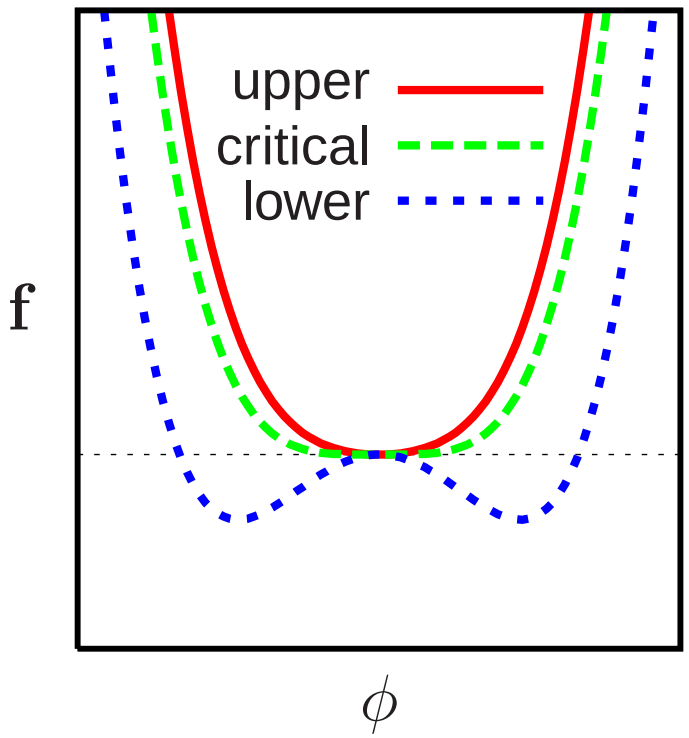
Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments.

Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = \phi_0$.

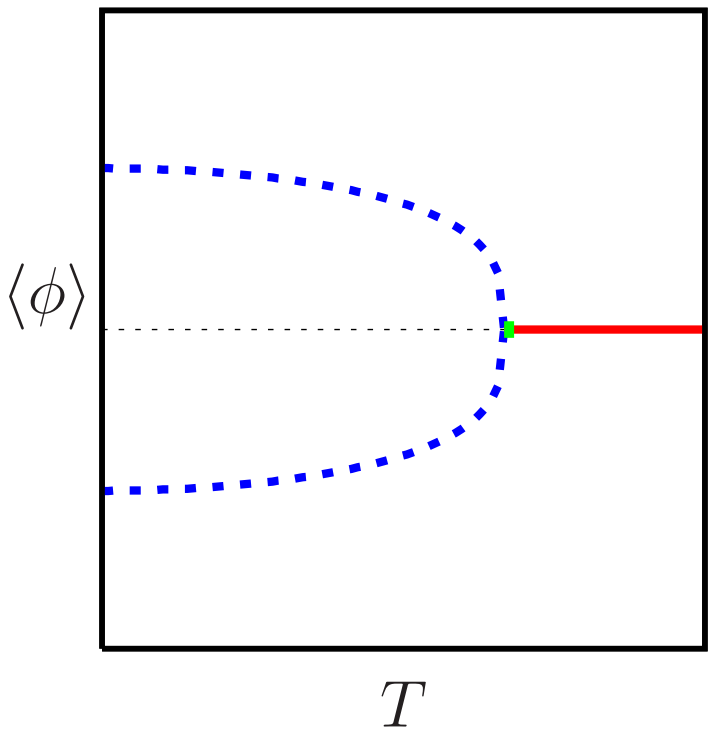
The free-energy is $\lim_{V \to \infty} f_V(\beta, J) = \mathbf{f}(\phi_{sp})$.

2nd order phase-transition

bi-valued equilibrium states related by symmetry, **e.g. Ising magnets**



Ginzburg-Landau free-energy



Scalar order parameter

Features

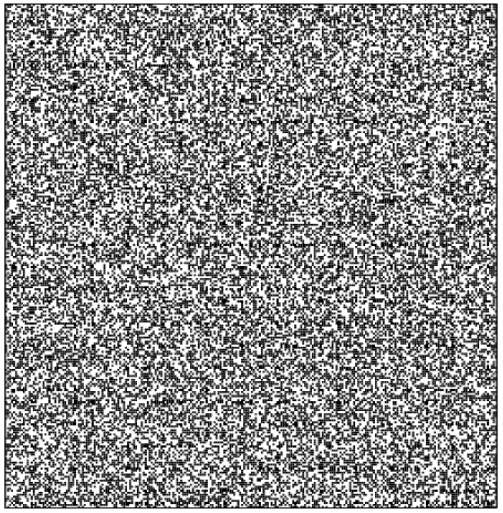
- **Spontaneous symmetry breaking** below T_c.
- Two equilibrium states related by **symmetry** $\phi \to -\phi$.
- The state is chosen by a **pinning field**.
- If the partition sum is performed over the whole phase space $\langle \phi \rangle = 0$ (a consequence of the symmetry of the action).
- **Restricted statistical averages** (*half* phase space) yield $\langle \phi \rangle \neq 0$.
- Under a magnetic field the free-energy landscape is tilted and one of the minima becomes a **metastable state**.
- The barrier in the **free-energy landscape** between the two states diverges with the size of the system implying **ergodicity breaking**.
- With $p > 2$-uplet interactions one finds **first order phase transitions**.

These results were not fully accepted as realistic at the time.

Equilibrium configurations

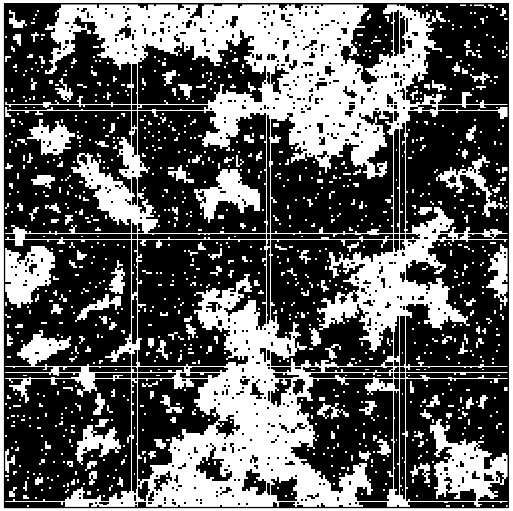
e.g. up & down spins in a $2d$ Ising model (IM)

$\langle \phi \rangle = 0$



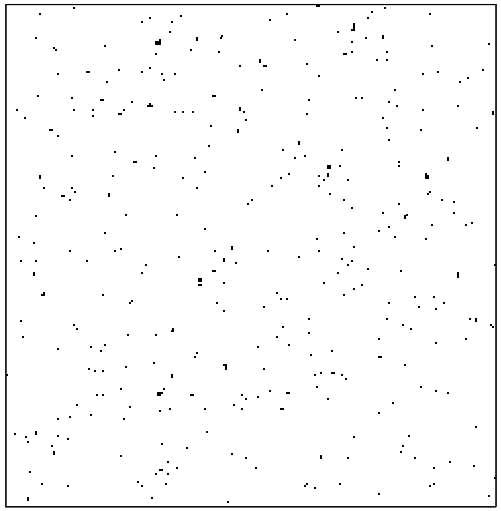
$T \to \infty$

$\langle \phi \rangle = 0$



$T = T_c$

$\langle \phi \rangle \neq 0$



$T < T_c$

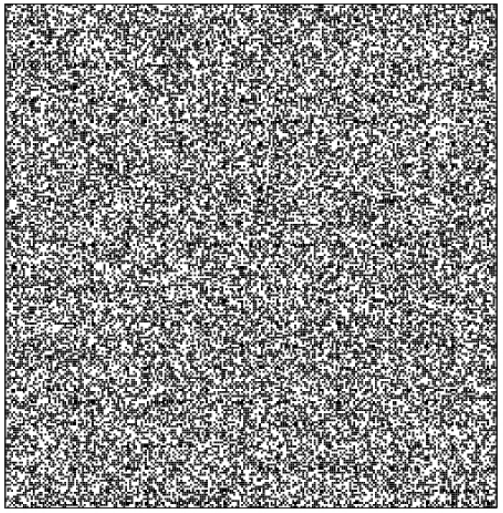
In a canonical setting the control parameter is T/J.

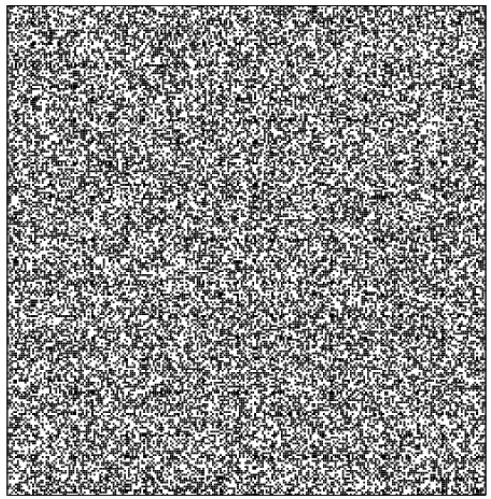
Spin-glasses

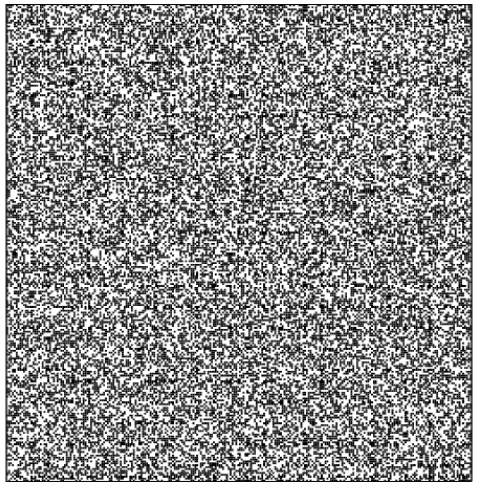
e.g. up & down spins in a $3d$ Edwards-Anderson model (EA)

$$H_J = \sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad [J_{ij}] = 0 \qquad [J_{ij}^2] = J^2$$

$\langle \phi \rangle = 0$ $\qquad$ $\langle \phi \rangle = 0$ $\qquad$ $\langle \phi \rangle = 0$







$T \to \infty$ $\qquad$ $T = T_c$ $\qquad$ $T < T_c$

Is there another order-parameter ?

MF theory for spin-glasses

Fully connected SG : Sherrington-Kirkpatrick model

$$\boxed{H = -\frac{1}{2\sqrt{N}} \sum_{i\neq j} J_{ij} s_i s_j - \sum_i h_i s_i}$$

with J_{ij} i.i.d. Gaussian variables, $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2 = \mathcal{O}(1)$.

One finds the naive free-energy landscape

$$\boxed{N\mathbf{f}(\{m_i\}) = -\frac{1}{2\sqrt{N}} \sum_{i\neq j} J_{ij} m_i m_j + T \sum_{i=1}^{N} \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \ .}$$

and the naive TAP equations

$$\boxed{m_{i\,sp} = \tanh(\beta \sum_{j(\neq i)} J_{ij} m_{j\,sp} + \beta h_i)}$$

that determine the restricted averages $m_i = \langle s_i \rangle = m_{i\,sp}$.

These should be corrected by the Onsager reaction term, that subtracts the self-response of a spin, **see notes**. **Thouless, Anderson, Palmer 77**.

MF theory for spin-glasses

A hint on the proof

The more traditional one assumes independence of the spins,

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

with $p_i(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1}$

and uses this form to express $\langle H \rangle - T\langle S \rangle$ with $S = \ln P(\{s_i\})$; **see notes**.

A more powerful proof expresses $\mathbf{f}$ as the Legendre transform of $-\beta F(h_i)$ with $m_i = N^{-1}\partial[-\beta F(h_i)]/\partial h_i = \langle s_i \rangle_h$.

Georges & Yedidia 91.

This proof is easier to generalize to dynamics (**Biroli 00**) and quantum systems (**Biroli & LFC 01**), **see later**.

Features

- Saddle-points $m_i \neq 0$ below T_c are **heterogeneous**.
- The fact that an **equilibrium state** is determined by $\{m_i\}$ can be made rigorous in MF.
- There are N **order parameters**, $m_i,\ i = 1, \ldots, N$.
- The TAP equations have an exponential in N number of solutions $\{{m_i}^\alpha\}$ that are extrema of the TAP free-energy landscape, *i.e.* saddles of all types, at low temperatures.
- For each solution $\{{m_i}^\alpha\}$ one also has $\{-{m_i}^\alpha_{sp}\}$ but apart from this trivial doubling, the remaining ones are not related by symmetry.
- One can study the **temperature-dependent free-energy landscape** and compute, *e.g.*, how many saddles of each kind exist and how many of these at each level of $\mathbf{f}$.
- One finds a hierarchy of **metastable states**, with high degeneracy, separated by diverging barriers (infinite life-time).

Statistical averages

The average of a generic observable is $\boxed{\langle O\rangle = \sum_\alpha w_\alpha \langle O\rangle_\alpha}$

In the **FM case**, each state $\langle\phi\rangle = \pm\phi_0$ has weigth $w_\pm = e^{-\beta N \mathbf{f}_\pm}\mathcal{Z}_N = 1/2$ and the sum is $\langle O\rangle = \frac{1}{2}\langle O\rangle_+ + \frac{1}{2}\langle O\rangle_-$. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to one of them.

Within the TAP approach $\boxed{w_\alpha^J = \frac{e^{-\beta N \mathbf{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathbf{f}_\gamma^J}}}$ and

$$\boxed{\langle O\rangle = \mathcal{Z}^{-1}(\beta, J)\int d\mathbf{f}\; e^{-\beta[N\mathbf{f} - T\ln\mathcal{N}_J(\mathbf{f},\beta)]}\; O(\mathbf{f})}$$

where $\mathcal{N}_J$ is the (possibly exponential in N) number of solutions to the TAP equations with free-energy density $\mathbf{f}$.

Statistical averages

Consequences

The **equilibrium free-energy** is given by the saddle-point evaluation of the partition sum that implies

$$\boxed{Nf = N\mathbf{f}_{min} - TS_c(\mathbf{f}_{min}, \beta) \qquad\qquad S_c(\mathbf{f}, \beta) = \ln \mathcal{N}_J(\mathbf{f}, \beta)}$$

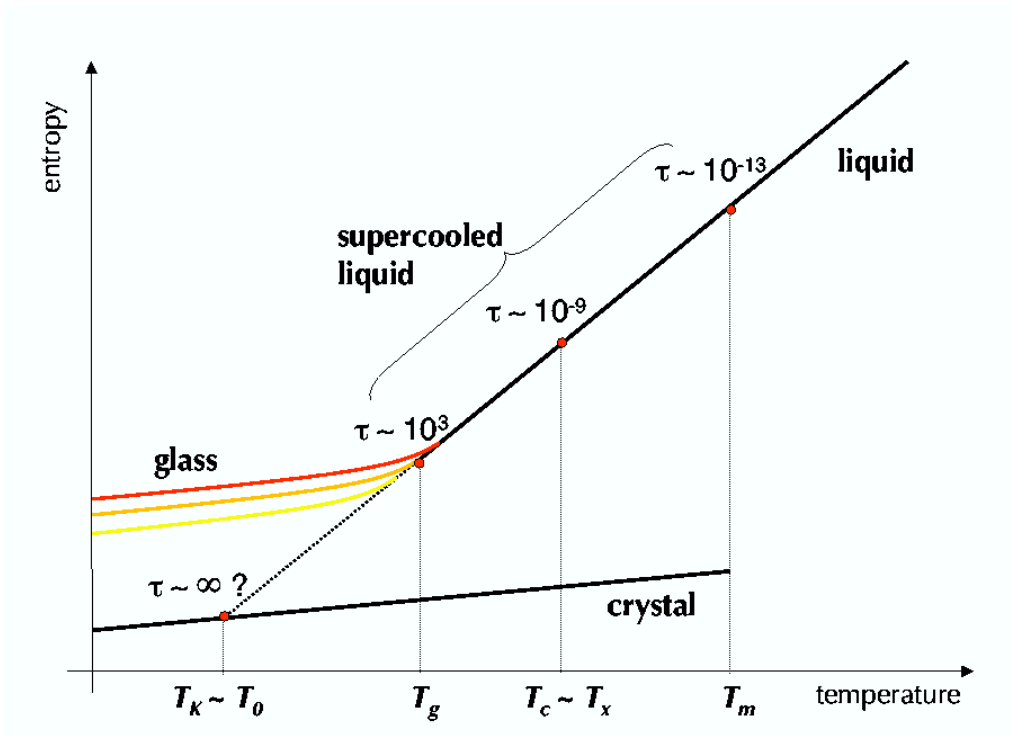
The rhs is the **Landau free-energy** of the problem, with $\mathbf{f}$ playing the role of the energy and $N^{-1} \ln \mathcal{N}_J(\mathbf{f}, \beta)$ the one of the entropy.

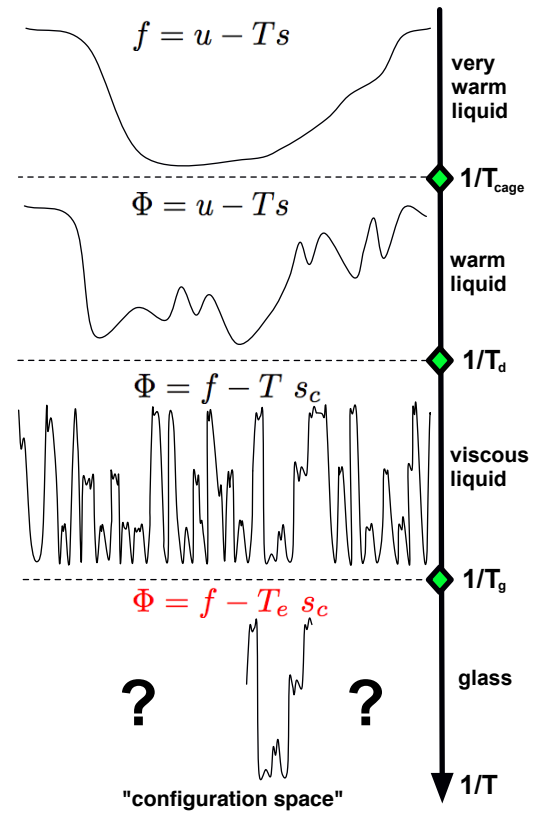
In the sum we do not distinguish the stability of the TAP solutions.

In some cases higher lying extrema (metastable states) can be so numerous to dominate the partition sum with respect to lower lying ones.

This feature is proposed to describe **super-cooled liquids**.

Structural glasses





The ruggedness of the free-energy landscape increases upon decreasing temperature until a configurational entropy crisis arises (at **Kauzmann** T_K).

Numerical simulations : one cannot access f but one can explore e : potential energy landscape. **D. J. Wales 03.**

Replica calculation

A sketch

$$\boxed{-\beta f = \lim_{N\to\infty} \ln \mathcal{Z}_N(\beta, J) = \lim_{N\to\infty} \lim_{n\to 0} \frac{[\mathcal{Z}_N^n(\beta, J)] - 1}{Nn}}$$

$\mathcal{Z}_N^n$ partition function of n independent copies of the system : **replicas**.

Average over disorder : coupling between replicas

$$\boxed{\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \;\Rightarrow\; N^{-1} \sum_{i\neq j} \left(\sum_a s_i^a s_j^a\right)^2}$$

Decoupling with the Hubbard-Stratonovich trick

$$\boxed{Q_{ab} N^{-1} \sum_i s_i^a s_i^b - \tfrac{1}{2} Q_{ab}^2}$$

Q_{ab} is a 0×0 matrix but it admits an interpretation in terms of **overlaps**.

The elements of Q_{ab} can be evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

Replica calculation

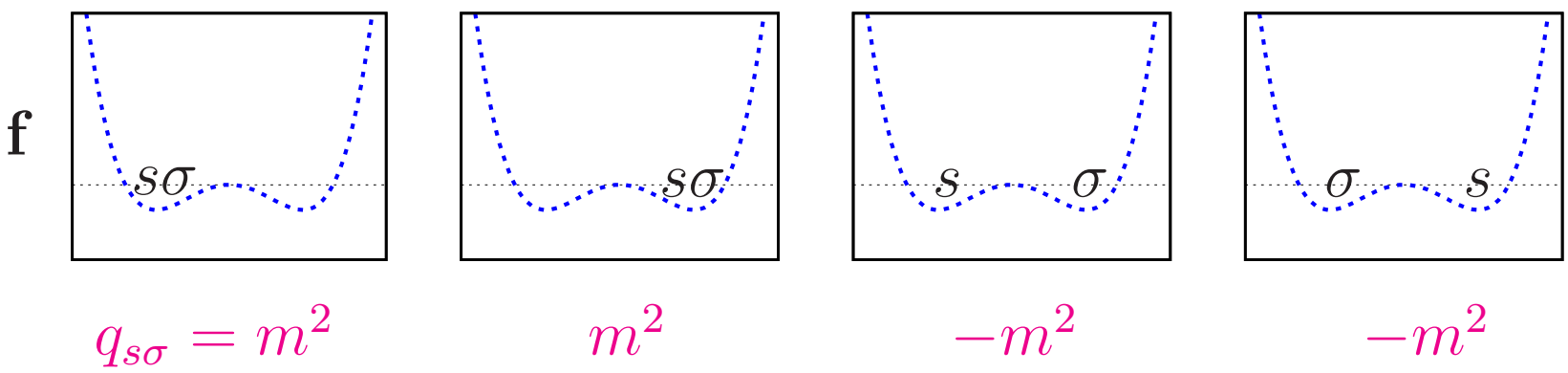
Overlaps

Take one sample and run it until it reaches **equilibrium**, measure $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it until it reaches **equilibrium**, measure $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **FM system** there are four possibilities

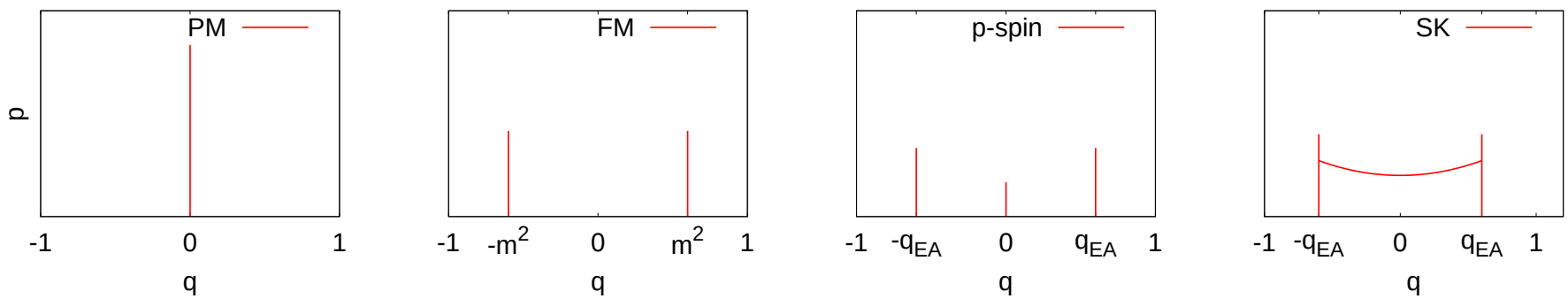


Many repetitions : $\boxed{P(q_{s\sigma}) = \frac{1}{2}\delta(q_{s\sigma} - m^2) + \frac{1}{2}\delta(q_{s\sigma} + m^2)}$

Replica calculation

Overlaps in disordered systems

Parisi 79-82 prescription for the replica symmetry breaking Ansatz yields



High temperature FM Structural glasses Spin-glasses

Thermodynamic quantities, in particular the equilibrium free-energy density, are expressed in terms of the **functional order parameter** $P(q)$.

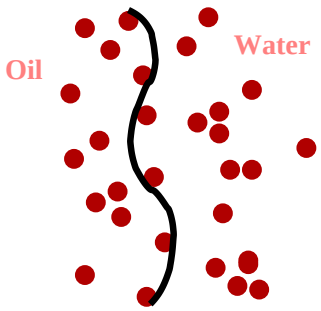
The equilibrium free-energy density predicted by the replica theory was confirmed by **Guerra & Talagrand 00-04** indepedent mathematical-physics methods.

Replica calculation

Applications to other problems

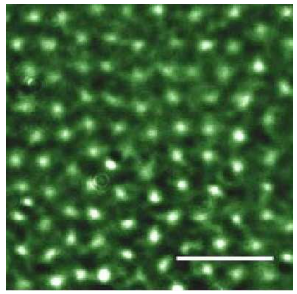
Random elastic manifolds, *e.g.* vortex systems, dirty interfaces, *etc.*

Models with self-generated disorder, *e.g.* Lennard-Jones particles.

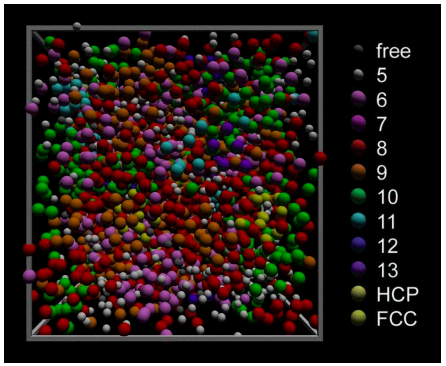


Interface, vortex

Abrikosov lattice



Lennard-Jones mixture



Mézard & Parisi 91, Giamarchi & Le Doussal 97, Mézard & Parisi 99.

Tricks are necessary to introduce quenched randomness in the calculation.

Droplet picture

A much simpler viewpoint for finite-d systems

Just two equilibrium states as in a FM, only that they look spatially disordered.

Compact excitations of linear size ℓ have energy $E(\ell) \simeq \Upsilon \ell^{\theta}$.

Proposition for $P(E_\ell)$.

$P(q)$ with two peaks at $\pm q_{EA}$.

A series of scaling laws lead to predictions for themodynamics, etc.

Also applicable to random manifold problems.

In a sense, a more conventional picture.

Fisher & Huse 87-89.

Long-standing debate, no consensus ; very hard to decide.

Plan

- **First lesson**.

  Introduction. Overview of disordered systems. Short presentation of methods.

- **Second lesson**.

  Statics of classical and **quantum disordered systems**.

- **Third lesson**.

  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson**.

  Quantum dynamics. Formalism and results for mean-field models.

Summary

Classical disordered systems

The statics and structure of metastable states is fully described in MF

Consistent results from **TAP, replicas, cavity for dilute systems and formal arguments**.

Attn ! the statistics of barriers is not fully known.

MF theory predicts a **functional ordered parameter** and three **universality classes** :

FM : 2nd order transition, two states below T_c. Curie-Weiss, GL

Structural glasses : metastable states combine to make the super-cooled liquid, **random first order transition p-spin**.

Spin-glasses : 2nd order transition, many states below T_c. **SK**.

Different scenario from droplet model.

Quantum spin-glasses

$$\boxed{\hat{H}_{syst} = -\sum_{ij} J_{ij}\hat{\sigma}_i^z\hat{\sigma}_j^z + \Gamma\sum_i \hat{\sigma}_i^x - \sum_i h_i\hat{\sigma}_i^z \ .}$$

$\hat{\sigma}_i^a$ with $a = 1, 2, 3$ the Pauli matrices, $[\hat{\sigma}^a, \hat{\sigma}^b] = 2i\epsilon_{abc}\hat{\sigma}^c$.

J_{ij} quenched random., e.g. Gaussian pdf conveniently normalized.

Γ transverse field. It measures quantum fluctuations.

In the limit $\Gamma \longrightarrow 0$ the classical limit should be recovered.

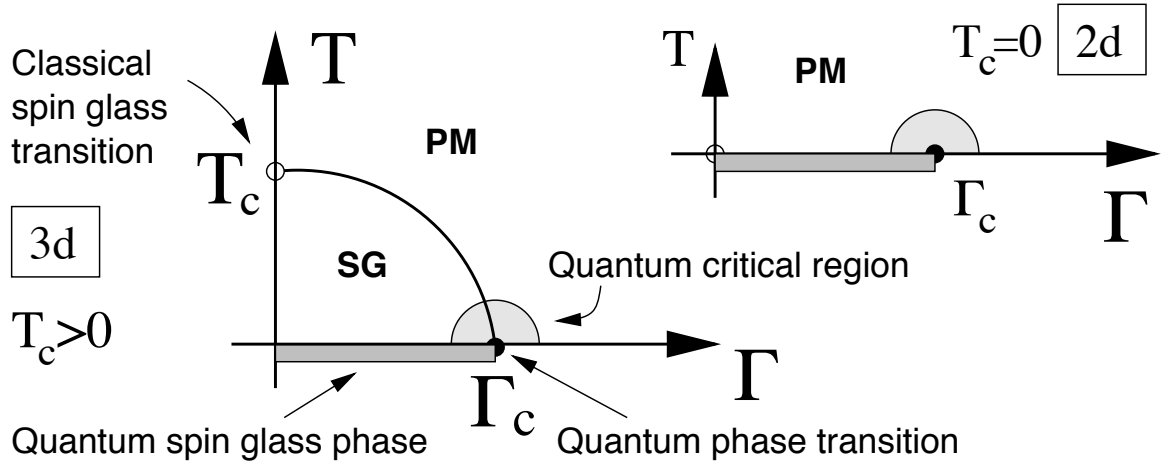
h_i longitudinal local fields

ij nearest neighbours on the lattice – finite d

or fully connected – mean-field

Quantum SG : finite d

Phase transitions in $d = 2,\ 3$



Γ_c Quantum critical point. T_c Classical critical point.

Quantum MC. Overviews in Rieger & Young 95, Kawashima & Rieger 03.

Many special features in $d = 1$ obtained with the Dasgupta-Ma RG decimation procedure. **Recall E. Altman's talk.** **D. S. Fisher 92.**

Mean-field methods

TAP method

Legendre transform of $f(\beta, h)$ with respect to $\{m_i(\tau)\}$ and $C(\tau - \tau')$ with $m_i(\tau) = \langle s_i(\tau) \rangle_h$ and $C(\tau - \tau') = N^{-1} \sum_i \langle s_i(\tau) s_i(\tau') \rangle_h$.

Biroli & LFC 01.

Replica trick

The partition function is a trace

$$\mathcal{Z}_{\mathcal{N}} = \text{Tr}\, e^{-\beta \hat{H}} = \int_{\{s_i(0)\}}^{\{s_i(\beta\hbar)\}} \mathcal{D}\{s_i(\tau)\}\, e^{-\frac{1}{\hbar} S^e_{syst}[\{s_i\}]}$$

with the Euclidean action

Mean-field methods

$$S^e_{syst}[\{s_i\}] = \int_0^{\beta\hbar} d\tau \left[\frac{M}{2} \left(\frac{\partial s_i(\tau)}{\partial \tau} \right)^2 + \sum_{ij} J_{ij} s_i(\tau) s_j(\tau) \right]$$

$s_i(\beta\hbar) = s_i(0)$ and the 'mass' given by $M = (\hbar\tau_0)/2\ln[\hbar/(\Gamma\tau_0)]$ as a function of the transverse field Γ.

Feynman-Matsubara construction of functional integral over imaginary time.

Overlap matrix-function : $Q_{ab}(\tau,\tau')N^{-1}\sum_i s_i^a(\tau)s_i^b(\tau') - \frac{1}{2}Q^2_{ab}(\tau,\tau')$

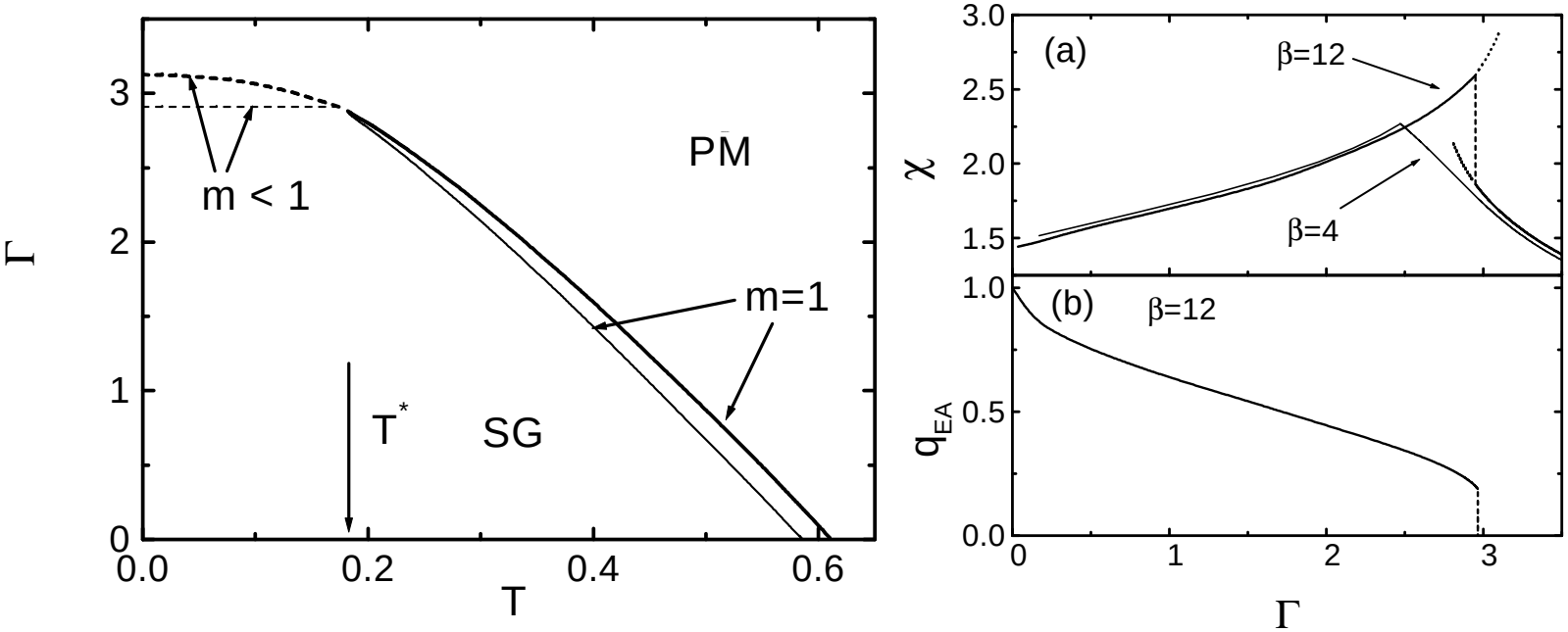
Slightly intricate imaginary-time & replica index structure. Recipes to deal with them

Bray & Moore 80.

Can an argument à la Guerra-Talagrand can be extended to quantum spin-glass models ?

1st order phase transitions

Quantum p-spin model



Jump in the susceptibility across the dashed part of the critical line.

LFC, Grempel & da Silva Santos 00 ; Biroli & LFC 01.

Many more examples, e.g. with **cavity method** in a model with a superfluid/ glass transition **Foini, Semerjian & Zamponi 11 ; see Yu's poster.**

Combinatorial optimization

Optimization problems consist in finding the configuration that renders minimal a **cost function**, *e.g.* the road traveled by a salesman to visit each of N cities once and only once.

The most interesting of these problems can be mapped onto a classical spin model on a random (hyper-)graph with the cost function its Hamiltonian.

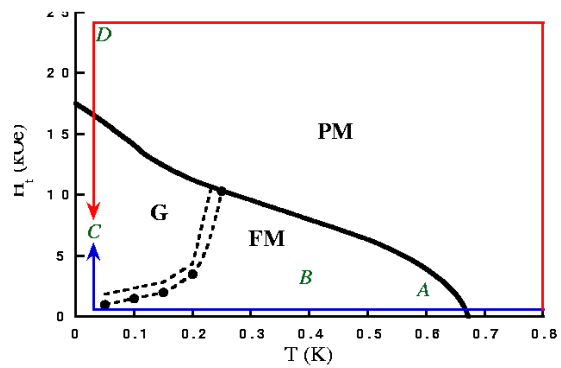
For instance, K**-satisfiability** is written in terms of $p(\leq K)$**- spin models on a random (hyper-)graph**.

Quantum annealing

Kadowaki & Nishimori 98

Dipolar spin-glass

G. Aeppli et al. 90s



1st order transitions : trouble for **quantum annealing** techniques.

Jorg, Krzakala, Kurchan, Maggs & Pujos 09.

Dissipative systems

Aim

Our interest is to describe the **statics** and **dynamics** of a **classical or quantum system** coupled to a **classical or quantum environment**.

The Hamiltonian of the ensemble is

$$\boxed{H = H_{syst} + H_{env} + H_{int}}$$

Syst.	x, p
Env.	$\{q_a, \pi_a\}$

What is the static and dynamic behaviour of the reduced system ?

Discuss it step by step :

1) equilibrium classical,

2) equilibrium quantum,

3) classical dynamics, 4) quantum dynamics.

Reduced system

Imagine the ensemble $H = H_{syst} + H_{env} + H_{int}$ is in equilibrium at inverse temperature β^{-1} :

Model the environment and the interaction

E.g., an ensemble of harmonic oscillators and a bi-linear coupling :

$$\boxed{H_{env} + H_{int} = \sum_{\alpha=1}^{\mathcal{N}} \left[\frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2 \right] + \sum_{\alpha=1}^{\mathcal{N}} c_\alpha q_\alpha x}$$

Either classical variables or quantum operators.

Reduced description

Statics of a classical system

The partition function of the coupled system is

$$\mathcal{Z}[\eta] = \sum_{osc,\, syst} e^{-\beta(H-\eta x)}$$

Integrating out the oscillator variables :

$$\mathcal{Z}_{red}[\eta] = \sum_{syst} e^{-\beta\left(H_{syst}+H_{count}+\eta x-\frac{1}{2}\sum_{a=1}^{\mathcal{N}}\frac{c_a^2\omega_a^2}{m_a}\,x^2\right)}$$

Choosing H_{count} to cancel the quadratic term in x^2 one recovers

$$\boxed{\mathcal{Z}_{red}[\eta] = \mathcal{Z}_{syst}[\eta]}$$

i.e., the partition function of the system of interest.

Reduced description

Statics of a quantum system

The density matrix of the coupled system is

$$\rho(x'', q''_a; x', q'_a) = \langle x'', q''_a | \, \hat{\rho} \, | x', q'_a \rangle$$
$$= \frac{1}{\mathcal{Z}} \int_{x(0)=x'}^{x(\hbar\beta)=x''} \mathcal{D}x(\tau) \int_{q_a(0)=q'_a}^{q_a(\hbar\beta)=q''_a} \mathcal{D}q_a(\tau) e^{-\frac{1}{\hbar} S^e[\eta]}$$

One integrates the oscillator's degrees of freedom to get the reduced density matrix

$$\rho_{red}(x'', x') = \mathcal{Z}_{red}^{-1} \int_{x'}^{x''} \mathcal{D}x(\tau) \, e^{-\frac{1}{\hbar}\left(S^e_{syst} - \int_0^{\hbar\beta} d\tau \int_0^{\tau} d\tau' \, x(\tau) K(\tau - \tau') x(\tau')\right)}$$

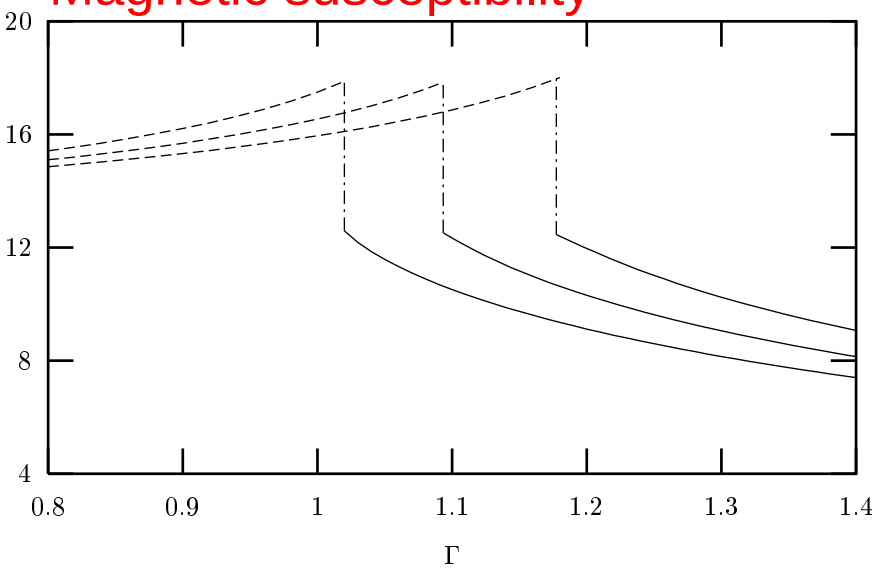
The counter-term was chosen to cancel a quadratic term in $x^2(\tau)$, $\mathcal{Z}_{red} = \mathcal{Z}/\mathcal{Z}_{osc}$ and $\mathcal{Z}_{osc}$ the partition function of isolated ensemble of oscillators, but a non-local interaction in the imaginary time with kernel

$K(\tau) = \frac{2}{\pi\hbar\beta} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\omega \, \frac{I(\omega)}{\omega} \, \frac{\nu_n^2}{\nu_n^2 + \omega^2} \, \exp(i\nu_n \tau)$ remains.

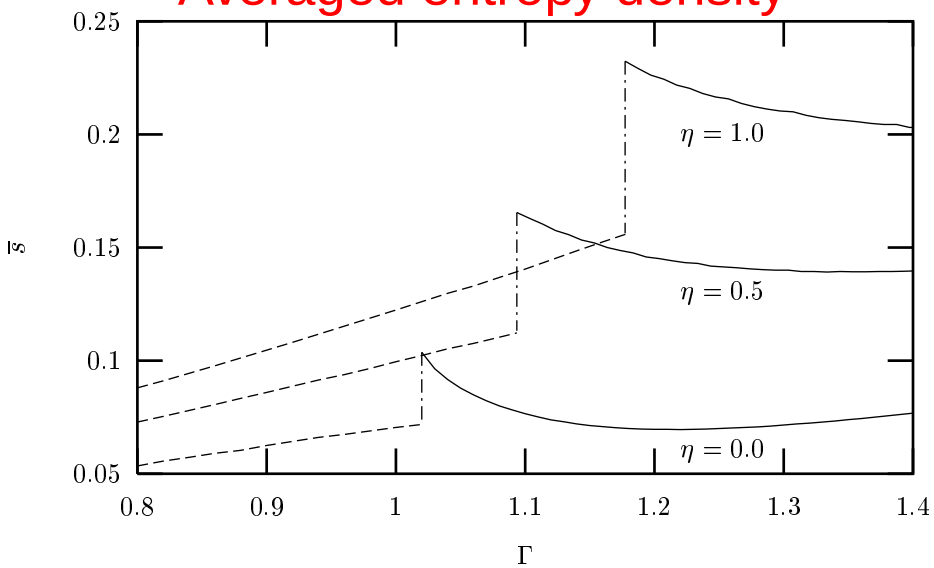
1st order phase transition

Dissipative Ising p-spin model with $p \geq 3$ at $T \approx 0$

Magnetic susceptibility



Averaged entropy density



$$\alpha = 0,\ 0.5,\ 1$$

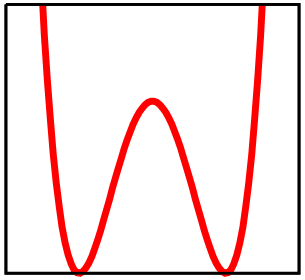
LFC, Grempel & da Silva Santos 00 ;

LFC, Grempel, Lozano, Lozza, da Silva Santos 04.

Localization

The Caldeira-Leggett problem

A quantum particle in a double-well potential coupled to a bath of quantum harmonic oscillators in equilibrium at $T = 0$.



Quantum tunneling for $0 < \alpha < 1/2$

'Classical tunneling' for $1/2 < \alpha < 1$

Localization in initial well for $1 < \alpha$

Bray & Moore 82, Leggett et al. 87.

More later.

Summary

Statics of quantum disordered systems

- We introduced quantum spin-glasses.
- We very briefly ‘explained’ that the TAP and replica approaches as well as the cavity method can be applied to them.
- We showed that models in the **random first order phase transition** class have **first order phase transitions** in the low temperature limit.
- We gave an argument as to why a quantum environment can have a highly non-trivial effect quantum mechanically.

Similar results for quantum Ising chains with FM and disordered interactions

LFC, Lozano & Lozza ; Chakravarty, Troyer, Voelker, Werner 04 (MC)

Schehr & Rieger 06 (decimation RG)

It was the end of the 2nd talk

Plan

- **First lesson**.

  Introduction. Overview of disordered systems. Short presentation of methods.

- **Second lesson**.

  Statics of classical and quantum disordered systems.

- **Third lesson**.

  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson**.

  Quantum dynamics. Formalism and results for mean-field models.

Phase ordering kinetics

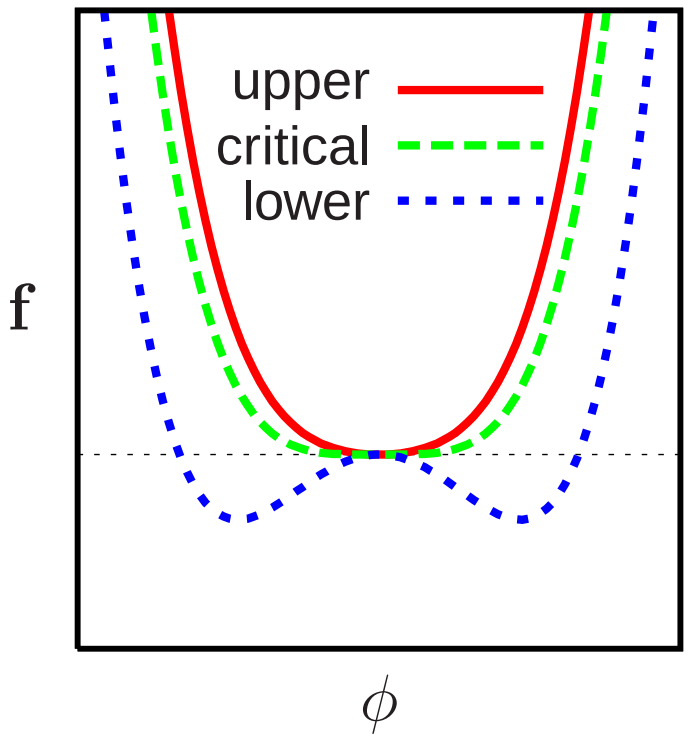
Dynamics across a phase transition

- **Equilibrium phases** are known on both sides of the transition.
- The **dynamic mechanism** can be understood.
- Interesting as a **theoretical problem**, beyond perturbation theory.
- To cfr. the observables to similar ones in problems for which we do not know the equilibrium phases nor the dynamic mechanisms.
- To investigate whether growth phenomena exist in problems with unknown dynamic mechanisms. **e.g. glasses**
- To unveil "generic" features of macroscopic systems out of equilibrium (classical or quantum).

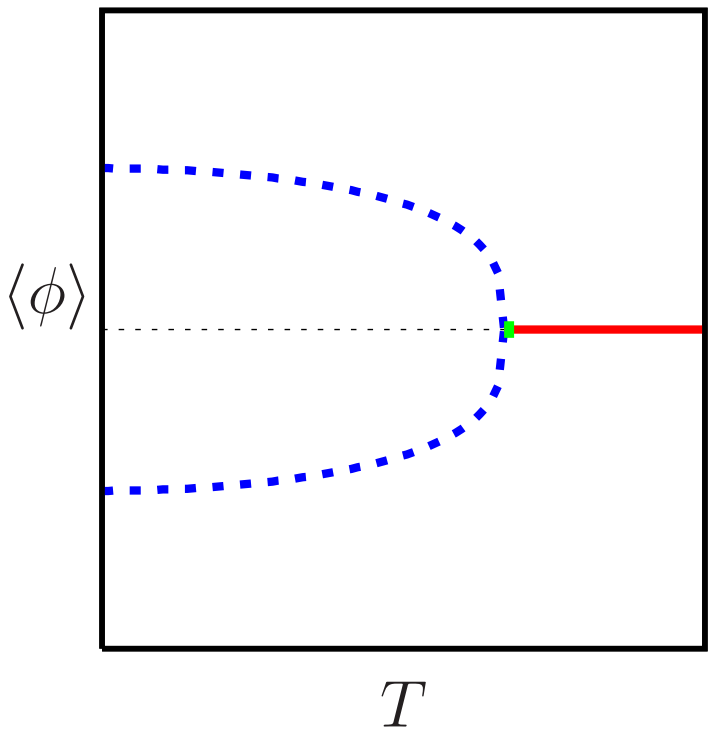
out of equilibrium statistical mechanics or thermodynamics

2nd order phase-transition

bi-valued equilibrium states related by symmetry, **e.g. Ising magnets**



Ginzburg-Landau free-energy



Scalar order parameter

Evolution

The system is in contact with a thermal bath

Thermal agitation

Non-conserved order parameter $\langle\phi\rangle(t,T) \neq$ ct

e.g. single spin flips with Glauber or Monte Carlo stochastic rules.

Development of magnetization in a ferromagnet.

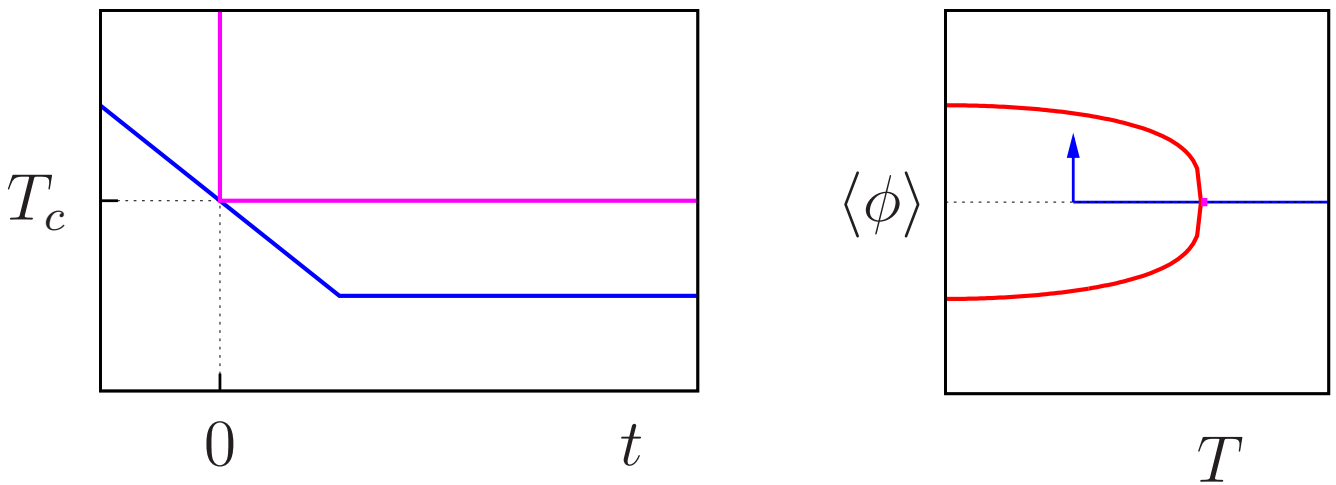
Conserved order parameter $\langle\phi\rangle(t,T) = \langle\phi\rangle(0,T) =$ ct

e.g. pair of antiparallel spin flips with stochastic rules.

Phase separation in binary fluids.

Evolution

A quench or an annealing across a phase transition

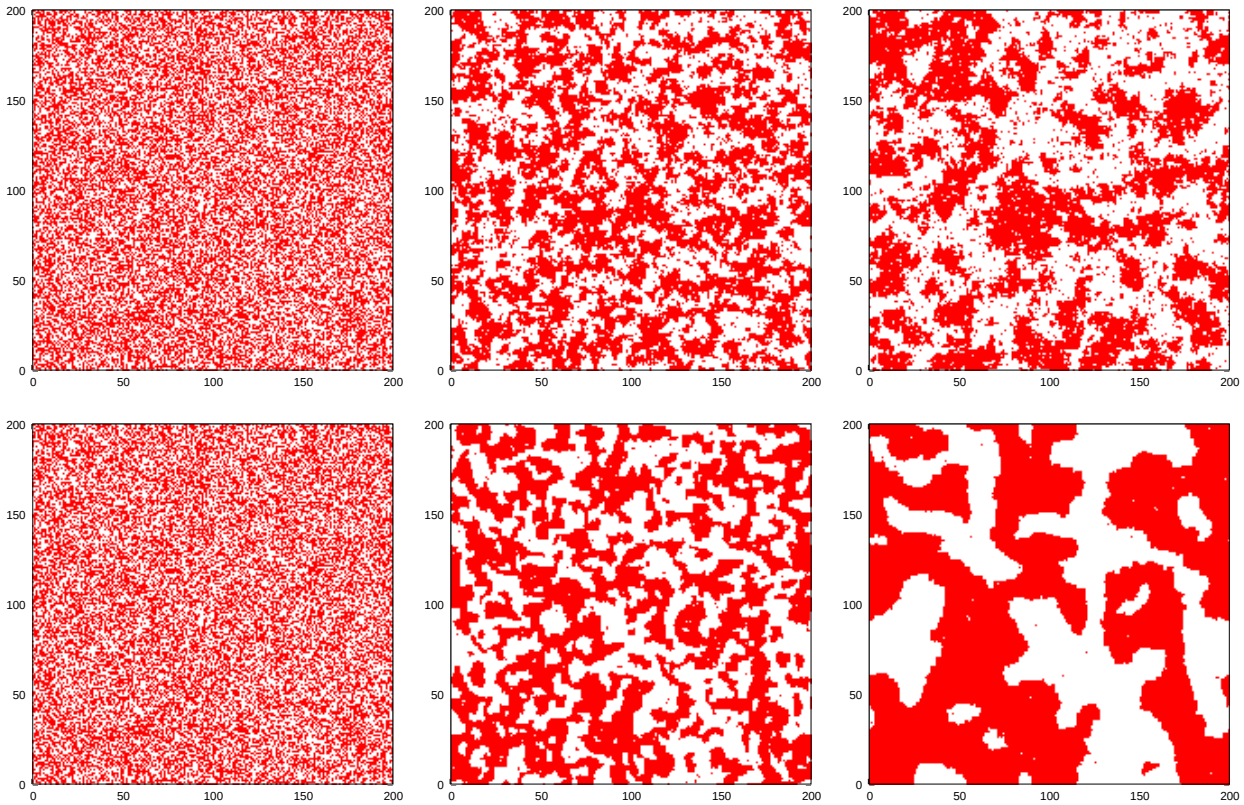


Non-conserved order parameter $\langle\phi\rangle(t,T) \neq$ ct

Development of magnetization in a ferromagnet after a quench.

The problem

e.g. up & down spins in a $2d$ Ising model (IM)



Question : starting from equilibrium at $T_0 \to \infty$ or $T_0 = T_c$ how is equilibrium at $T_f = T_c$ or $T_f < T_c$ attained ?

Growth kinetics

- At $T_f = T_c$ the system needs to grow structures of all sizes.

  Critical coarsening.

  At $T_f < T_c$: the system tries to order locally in one of the two competing equilibrium states at the new conditions.

  Sub-critical coarsening.

  In both cases the **linear size of the equilibrated patches increases in time**.

In both cases one extracts a **growing linear size of equilibrated patches**

$\boxed{\mathcal{R}(t,g)}$ from $\boxed{C(r,t) = \frac{1}{N}\sum_{i,j=1}^{N} \langle \delta s_i(t) \delta s_j(t) \rangle_{|\vec{r}_i - \vec{r}_j| = r}}$

- The relaxation time t_r needed to reach $\pm|\langle \phi \rangle_{eq}(T)|$ diverges with the size of the system, $t_r(T,L) \to \infty$ when $L \to \infty$ for $T \leq T_c$.

Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $\mathcal{R}(T,t)$*, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by* $\mathcal{R}(T,t)$*, e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t) \,\rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim m_{eq}^2(T)\; f\left(\frac{r}{\mathcal{R}(T,t)}\right),$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w) \,\rangle \sim m_{eq}^2(T)\; f_c\left(\frac{\mathcal{R}(T,t)}{\mathcal{R}(T,t_w)}\right),$$

etc. when $r \gg \xi(T)$, $t, t_w \gg t_0$ and $C < m_{eq}^2(T)$.

Suggested by experiments and numerical simulations. Proved for

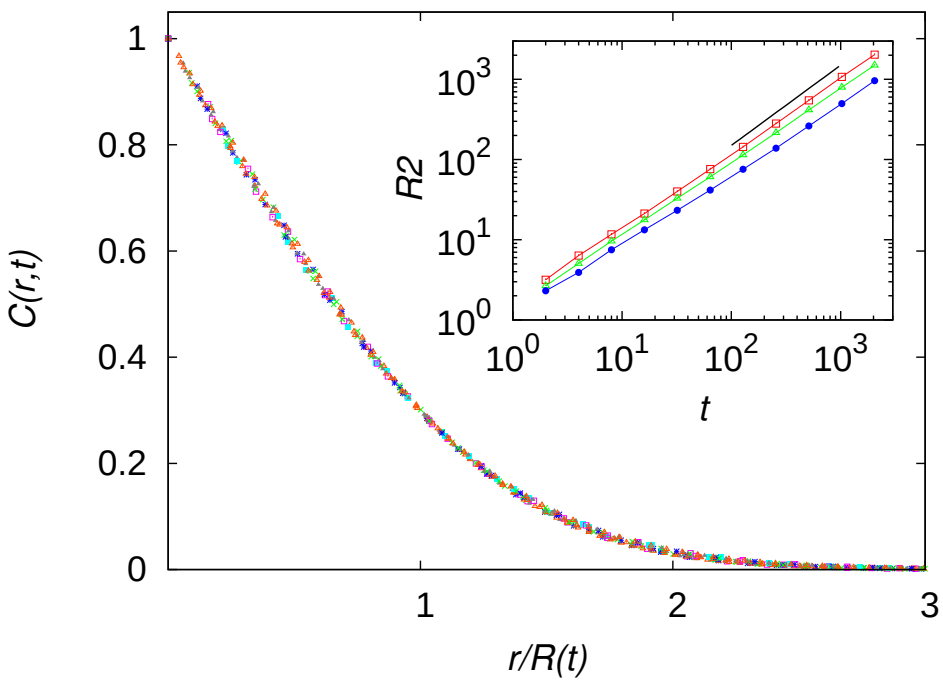
- Ising chain with Glauber dynamics.
- Langevin dynamics of the $O(N)$ model with $N \to \infty$, and the spherical ferromagnet. **Review Bray, 1994.**
- Distribution of hull-enclosed areas in $2d$ curvature driven coarsening.

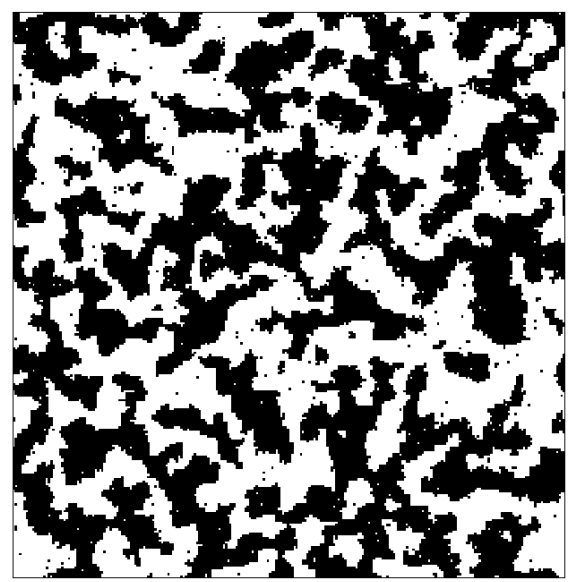
Space-time correlation

Magnetic model

Scaling regime $a \ll r \ll L, \qquad \frac{r}{\mathcal{R}(t,T)}, \qquad \mathcal{R}(t,T) \simeq \lambda(T) t^{1/z_d}$

$$C(r,t) = N^{-1} \sum_{ij/r_{ij}=r} s_i(t) s_j(t) \simeq m_{eq}^2(T)\, f_c\left(\frac{r}{\mathcal{R}(t,T)}\right)$$





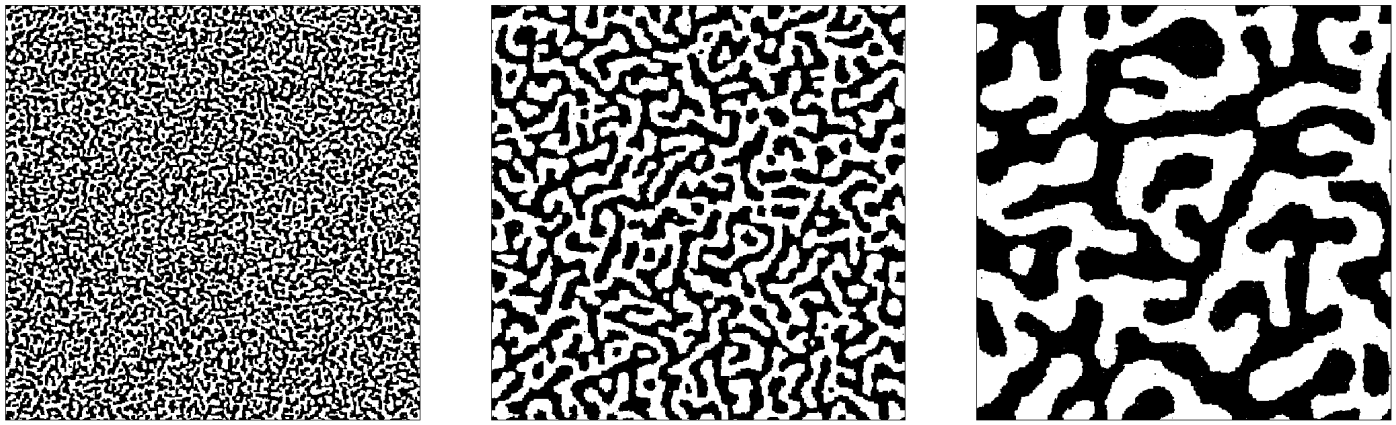
Phase separation

Spinodal decomposition in binary mixtures

A spieces $\equiv$ spin up ; B species $\equiv$ spin down

$2d$ Ising model with Kawasaki dynamics at T

locally conserved order parameter

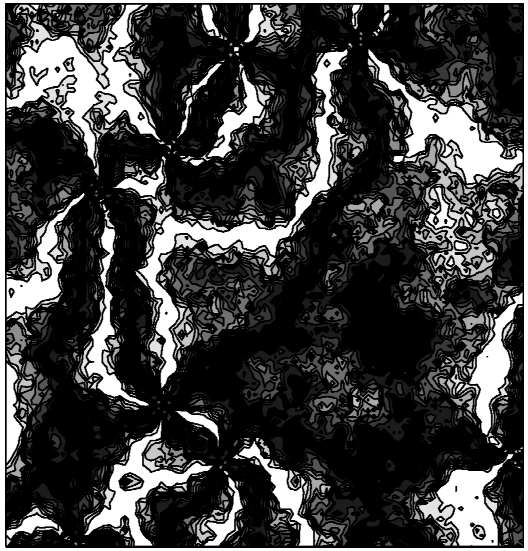


50 : 50 composition ; Rounder boundaries

Dynamics in the $2d$ XY model

Schrielen pattern : gray scale according to $\sin^2 2\theta_i(t)$

Defects are vortices (planar spins turn around these points)



After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t,T) \simeq \lambda(T)\{t/\ln[t/t_0(T)]\}^{1/2}$$

Yurke *et al* **93, Bray & Rutenberg 94.**

Growing lengths

Universality classes

$$\mathcal{R}(t,T) \simeq \begin{cases} \lambda(T)\, t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\ \lambda(T)\, t^{1/3} & \text{scalar COP} \quad z_d = 3 \\ \lambda(T) \left(\dfrac{t}{\ln t/t_0} \right)^{1/2} & 2\text{-comp. vector NCOP in} \quad d = 2 \\ \text{etc.} & \end{cases}$$

Defined by the time-dependent. Temperature and other parameters appear in the prefactor.

Super-universality ?

Are scaling functions independent of temperature and other parameters ?

Review Bray 94

Weak disorder

e.g., random ferromagnets

At short time scales the dynamics is relatively fast and independent of the quenched disorder ; domain walls accomodate in places where the disorder is the weakest, thus

$$\mathcal{R}(t,T) \simeq \lambda(T) t^{1/z_d}$$

At longer time scales domain-wall pinning by disorder becomes important.

Assume that a length-dependent barrier $B(\mathcal{R}) \simeq \Upsilon \mathcal{R}^{\psi}$

The Arrhenius time needed to go over such a barrier is $t \simeq t_0 \, e^{\frac{B(\mathcal{R})}{k_B T}}$

This implies

$$\boxed{\mathcal{R}(t,T) \simeq \left(\frac{k_B T}{\Upsilon} \ln t/t_0 \right)^{1/\psi}}$$

Weak disorder

Still two ferromagnetic states related by symmetry

$$\mathcal{R}(t,T) \simeq \begin{cases} [\lambda(T)t]^{1/z_d} & \mathcal{R} \ll L_c(T) \quad \text{curvature-driven} \\ L_c(T)(\ln t/t_0)^{1/\psi} & \mathcal{R} \gg L_c(T) \quad \text{activated} \end{cases}$$

with $L_c(T)$ a growing function of T.

Inverting times as a function of length $t \simeq [\mathcal{R}/\lambda(T)]^{z_d}\, e^{\mathcal{R}/L_c(T)}$

At short times this equation can be approximated by an effective power law with a T-dependent exponent :

$$t \simeq \mathcal{R}^{\overline{z}_d(T)} \qquad \overline{z}_d(T) \simeq z_d\,[1 + \text{ct}/L_c(T)]$$

Bustingorry et al. 09.

End of 3rd lecture

Plan

- **First lesson**.

  Introduction. Overview of disordered systems. Short presentation of methods.

- **Second lesson**. Static methods.

- **Third lesson**.

  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson**.

  Quantum dynamics. Formalism. Mean-field models.

Real-time dynamics

Two-time dependence

$t = 0$ initial time t_w waiting-time t measuring time.

Correlation

$$C(t, t_w) = \langle [\hat{O}(t), \hat{O}(t_w)]_+ \rangle$$

Symmetrized correlator

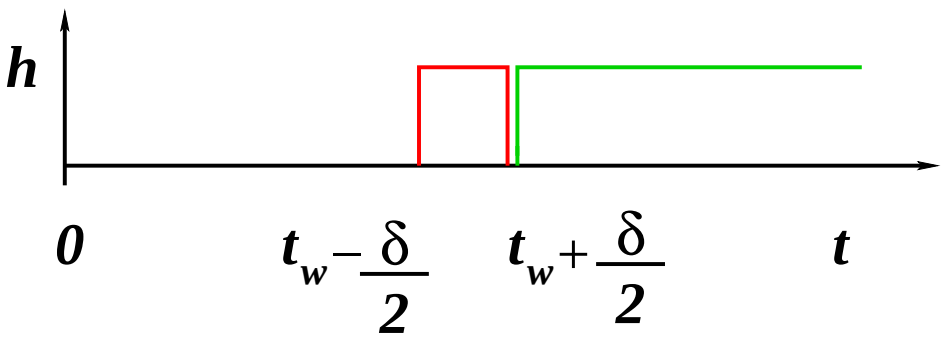
Linear response

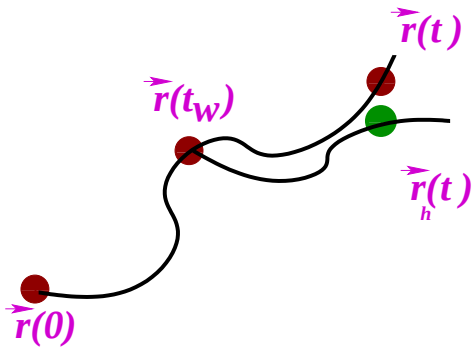
$$R(t, t_w) = \left. \frac{\delta \langle \hat{O}(t) \rangle}{\delta h(t_w)} \right|_{h=0} = \langle [\hat{O}(t), \hat{O}(t_w)]_- \rangle$$

Antisymmetrized correlator

Linear response

To a kick and to a step





The perturbation couples linearly to the observable $H \quad \longrightarrow \quad H - hB(\{\vec{r}_i\})$

The linear instantaneous response of another observable $A(\{\vec{r}_i\})$ is

$$R_{AB}(t, t_w) \equiv \left\langle \left. \frac{\delta A(\{\vec{r}_i\})(t)}{\delta h(t_w)} \right|_{h=0} \right\rangle$$

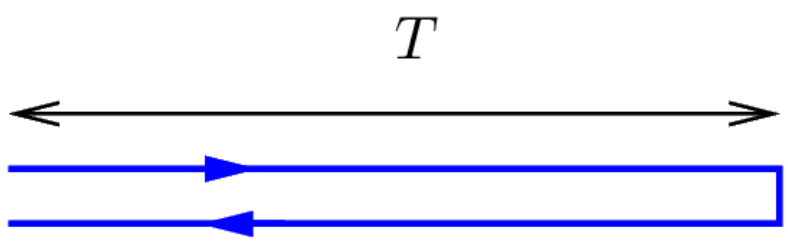
The linear integrated response or dc susceptibility is

$$\chi_{\mathbf{AB}}(\mathbf{t}, \mathbf{t_w}) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t')$$

Formalism

- **Path-integral Schwinger-Keldysh** formalism.

$$\mathcal{T} > t$$



Closed-time path to allow for $\langle \hat{O}(t_w)\hat{O}(t)\rangle$ with $t > t_w$.

- Connect system to reservoirs at time $t = 0$, factorize density matrix :

$$\boxed{\hat{\rho}(0) = \hat{\rho}_{syst}(0) \otimes \hat{\rho}_{env}(0)}$$

and take the reservoir in equilibrium at its own β (and μ).

- Integrate out the bath – assumed to be in equilibrium.

- Obtain an effective action $\boxed{S = S_{syst} + S_{bath-syst}}$

Formalism

Some important technical remarks

- Integration over bath variables $\Rightarrow$

  Two **real-time long-range interactions** (**recall static cases**).

- The vanishing quantum fluctuations limit ($\hbar \longrightarrow 0$) of the Schwinger-Keldysh generating functional is :

  if no environment, **Newton dynamics**.

  if environment on, **Langevin dynamics with coloured noise**.

How to prove it : linear combinations of the forward and backward variables $x_+(t)$ and $x_-(t)$ combine into $x(t)$ and $i\hat{x}(t)$. For $\hbar \longrightarrow 0$ the quantum bath kernels become the ones of a colored classical Langevin process.

Formalism

The rôle played by the initial condition & disorder

- Typical initial conditions : $\hat{\rho}_{syst}(0) = I$ 'random' (quench from PM)

  no need of replica trick to average over disorder !

  In the classical limit $\mathcal{Z}[\eta = 0] = \int \mathcal{D}\xi P[\xi] = 1$, independently of disorder

  de Dominicis 78.

  In the quantum model $\hat{\rho}(0)$ is independent of disorder.

  LFC & Lozano 98.

  We simply average $\mathcal{Z}$ or $\hat{\rho}$.

- Derive Schwinger-Dyson equations for correlations and responses with saddle-point in the **large N limit or a variational approximation**.

- Solve these equations.

Formalism

An example : rotors with pair intereactions

$$
\begin{aligned}
S_{syst} &= \sum_{a=\pm} a \int \mathrm{d}t \left[\frac{\hbar^2}{2\Gamma} \sum_i (\dot{\mathbf{n}}_{ia}(t))^2 + \sum_{i<j} J_{ij} \mathbf{n}_{ia}(t) \mathbf{n}_{ja}(t) \right] . \\
S_{bath-syst} &= -\frac{1}{2} \sum_{ab=\pm} \int \mathrm{d}t \mathrm{d}t' \, \Sigma^B_{ab}(t,t') \sum_i \mathbf{n}_{ia}(t) \mathbf{n}_{ib}(t') \, , \\
S_\lambda &= \sum_{a=\pm} \frac{a}{2} \int \mathrm{d}t \sum_i \lambda_{ia}(t) (\mathbf{n}^2_{ia}(t) - \mathcal{M})
\end{aligned}
$$

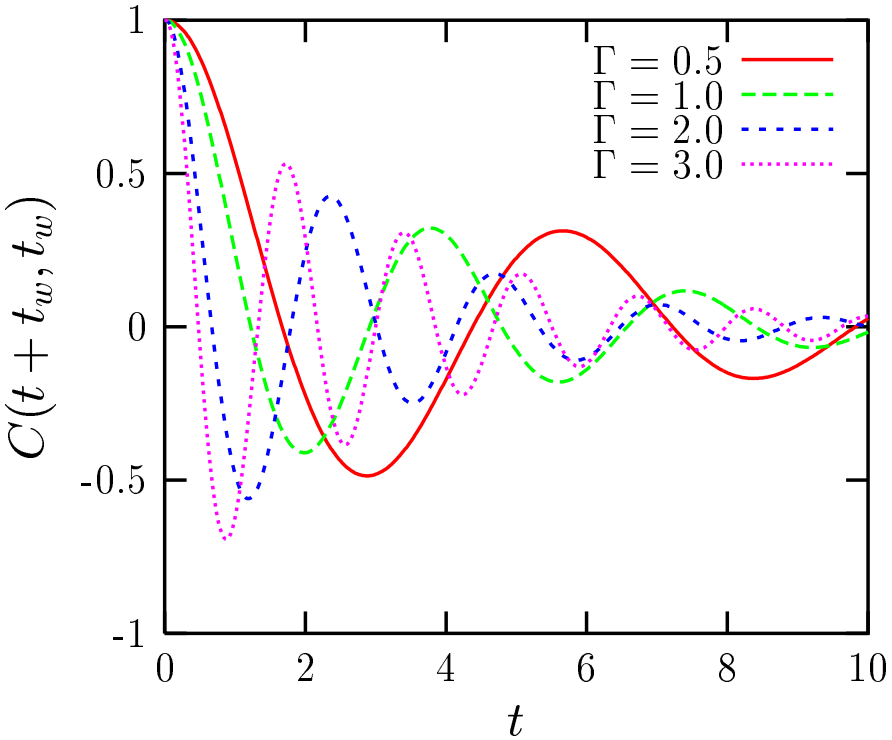
with the bath induced kernels

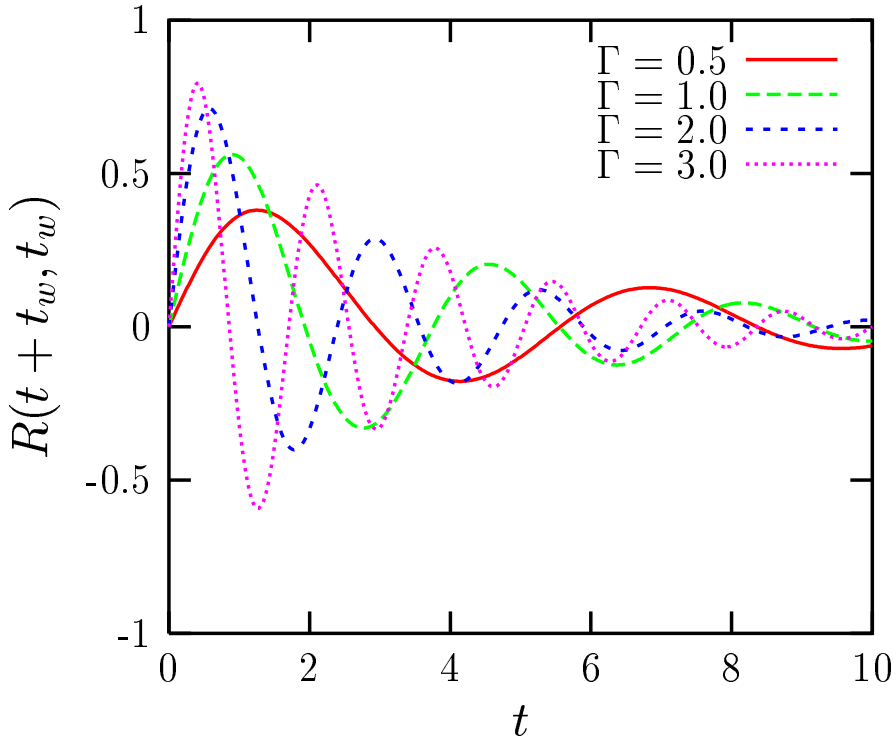
$$\Sigma^B_{ab}(t,t')$$

that take different forms for different baths, *e.g.* oscillators, leads, etc.

Real-time dynamics

Paramagnetic phase





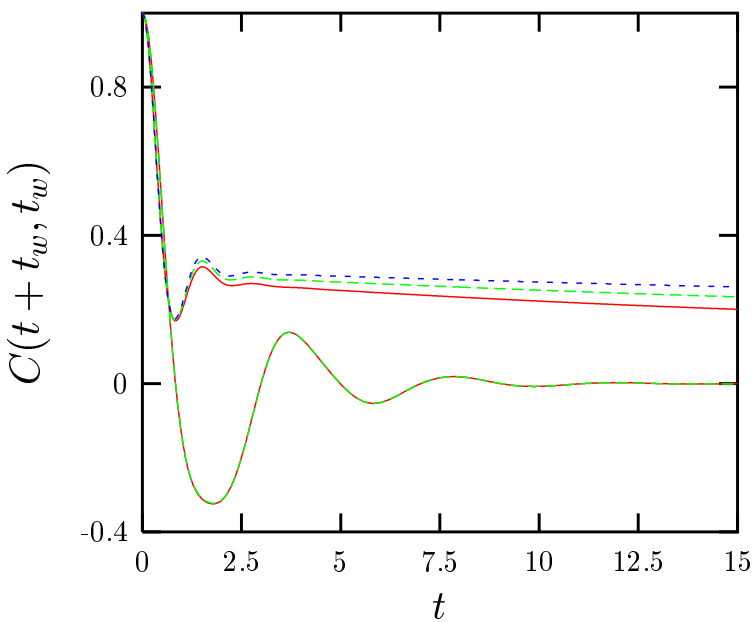
Dependence on the quantum parameter Γ ($T = 0$, α fixed.)

LFC & G. Lozano 98-99.

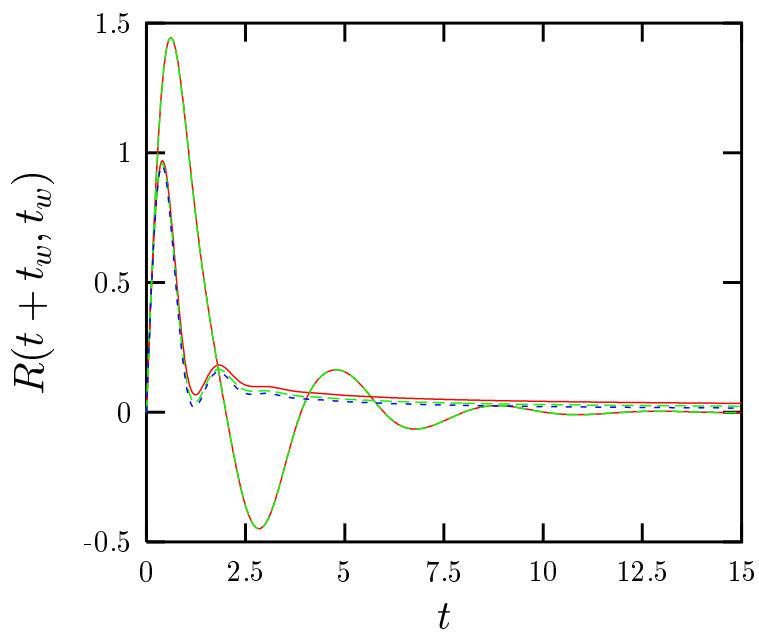
Real-time dynamics

Dependence on the coupling to the bath

Symmetric correlation



Linear response



Comparison between $\alpha = 0.2$ (PM) and $\alpha = 1$ (SG)

LFC, Grempel, Lozano, Lozza & da Silva Santos 02.

The effect of the bath

Summary

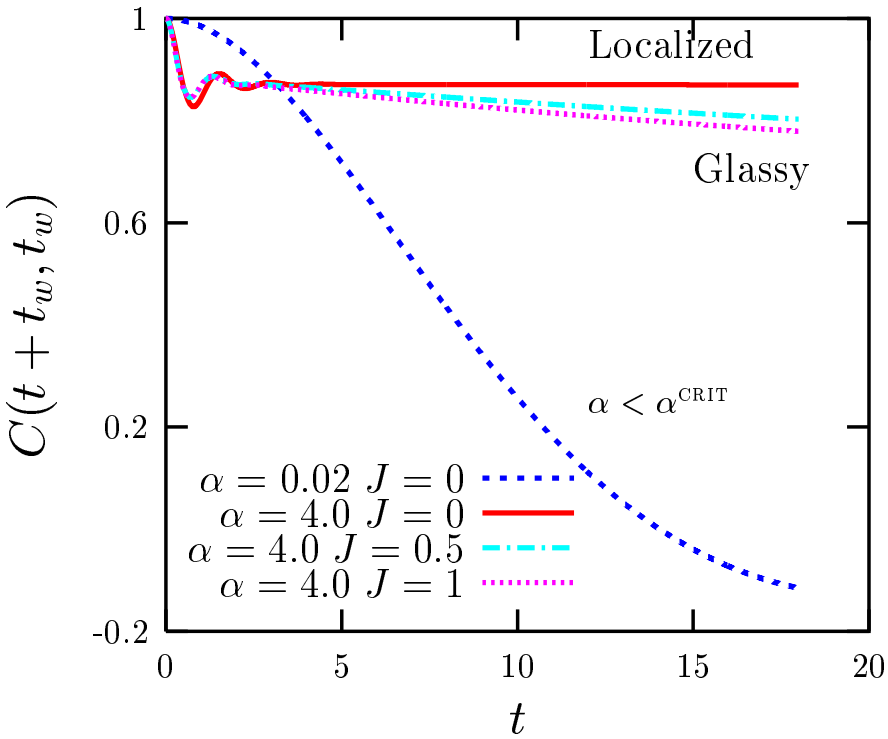
- Classical statics is not altered by the bath.
- Quantum statics **is** altered by the bath.
- Classical dynamics becomes a coloured noise Langevin equation and, although the evolution can depend on the bath, the target equilibrium does not.
- Quantum dynamics **is** very importantly altered by the bath, e.g. localization and modification of the phase transition line.

Quantum aging

Other examples : SK (Chamon, Kennett & Yu), SU(N) in large N (Biroli & Parcollet), rotors (Rokni & Chandra ; Aron et al), Wigner crystals (LFC, Giamarchi & Le Doussal), etc.

Real-time dynamics

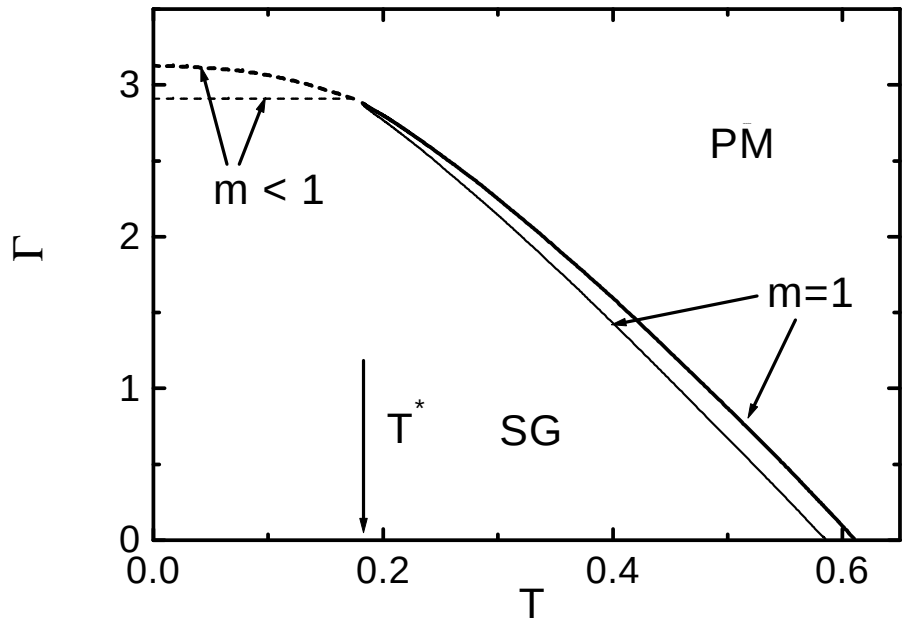
Interactions against localization



LFC, Grempel, Lozano, Lozza & da Silva Santos 02

Dynamic vs static phase diagram

Quantum p-spin model



Dynamic evidence of high-lying metastable states !

The relaxational dynamics gets trapped in a region of phase space named **threshold**.

Rue de Fossés St. Jacques et rue St. Jacques

Paris 5ème Arrondissement.

LFC

Fluctuation-dissipation

Equilibrium spontaneous (C) and induced (R) fluctuations

If $$p(\{\vec{r}\}), t_w) = p_{eq}(\{\vec{r}\})$$

- The dynamics is stationary, $C \to C(t - t_w)$ and $R \to R(t - t_w)$.
- The fluctuation-dissipation th. $R(t - t_w) = -\frac{1}{k_B T} \frac{\partial C(t-t_w)}{\partial t} \, \theta(t - t_w)$

  holds and implies

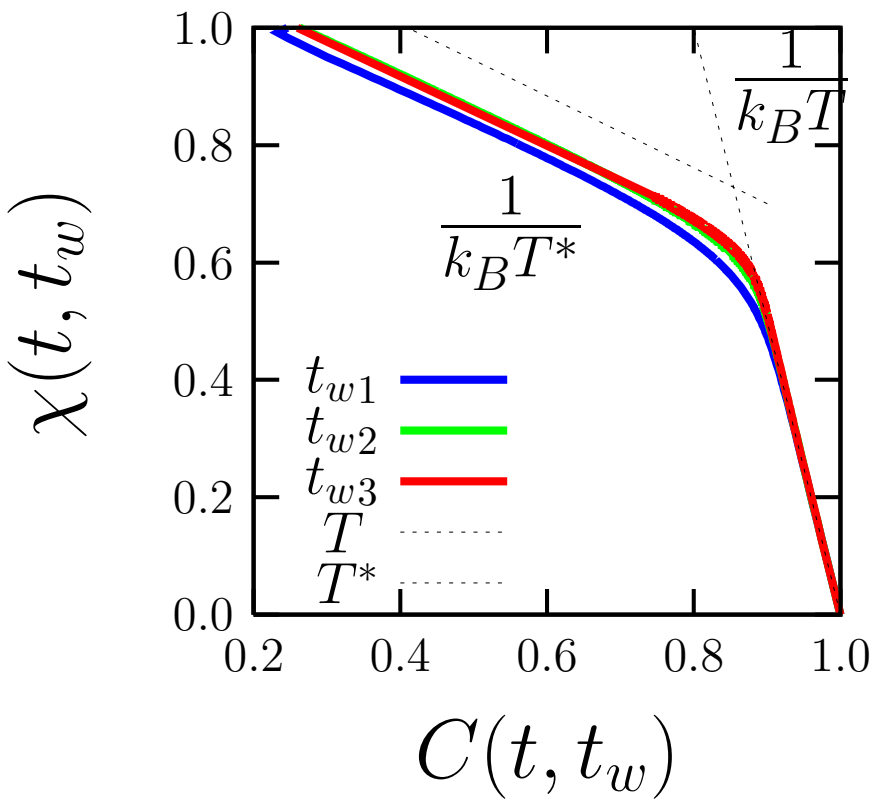
$$\boxed{\chi(t - t_w) \equiv \int_{t_w}^{t} dt' \, R(t, t') = \frac{1}{k_B T} [C(0) - C(t - t_w)] \, .}$$

In glassy systems below T_g : breakdown of stationarity & FDT.

$\chi(t, \mathbf{t_w}) \equiv \int_{\mathbf{t_w}}^{t} dt' \, R(t, t')$ and $C(t, \mathbf{t_w})$ not obviously related.

Fluctuation-dissipation

Solvable cases : p spin-models



Parametric construction

t_w fixed

$t : t_w \to \infty$ or

$dt : 0 \to \infty$.

LFC & Kurchan 93.

Fluctuation-dissipation

Proposal

For non-equilibrium systems, relaxing slowly towards an **asymptotic** limit (*cfr.* threshold in p spin models) such that one-time quantities [*e.g.* the energy-density $\mathcal{E}(t)$] approach a finite value [*e.g.* $\mathcal{E}_\infty$]

$$\boxed{\lim_{\substack{t_w \to \infty \\ C(t,t_w)=C}} \chi(t,t_w) = f_\chi(C)}$$

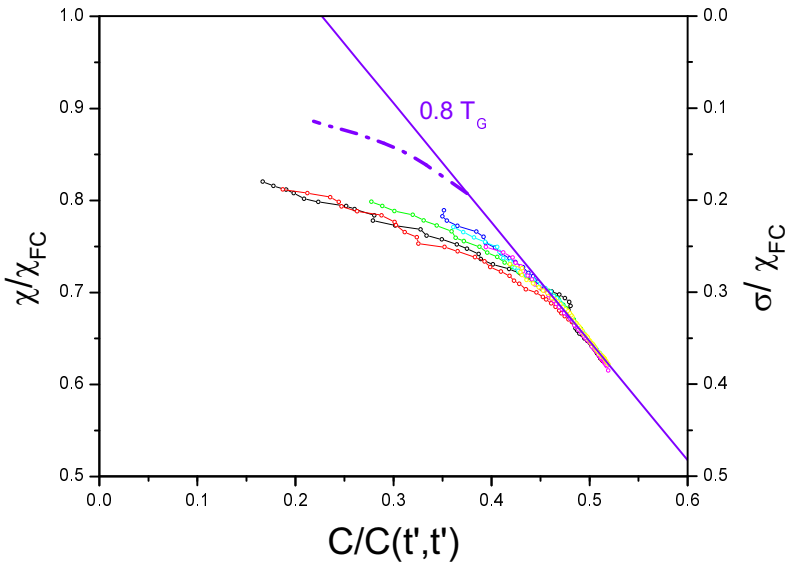
LFC & Kurchan 94.

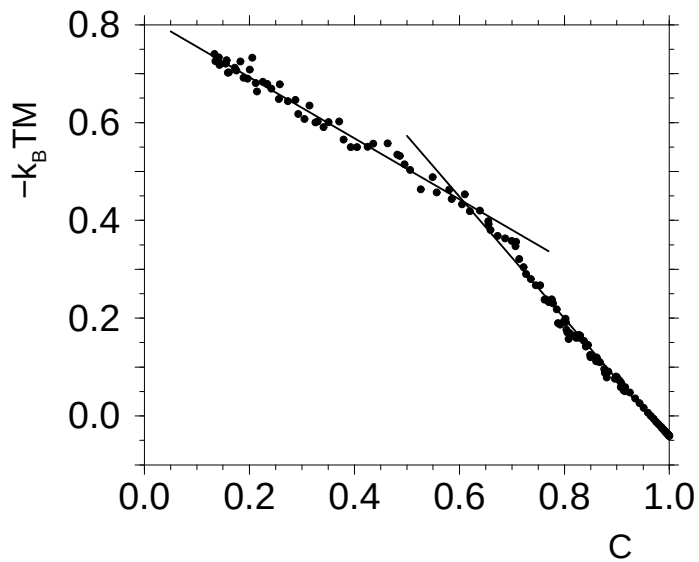
For weakly forced non-equilibrium systems in the limit of **small work**

$$\lim_{\substack{\epsilon \to 0 \\ C(t,t_w)=C}} \chi(t,t_w) = f_\chi(C)$$

FDT in relaxing glasses

Experiments and simulations





Spin-glass (thiospinel)

Lennard-Jones binary mixture

Hérisson & Ocio 02.

J-L Barrat & Kob 99.

also in glycerol (**Grigera & Israeloff 99**) and after this paper many others in colloidal suspensions & polymer glasses (exps.), silica, vortex glasses, dipolar glasses, *etc.*

Phenomenology

Times scales

In all these systems the dynamics occur

- In quasi-equilibrium [$\chi = \frac{1}{k_B T}(1 - C)$] when

$$t \gtrsim t_w \qquad \text{or} \qquad q_{ea} \leq C \leq 1.$$

- Clearly out of equilibrium when

$$t > t_w \qquad \text{or} \qquad C \leq q_{ea}.$$

In structural glassy systems one finds

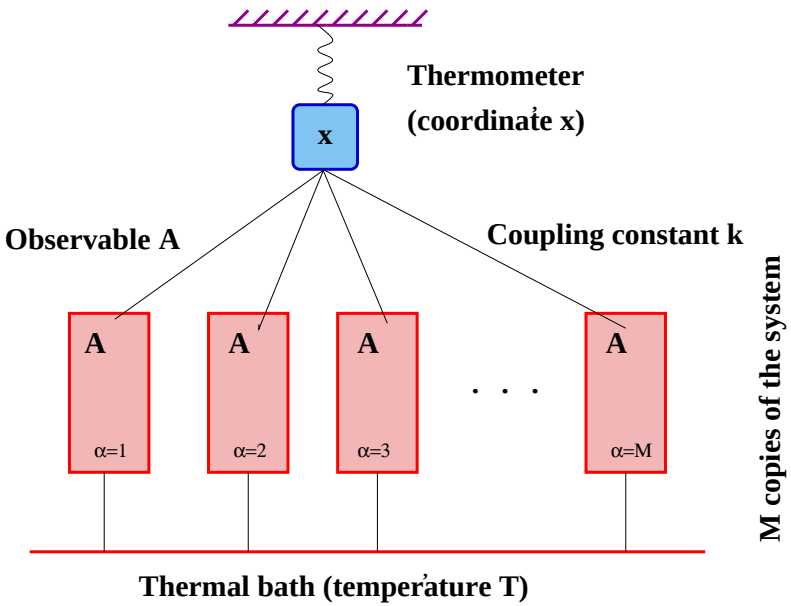
$$\chi = \frac{1}{k_B T^*}\left(q_{ea} - C\right) + \frac{1}{k_B T}\left(1 - q_{ea}\right)$$

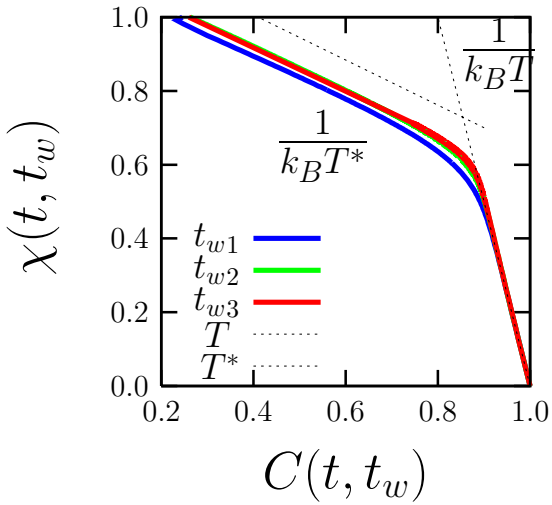
Interpretation

- In particle systems, rattling within cages *vs.* structural relaxation.
- In magnetic coarsening, thermal fluctuations within domains *vs.* domain wall motion.

FDT & effective temperatures

Can one interpret the slope as a temperature ?





(1) Measurement with a **thermometer** with

- Short internal time scale τ_0, fast dynamics is tested and T is recorded.
- Long internal time scale τ_0, slow dynamics is tested and T^* is recorded.

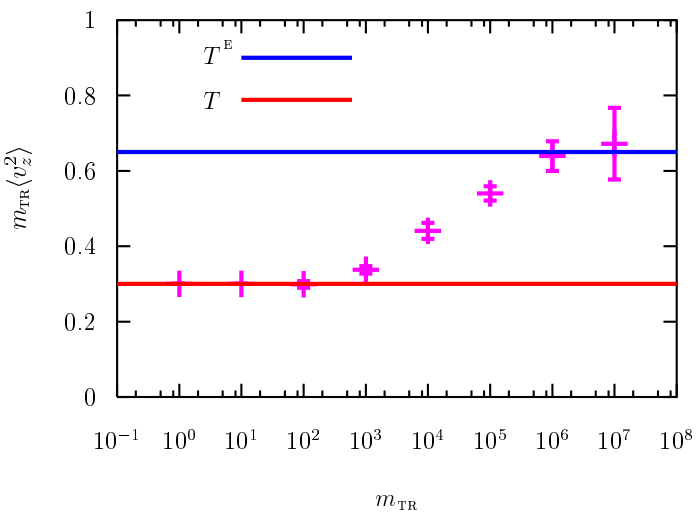
(2) **Partial equilibration**

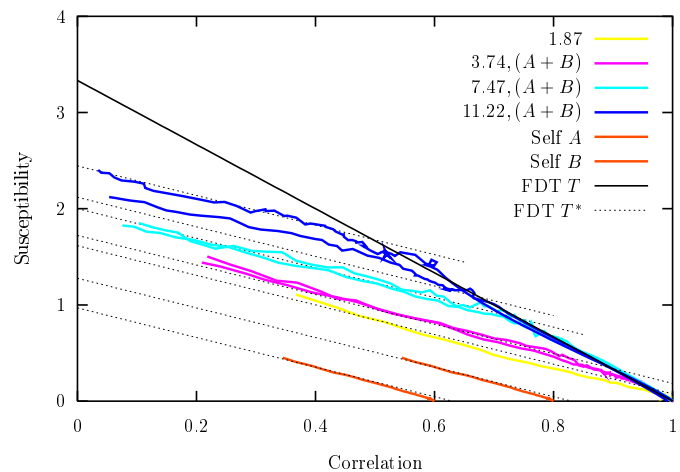
(3) **Direction of heat-flow**

LFC, Kurchan & Peliti 97.

FDT & effective temperatures

Sheared binary Lennard-Jones mixture





Left : The kinetic energy of a tracer particle (the thermometer) as a function of its mass ($\tau_0 \propto \sqrt{m_{tr}}$) $\frac{1}{2}m_{tr}\langle v_z^2\rangle = \frac{1}{2}k_B T_{eff}$.

Right : $\chi_k(C_k)$ plot for different wave-vectors k : partial equilibrations.

J-L Barrat & Berthier 00.

FDT & effective temperatures

Dissipative quantum glassy models

The quantum equilibrium FDT

$$R(t,t_w) \;=\; \frac{i}{\hbar}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\, e^{-i\omega(t-t_w)}\,\tanh\left(\frac{\beta\hbar\omega}{2}\right) C(\omega,t_w)$$

becomes

$$\boxed{\chi(t,t_w) \approx -\tfrac{1}{T_{\mathtt{eff}}} C(t,t_w) \qquad t \gg t_w}$$

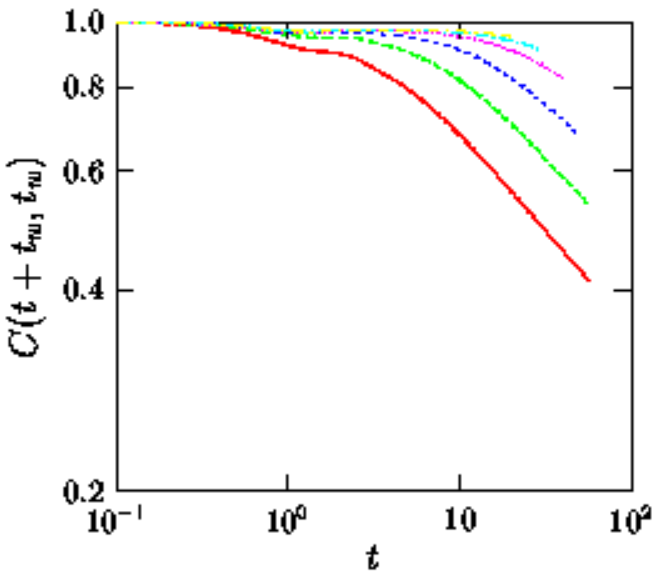
if the integral is dominated by $\omega(t-t_w) \ll 1$ and $T \to T_{\mathtt{eff}} > 0$ such that $\beta_{\mathtt{eff}}\hbar\omega \to 0$.

LFC & G. Lozano 98-99.

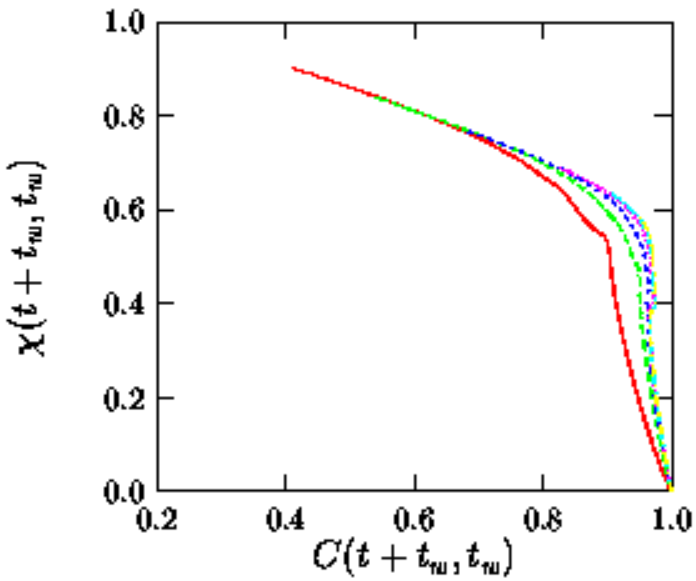
Real-time dynamics

Glassy phase

Symmetric correlation



Linear response



LFC & G. Lozano 98-99.

Summary I

- We analyzed **fully-connected classical and quantum spin models with quenched random interactions**.
- The **TAP method** yields the free-energy landscape in phase space.
- The **replica trick** yields complementary information on equilibrium states.
- The **out of equilibrium relaxation** from a random initial condition captures a slow aging decay of the correlation and linear response. Generation of an **effective temperature**.

These methods can be adapted to deal with particles in interaction moving in a continuous space.

Summary II

Suggest the existence of three classes of systems :

- **FM-like** two-equilibrium states, coarsening. Droplet picture of spin-glasses but the response decays very fast.
- **p-spin-like** random first order phase transition. Have many metastable states that block the relaxation (super-cooled liquids, structural glasses) Different dynamic and static transitions. The effective temperature is finite and takes the 'microcanonic' value where the configuration-al entropy is the relevant one.
- **SK-like**. 2nd order phase transition, exponentially large number of equilibrium states, all kinds of overlaps between them, very slow dynamics.

A many morepeculiar features that I have not discussed here !

Summary III

What is special of quantum models ?

- **First order phase transitions** in models of the p-psin/random first order kind.
- **The phase transition depends on the bath**.
- For the out of equilibrium relaxation, **interactions** go against **localization**.
- **Effective temperature 'driven' decoherence** at long time or spatial scales ; *e.g.*

  for coarsening systems the time-dependence of the growing length, $\mathcal{R}(t;T,\Gamma)$, should be the same as for the classical counterpart ;

  in all these cases the FDT becomes classical.

Challenges

The main one

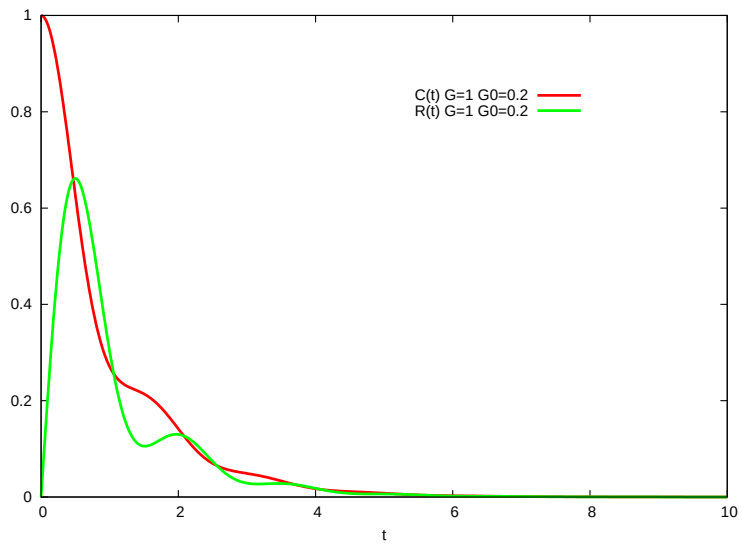
Get a convincing real-space translation of the complex phase space and real-time information.

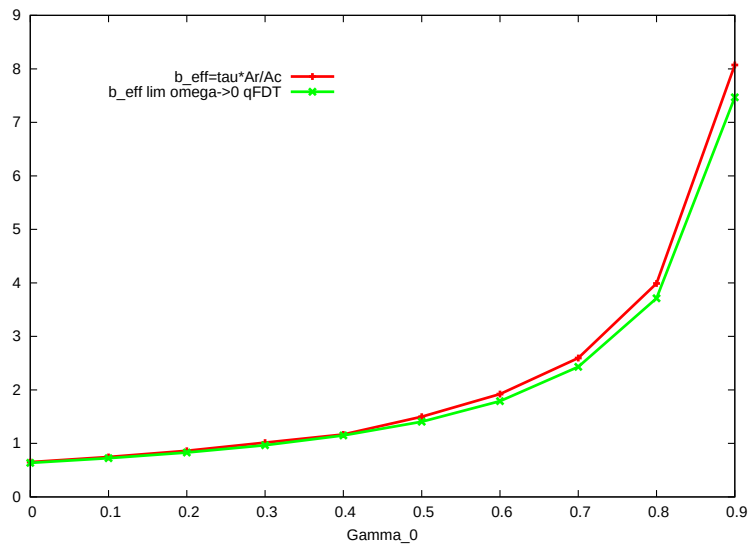
Two problems in progress

T_{eff} from FDT

A quantum quench $\Gamma_0 \to \Gamma$ of the isolated Ising chain

Here : to its critical point $\Gamma = 1$





Dissipative dynamics of the quantum Ising chain

Part of Laura Foini's PhD project.

The end