Two dimensional spin-ice and the sixteen-vertex model

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Paris, October 26th 2012

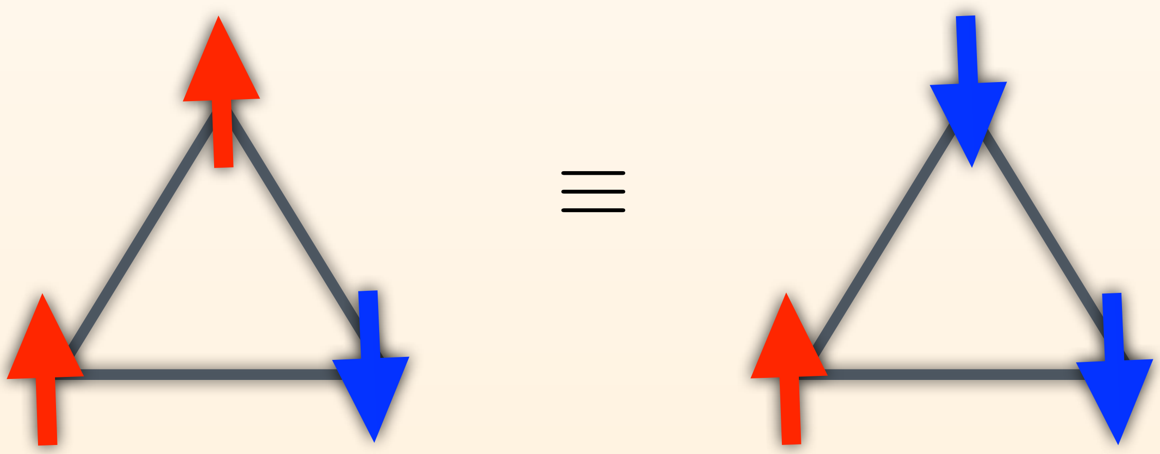
PLAN of the talk

- **Introduction:**
  - Geometrical frustration: Water Ice, Pyrochlore magnets (3D spin ice)
  - Generic model: 2D ***16-vertex model***
  - Experiments: Artificial nano-arrays (2D spin ice)
- **Equilibrium phases:**
  - Using CTMC and Bethe-Peierls approx. (cavity method). Phase diagram and critical properties of the 16V model.
  - Theory vs. experiments in `a-thermal' Artificial spin ice: Edwards' measure vs. Canonical.
- **Stochastic dynamics:**
  - Evolution after a quench into the PM, FM and AF phases.
  - Dynamical arrest , mechanisms leading the evolution and topological defects.
- **Conclusion & perspectives**

Geometrical frustration

Prototypical example: Ising AF on a triangular lattice

$$H = J \sum_{<i,j>} S_i S_j \qquad S_i = \pm 1 \ \forall i \qquad J > 0$$



Ground state:

All configurations such that $\sum_{i=1}^{3} S_i = \pm 1$

→ *Macroscopic degeneracy* $\Omega_0 \sim \exp(N)$

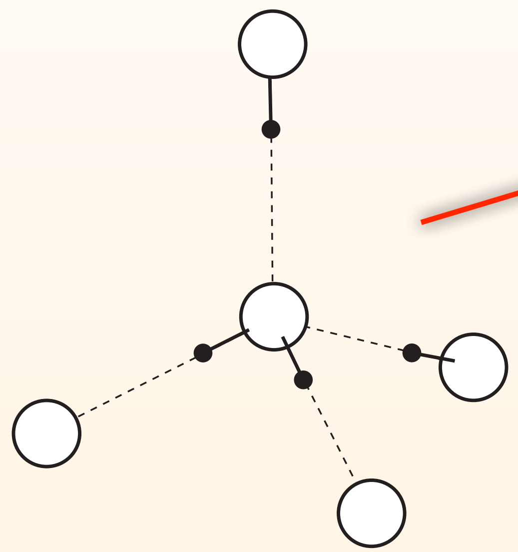
→ *Extensive zero-point entropy* $S_0 = \ln \Omega_0$

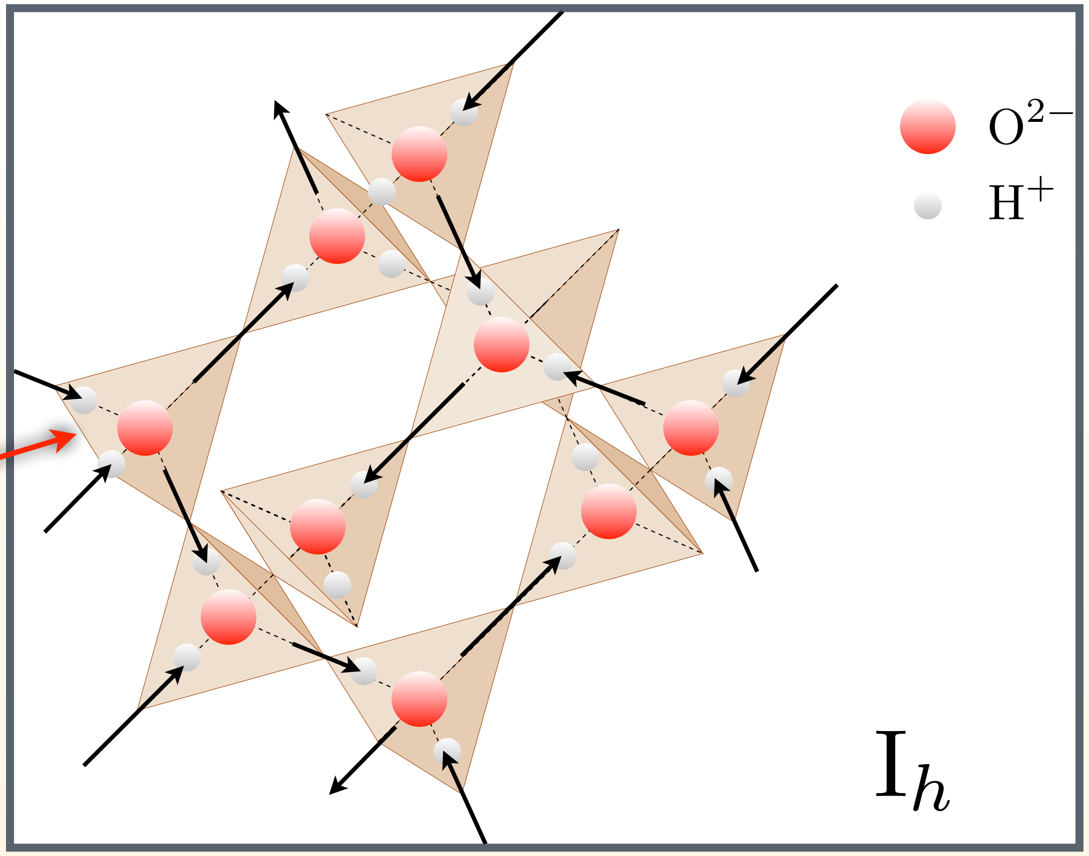
Frustrated because of the geometry of the lattice and the antiferromagnetic nature of the interactions.

Geometrical frustration

Crucial example: Water Ice

Single unit:





Experimental evidence of an extensive entropy at T=0 $\Longleftarrow$ Predicted by Pauling's model

Giauque & Stout 1936 **Pauling 1935**

each bond carries a dipolar moment $\vec{\mu}$ verifying the ***ice-rules*** $=$ at each ***vertex*** (oxygen atoms) *two in - two out*

$\Rightarrow$ **SIX** vertices

Ground state : Ice-rule vertices are favoured $\Rightarrow$ Geometrically frustrated

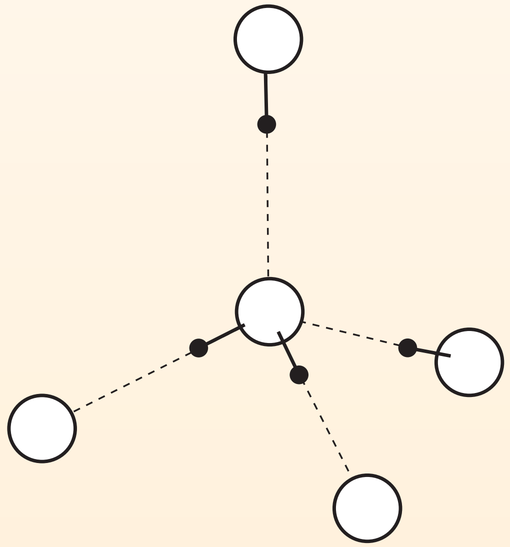
Geometrical frustration

Harris, Bramwell, McMorrow, Zeiske & Godfrey 1997

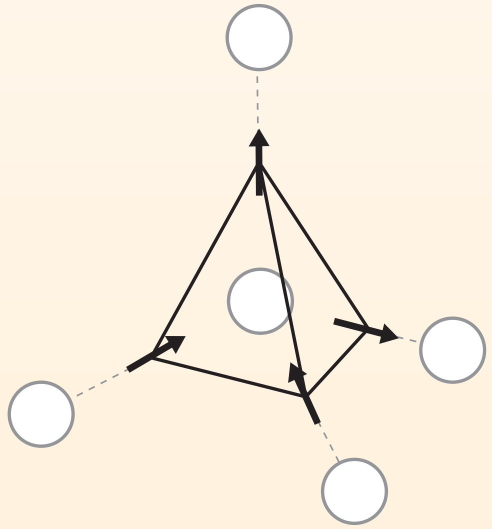
Water Ice $\Longleftrightarrow$ ***Spin Ice***

electric dipoles

magnetic moments



$\Longleftrightarrow$



$$S_w \approx 0.41R$$

$$S_{si} \approx 0.46R$$

zero point entropy
measured in $\mathrm{Dy_2Ti_2O_7}$

Ramirez, Hayashi, Cava, Siddharthan & Shastry 1999

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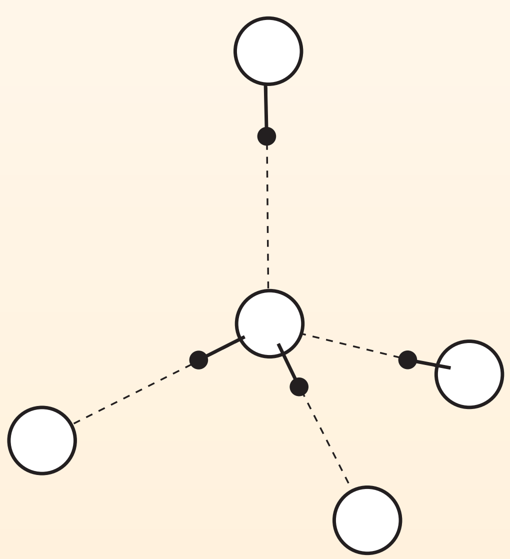

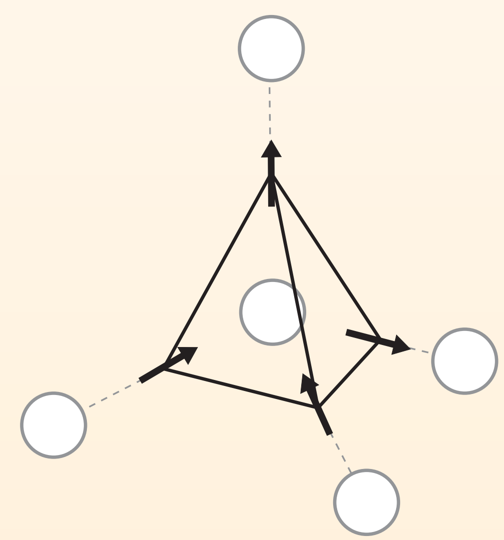

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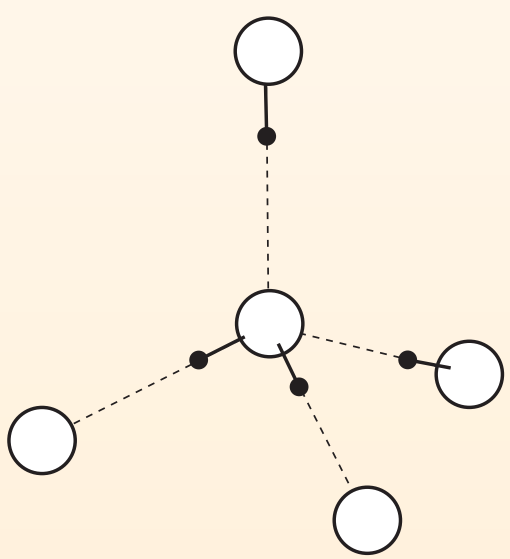

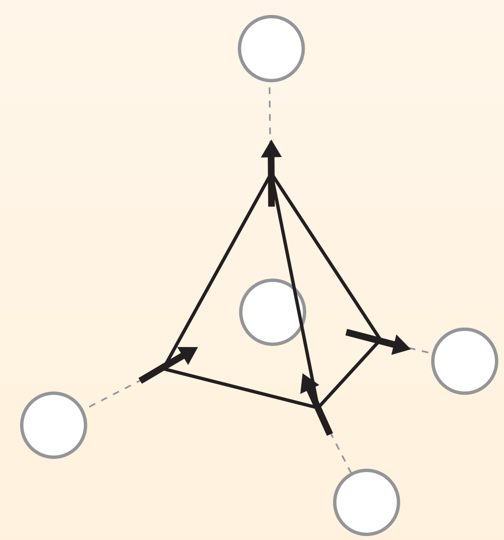

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Why??

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Geometrical frustration

in spin ice

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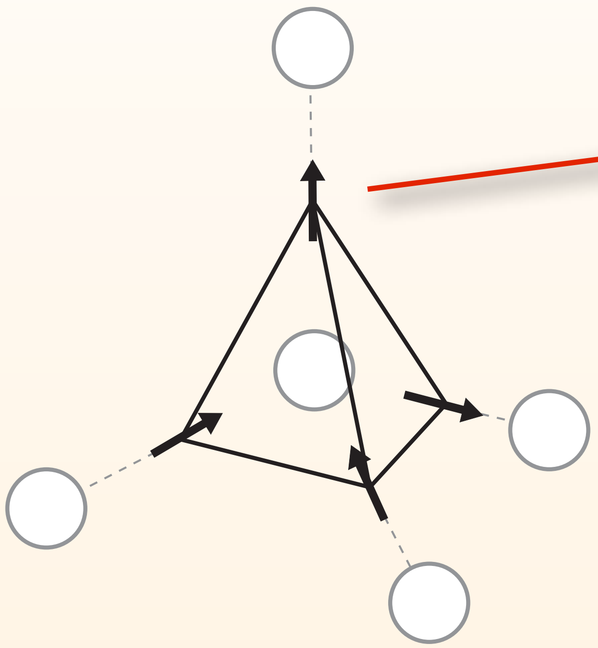


classical Ising spins

DSI: nn **FM** + *dipolar interactions*

$$H = -J \sum_{<i,j>} \vec{S}_i.\vec{S}_j + Da^3 \sum_{i<j} \frac{\vec{S}_i.\vec{S}_j}{r_{ij}^3} - 3\frac{(\vec{S}_i.\vec{r}_{ij})(\vec{S}_j.\vec{r}_{ij})}{r_{ij}^5}$$

NN contribution: $H \approx \frac{J+5D}{3} \sum_{<i,j>} \sigma_i \sigma_j$ with: $\sigma_i = \pm 1$

AF

$\Rightarrow$ GS: 2 in-2 out (ice-rules)

Geometrically frustrated *Ferromagnet*

Experiments 3D		**Exact 2D**		**Approx.**
$S_w \approx 0.41R$ $S_{si} \approx 0.46R$	$\simeq$	$S_{2D} = 3/2 \ln(4/3) \approx 0.43R$ **Lieb 1967**	$\simeq$	$S_\infty \approx 0.40R$

Thermal excitations in spin-ice

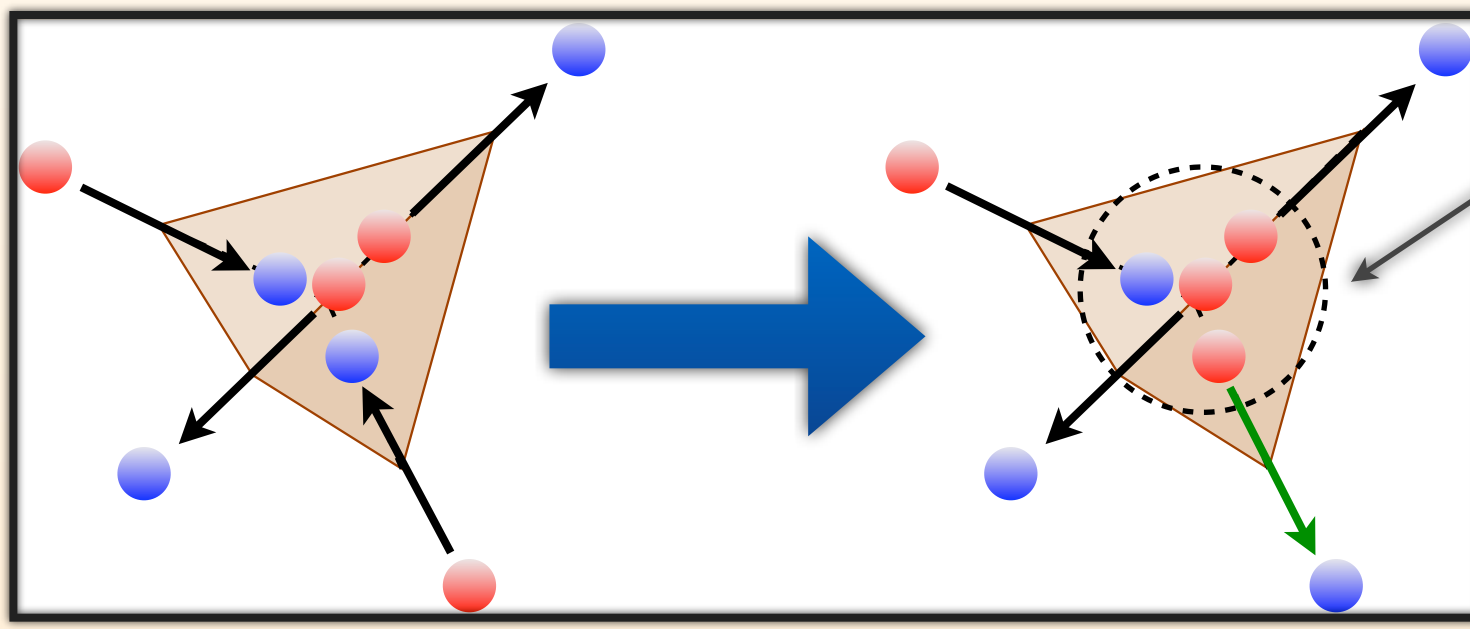
$+q$ $-q$

"Dumbbell picture"

$\Rightarrow$ Ice configurations $\equiv \{q_\alpha = 0\}$

Thermal excitations violating the ice rules =

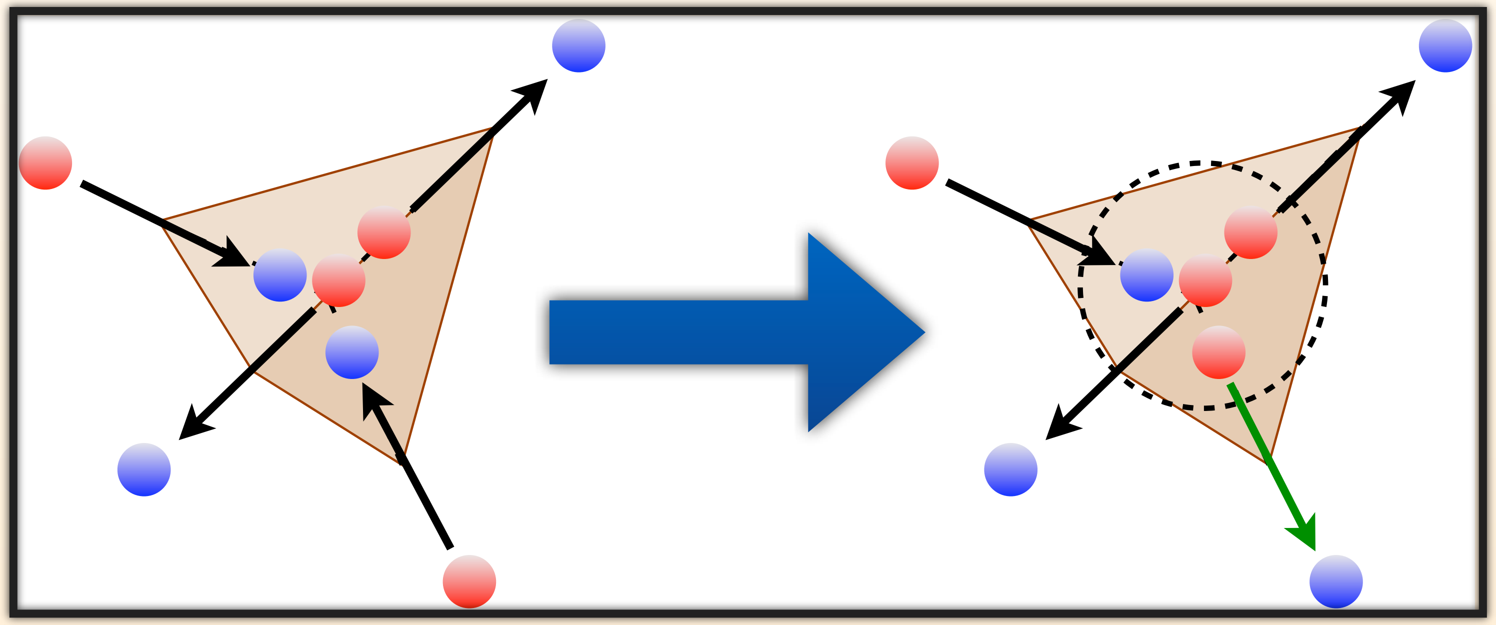
magnetic monopoles



Castelnovo, Moessner & Sondhi 2008

Thermal excitations in spin-ice

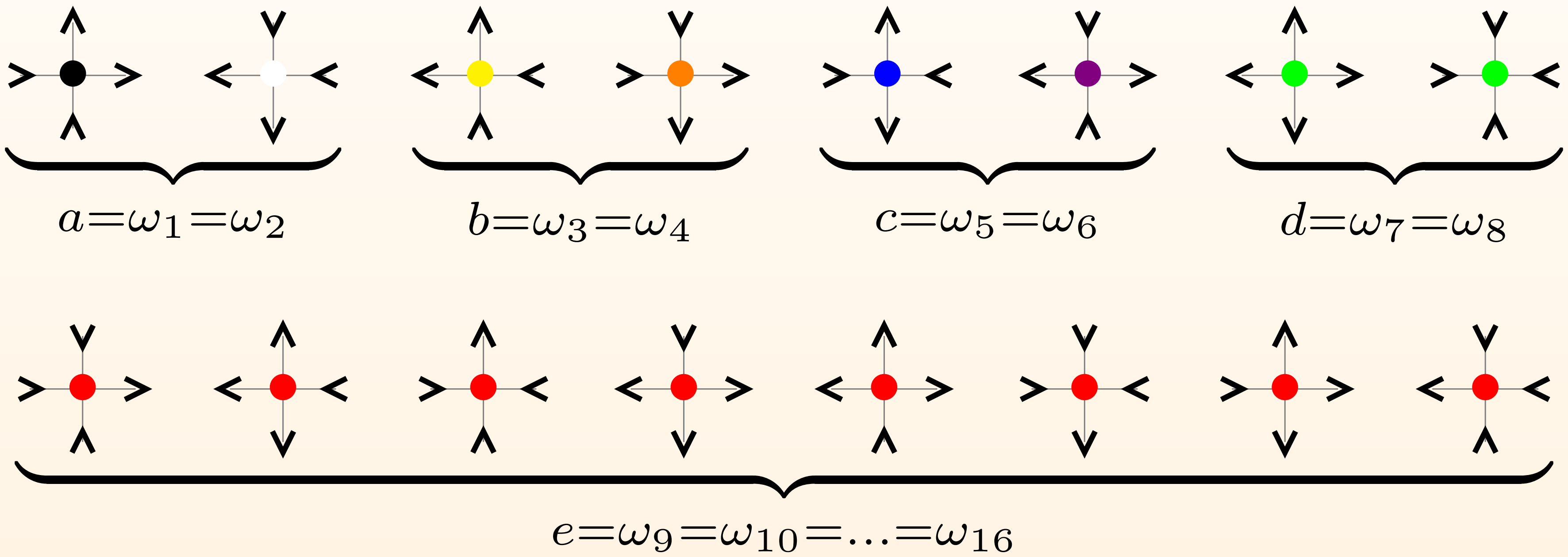
2D projection



$$\Longrightarrow \quad 2^4 = 16 \text{ - vertex model}$$

2D Spin-Ice theory: the sixteen-vertex model

Fix the Boltzmann weight of each vertex: $\omega_k = e^{-\beta \epsilon_k}$



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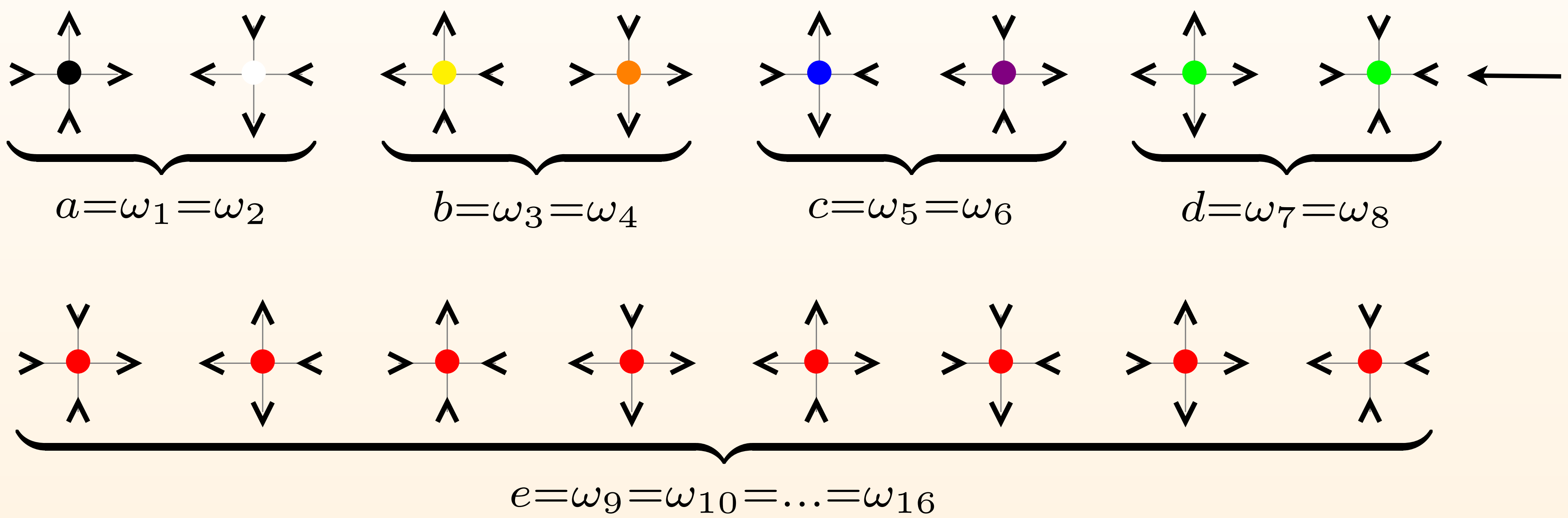

Integrability

(*six*- and *eight-vertex models*)

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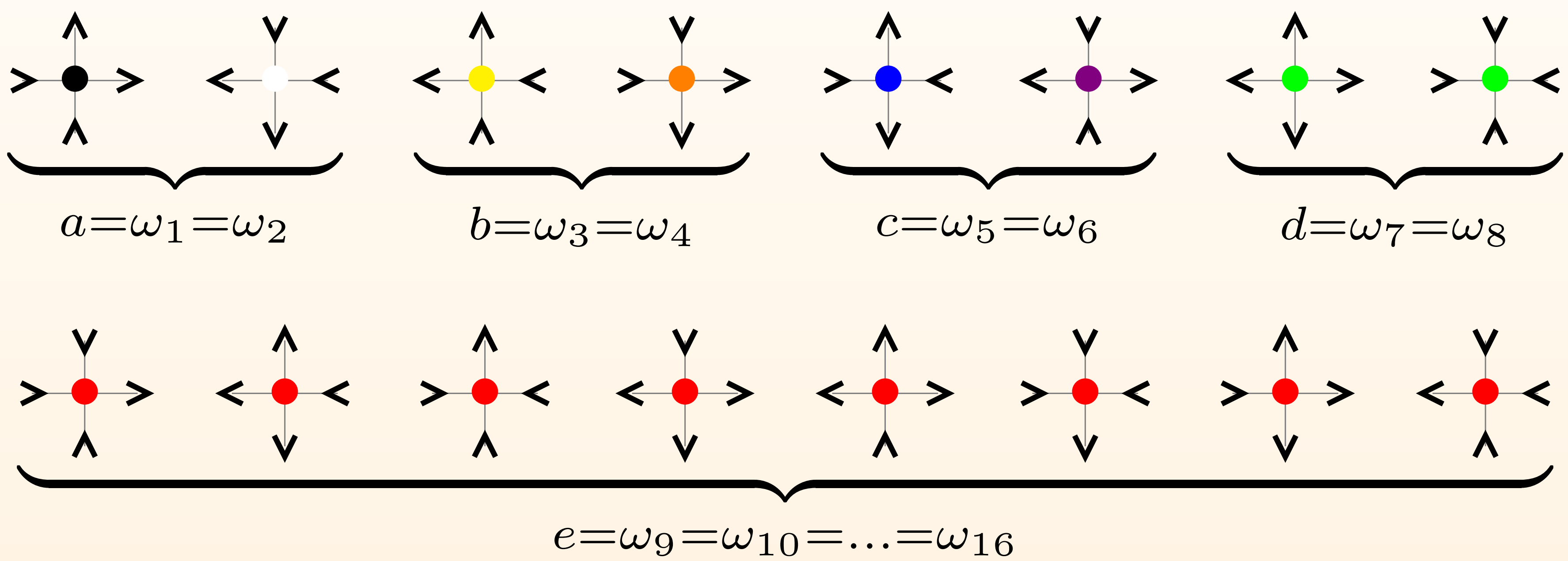

$\longleftarrow$ Integrability
(*six*- and *eight*-*vertex models*)

$\longleftarrow$ Non-Integrability
Monte Carlo simulations
& Cavity calculation

Thermal excitations:

Vertices breaking the ice-rule are allowed but un-favoured = *defects*

Dynamics $\Longrightarrow$ *Stochastic extension of a general integrable model*

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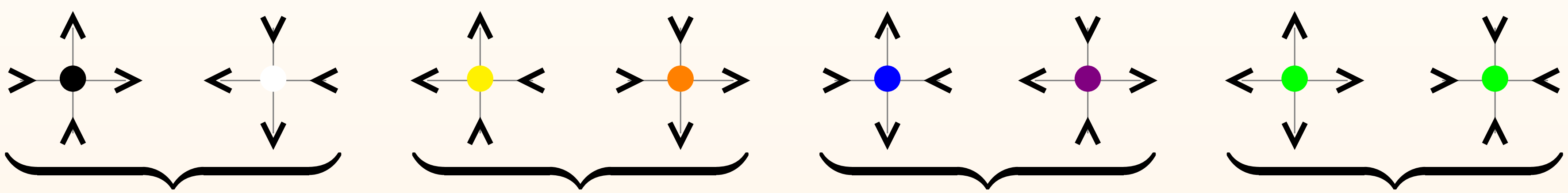
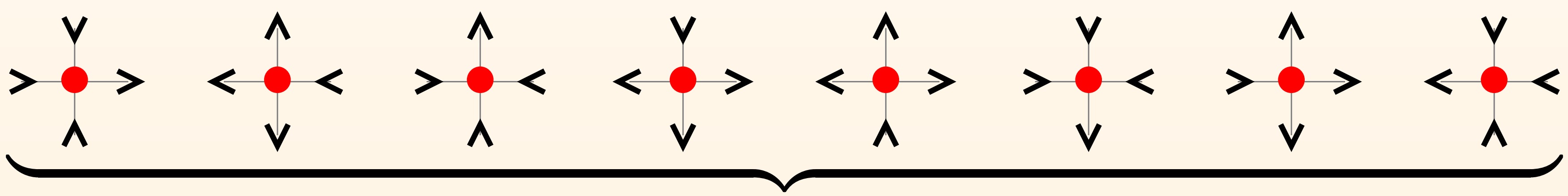


$a=\omega_1=\omega_2$ $b=\omega_3=\omega_4$ $c=\omega_5=\omega_6$ $d=\omega_7=\omega_8$ $\longleftarrow$

Integrability
(*six*- and *eight*-*vertex models*)



$e=\omega_9=\omega_{10}=\ldots=\omega_{16}$ $\longleftarrow$

Non-Integrability
Monte Carlo simulations
& Cavity calculation

Equilibrium phases of the general model ?

Thermal excitations:
Vertices breaking the ice-rule are allowed but un-favoured = *defects*

Dynamics $\Longrightarrow$ *Stochastic extension of a general integrable model*

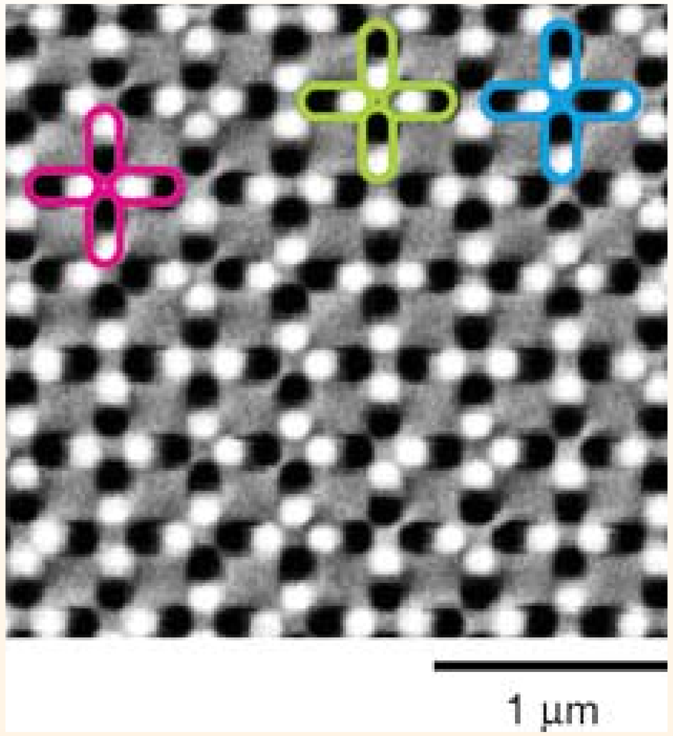
2D experiments: Artificial Spin-Ice

1. Array of elongated ferromagnetic islands behave as classical Ising spins.

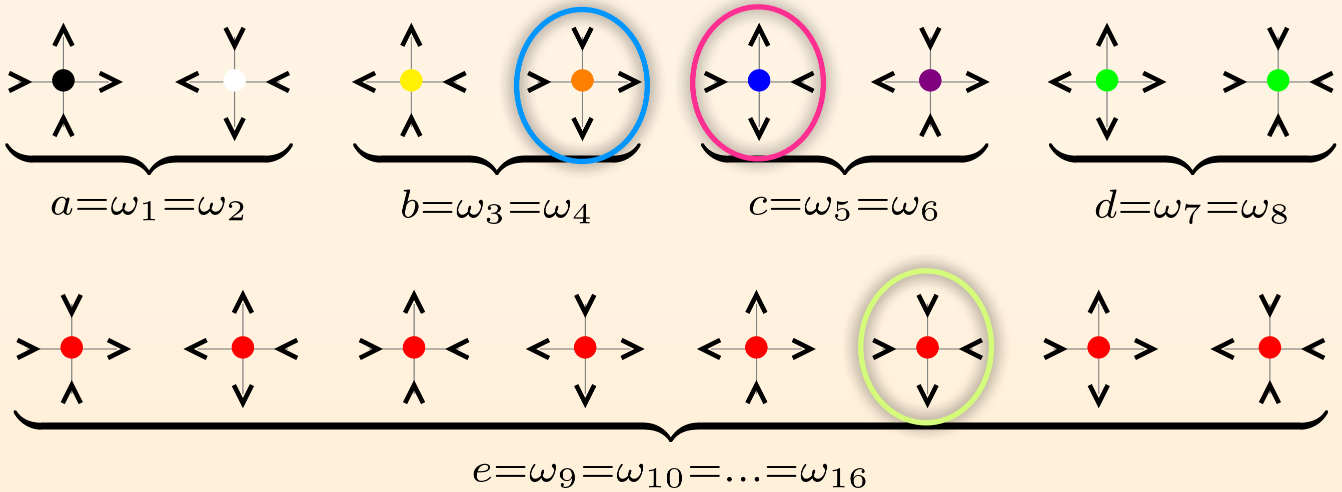
2. Dipolar interactions $\longrightarrow c > a = b$

3. Direct visualisation of defects' dynamics

Ladak, Read, Perkins, Cohen, Branford 2010 ;
Mengotti, Heyderman, Rodríguez, Nolting, Hügli, Braun 2011.



from **Wang et al., Nature** 2008

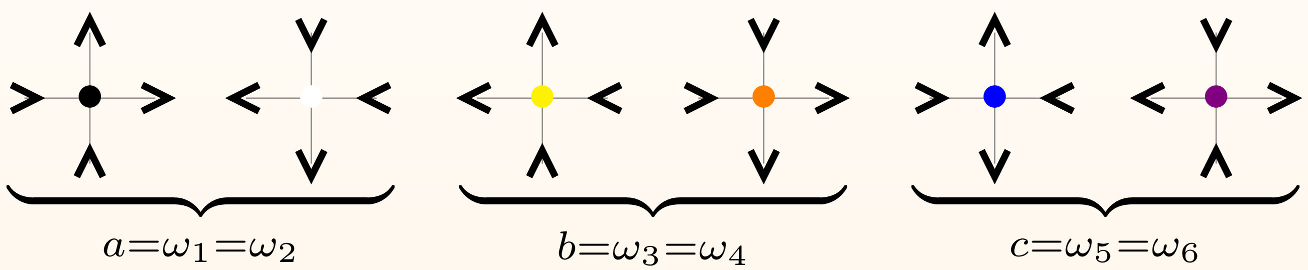


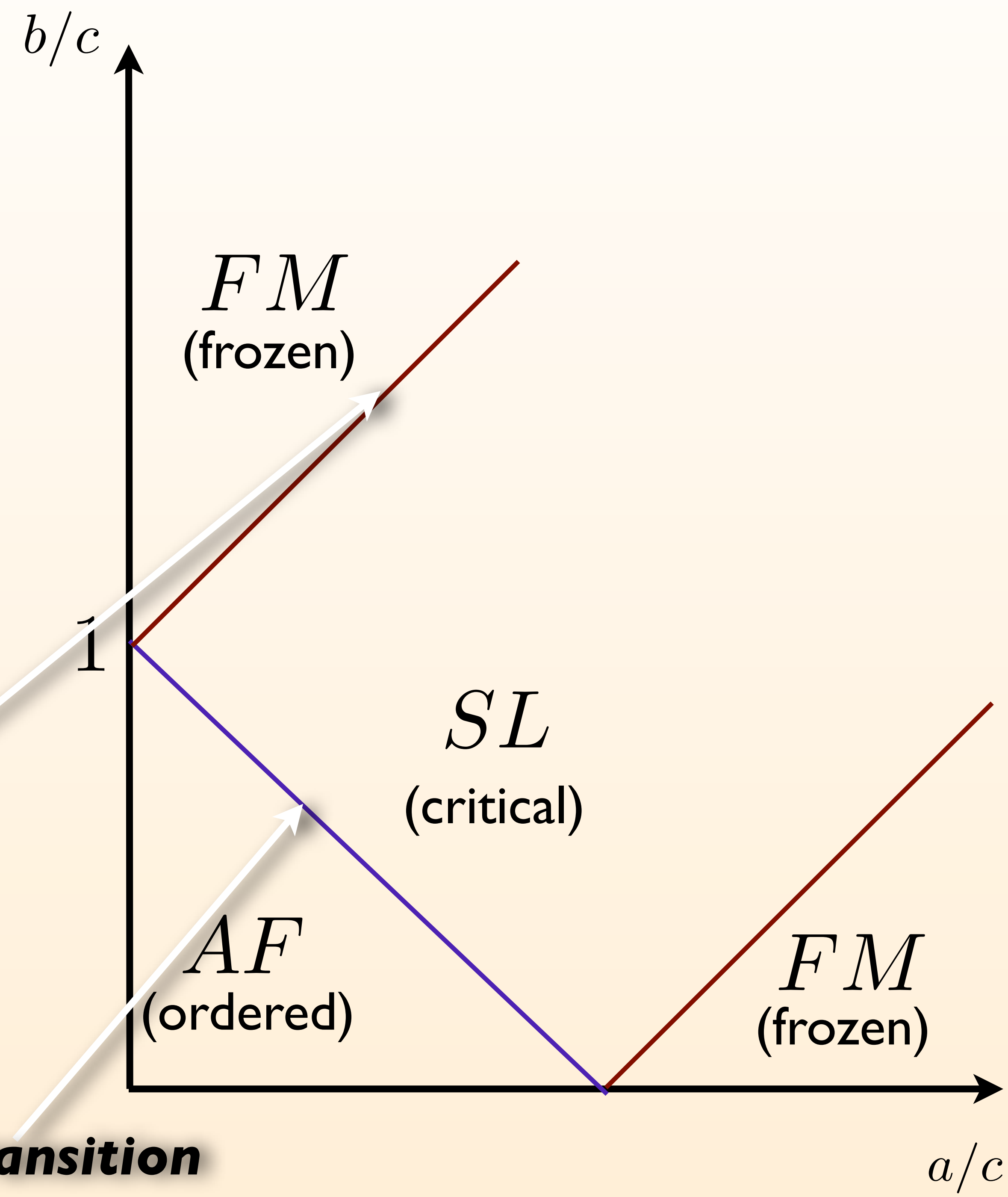
Wang, Nisoli, Freitas, Li, McConville, Cooley, Lund, Samarth, Leighton, Crespi, Schiffer 2006.

Equilibrium.

Six-vertex model (exact results)

Ice-rules $\Rightarrow \quad d = e = 0$





Equilibrium phases:

$$\boxed{\Delta_6 = \frac{a^2 + b^2 - c^2}{2ab}}$$

$\Delta_6 > 1$ *frozen* FM phase

`Frozen-to-critical' phase transition

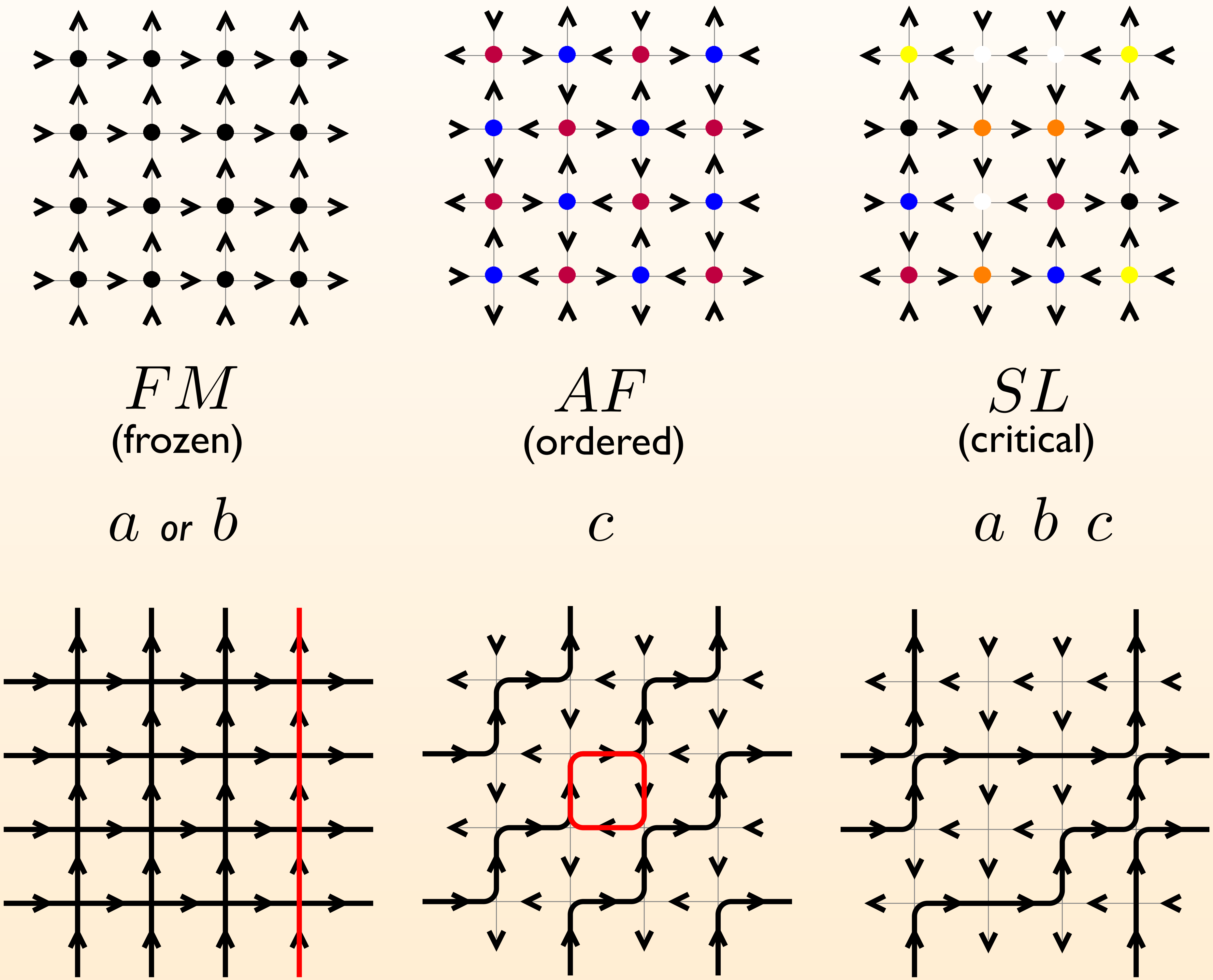
$1 > \Delta_6 > -1$ *quasi long-range ordered* SL phase

Kosterlitz-Thouless phase transition

$\Delta_6 < -1$ *ordered* AF phase with low energy excitations

Baxter, *Exactly Solvable Models in Statistical Mechanics* 1982

Six-vertex model (exact results)



Sixteen-vertex model (numerics)

Allow defects $\Rightarrow \quad d = e \neq 0$

Conjecture: (numerics)

$$\boxed{\Delta_{16} = \frac{a^2 + b^2 - c^2 - (3e+d)^2}{2(ab + c(3e+d))}} \quad \xrightarrow{e=0} \quad \Delta_8 = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

1. All the transitions become continuous
2. The PM phase is not critical, the FM phase is not frozen.
3. Finite size scaling: Critical exponents depend on the value of the parameters: a, b, c, d, e .
4. Phase diagram characterised by a generalised 'anisotropy parameter'.

$\Rightarrow$ consistent with the exact results on the eight-vertex model **Baxter** 1971, 1982
and (the very few) the sixteen-vertex model **Wu** 1969

Levis & Cugliandolo 2011

Foini, Levis, Tarzia & Cugliandolo 2012

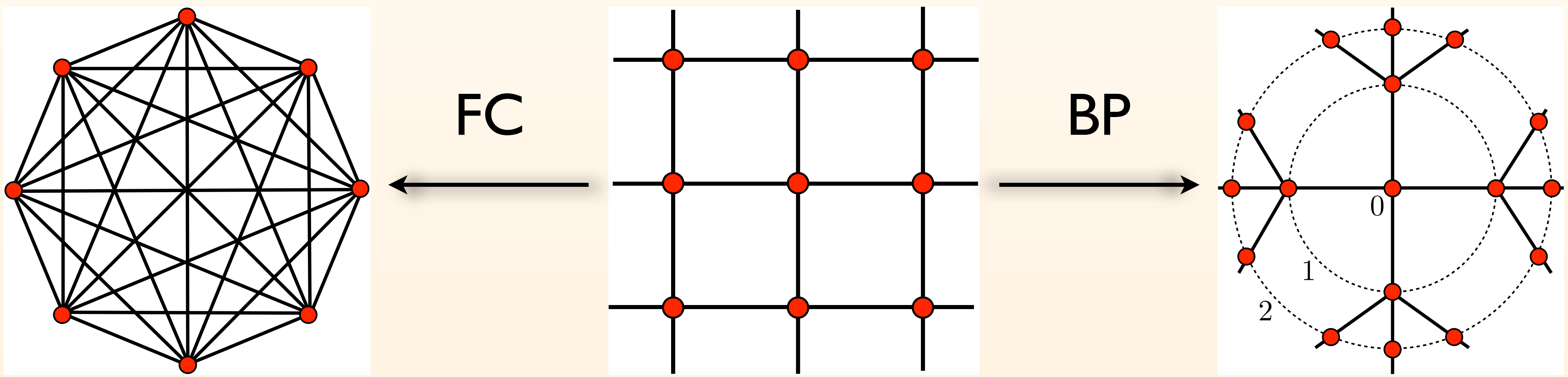
Spin-ice on a tree graph (cavity method)

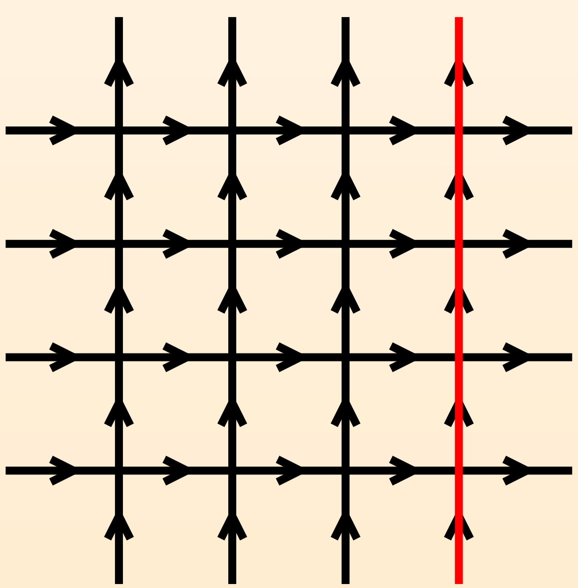
Bethe-Peierls approximation:

In the absence of a given site, the neighbours are de-correlated.

→ The connectivity of the original model is preserved BUT no loops

→ Allows for a recursive treatment to be solved self-consistently



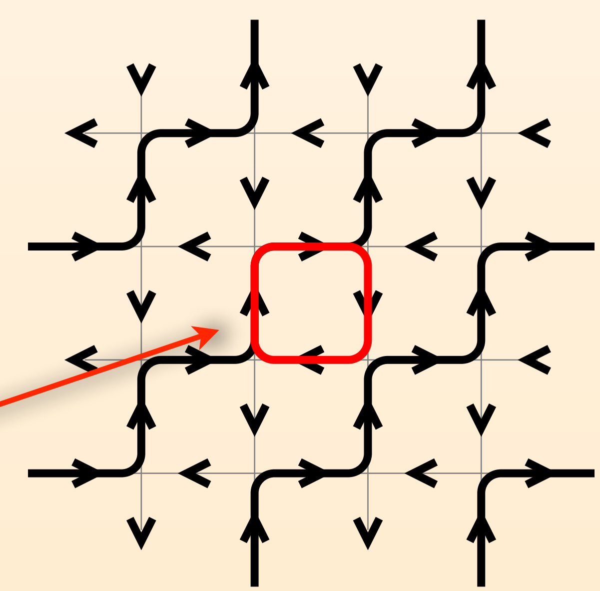


⟹ Good approximation for FM order

No loops

AF order ?

Defects ?



Loop fluctuations

Cavity method.

Spin-ice on a tree graph (cavity method)

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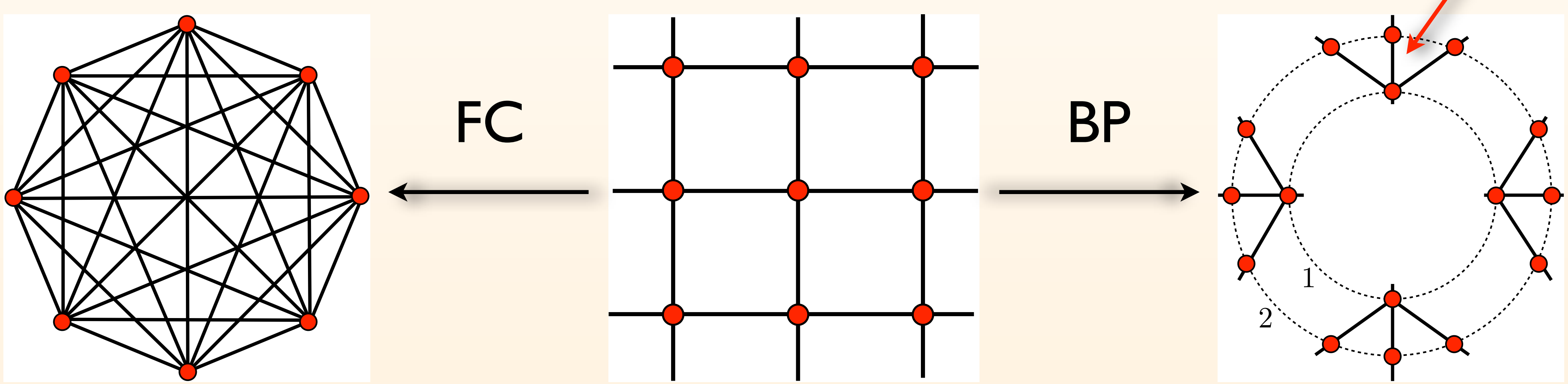
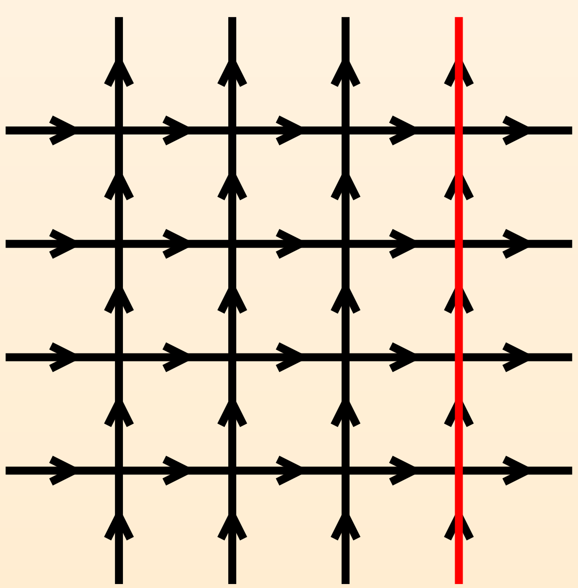
4 disconnected rooted trees





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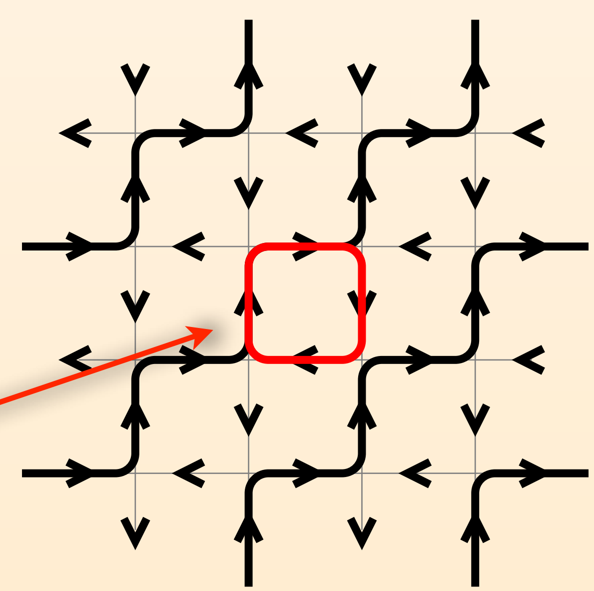


Loop fluctuations

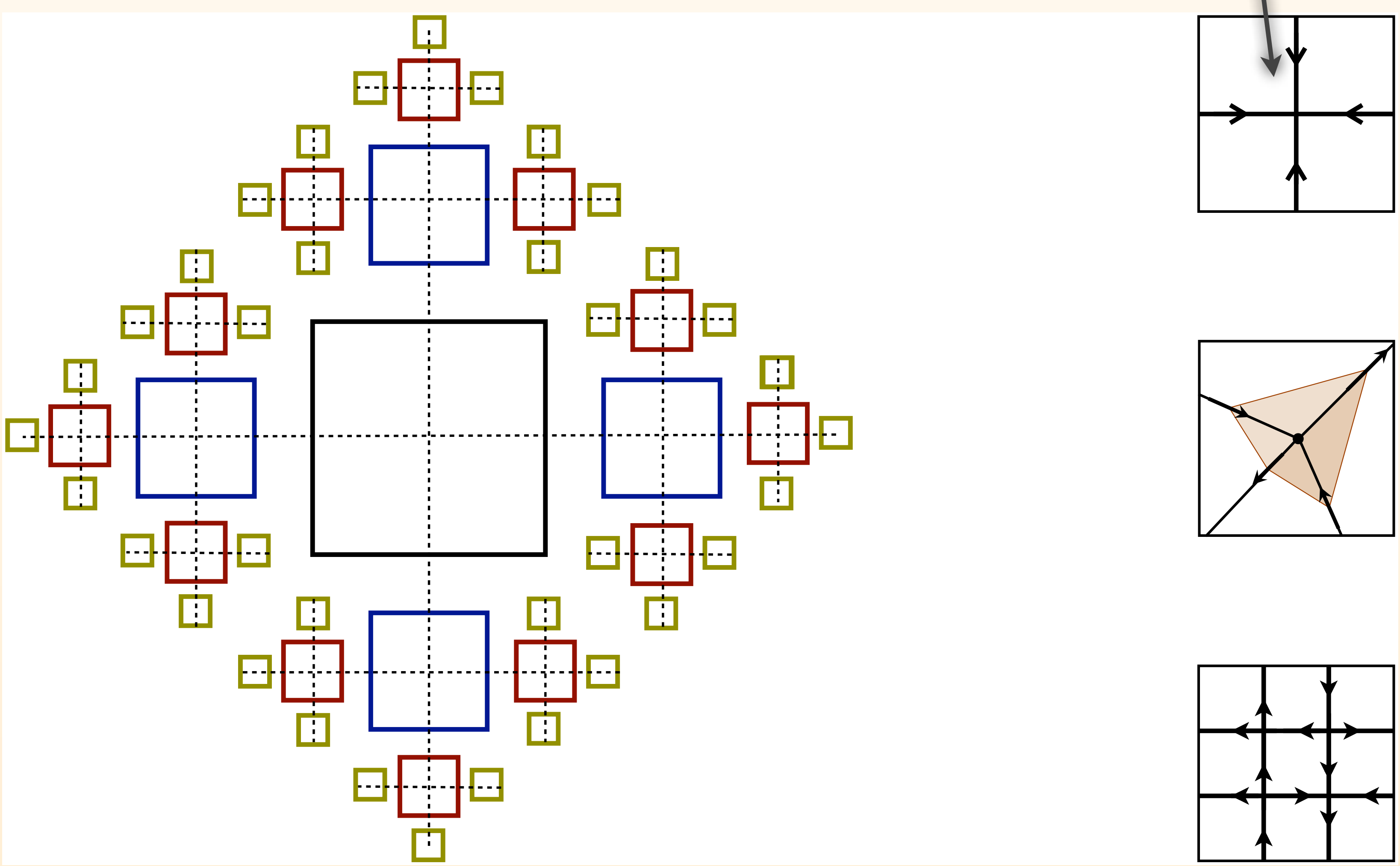
Cavity method.

Spin-ice on a tree graph (cavity method)

Two models:

1. An oriented tree of vertices 'single vertex tree'
   No loop fluctuations

2. A tree of plaquettes made of four vertices 'plaquette tree'
   Include elementary loop fluctuations



Foini, Levis, Tarzia & Cugliandolo 2012

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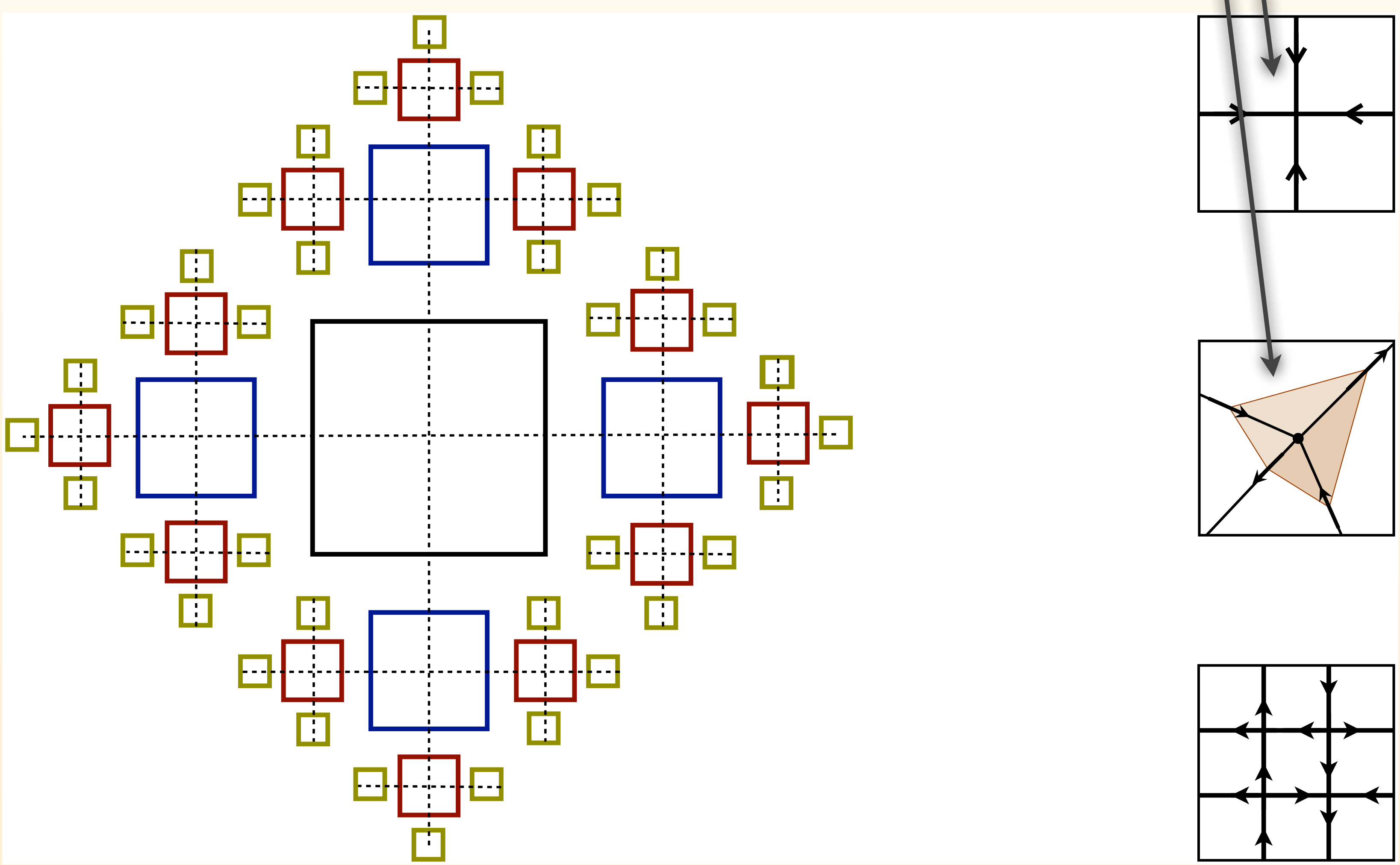

Jaubert, Chalker, Holdsworth & Moessner 2008

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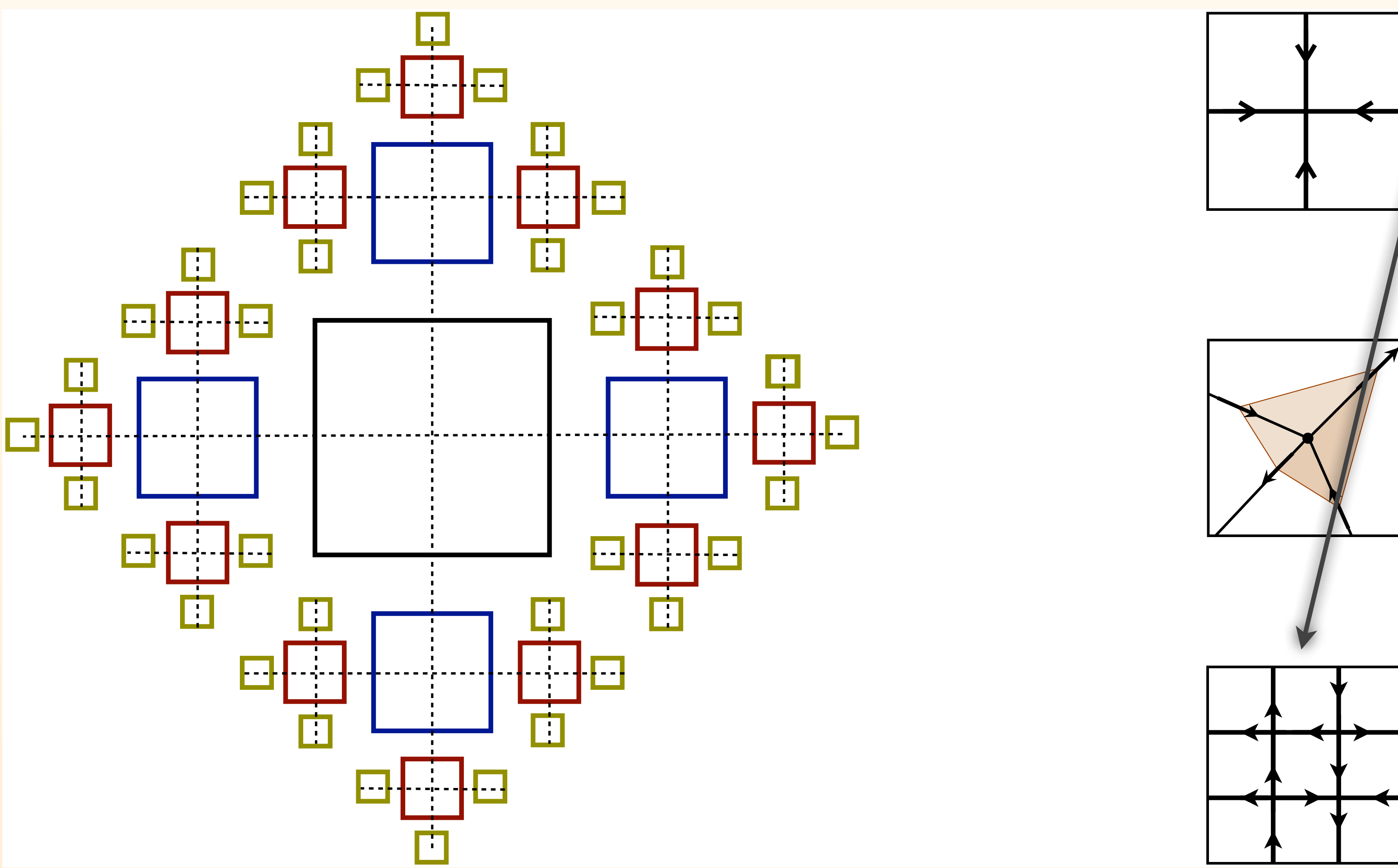
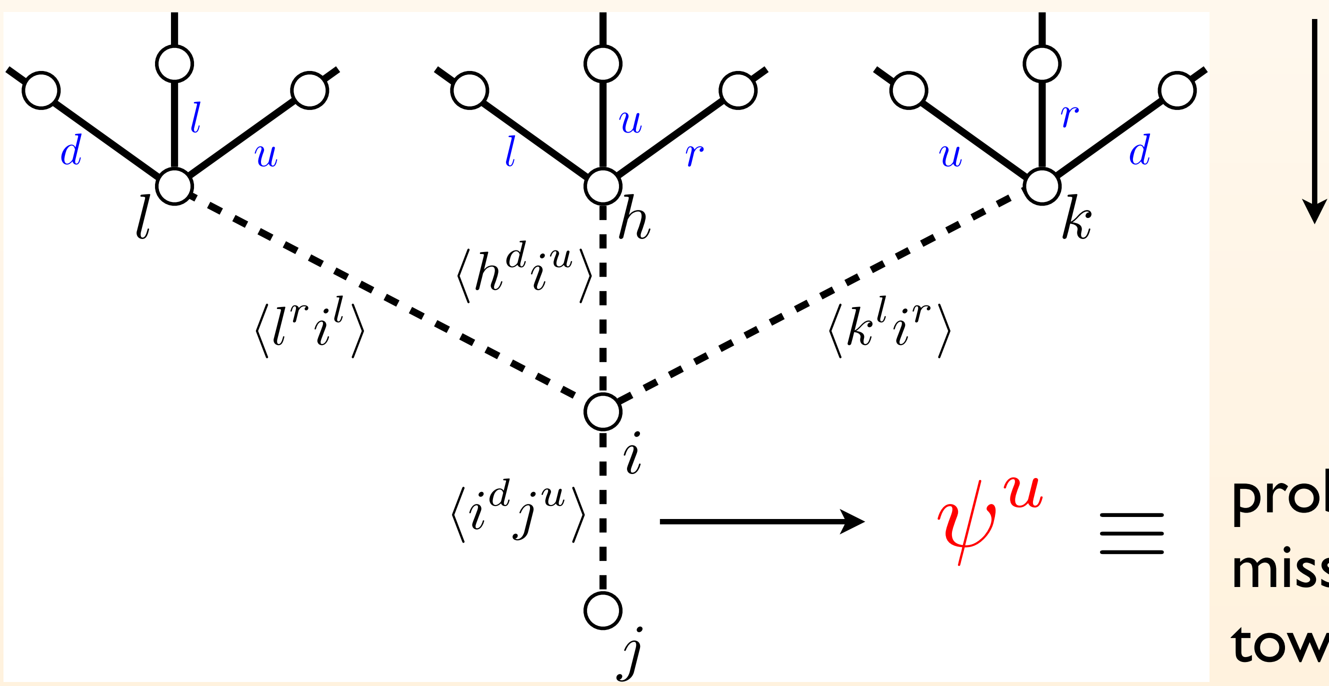


Foini, Levis, Tarzia & Cugliandolo 2012

Spin-ice on a tree graph (cavity method)

Strategy: 1. Define an oriented tree (up,left,down,right) where vertices can be identified.

2. Write the self-consistent equations satisfied by the 'cavity probabilities': *n-th* shell as a function of the *(n+1)th* shell.



probability that the spin in the missing edge $\langle i^d j^u \rangle$ points towards i (up).

up-rooted tree from merging a right, up and left rooted tree

Foini, Levis, Tarzia & Cugliandolo 2012

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The simplest example: eight-vertex model in the single vertex tree

$$\psi^u = \frac{1}{z^u}\Big[a\psi^l\psi^u\psi^r + b(1-\psi^l)\psi^u(1-\psi^r) + c(1-\psi^u)(1-\psi^l)\psi^r + d\psi^l(1-\psi^u)(1-\psi^r)\Big] \qquad \times 4 \ : \ \psi^{l,d,r}$$

$$\boxed{\psi^\alpha = \hat{\Psi}^\alpha[a,b,c,d,e,\psi^u,\psi^d,\psi^l,\psi^r]}$$

Self-consistent equations

$\alpha = u,\ l,\ d,\ r$

Foini, Levis, Tarzia & Cugliandolo 2012

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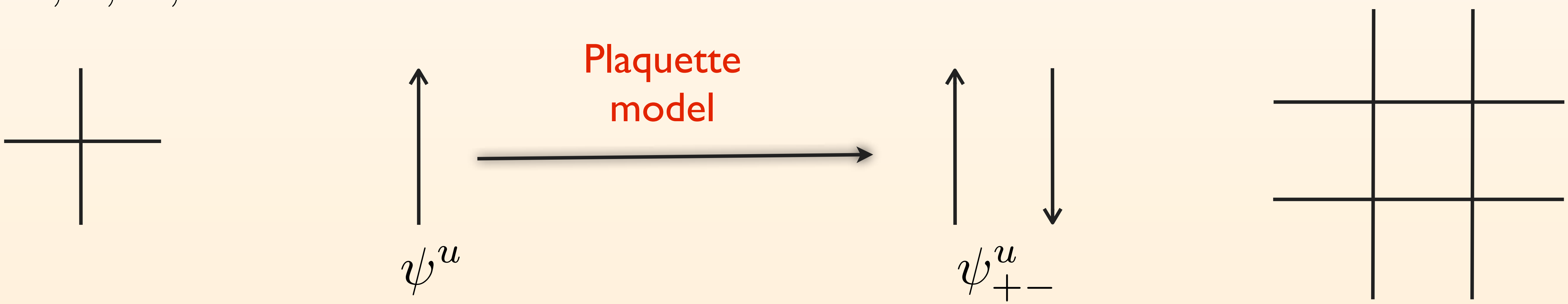

$$\boxed{\psi^\alpha_{s_1 s_2} = \hat{\Psi}^\alpha_{s_1 s_2}[a, b, c, d, e, \boldsymbol{\psi}^u, \boldsymbol{\psi}^d, \boldsymbol{\psi}^l, \boldsymbol{\psi}^r]}$$

4×4

Foini, Levis, Tarzia & Cugliandolo 2012

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3. Find the fixed points of the self-consistent equations.

$\longrightarrow$ Equilibrium phases

4. Compute thermodynamic observables: free energy, magnetisation, etc.

5. Study the stability of the solutions

$$M_{\alpha,\beta} = \left.\frac{d\hat{\Psi}^{\alpha}}{d\psi^{\beta}}\right|_{fp} \longrightarrow$$

Phase transitions and nature of the phases.

Eigenvalues: $E_1, E_2, \dots$

$\longrightarrow$ Try to find an anisotropy parameter $\Delta(E_1, E_2, \dots)$ which characterises the phase diagram

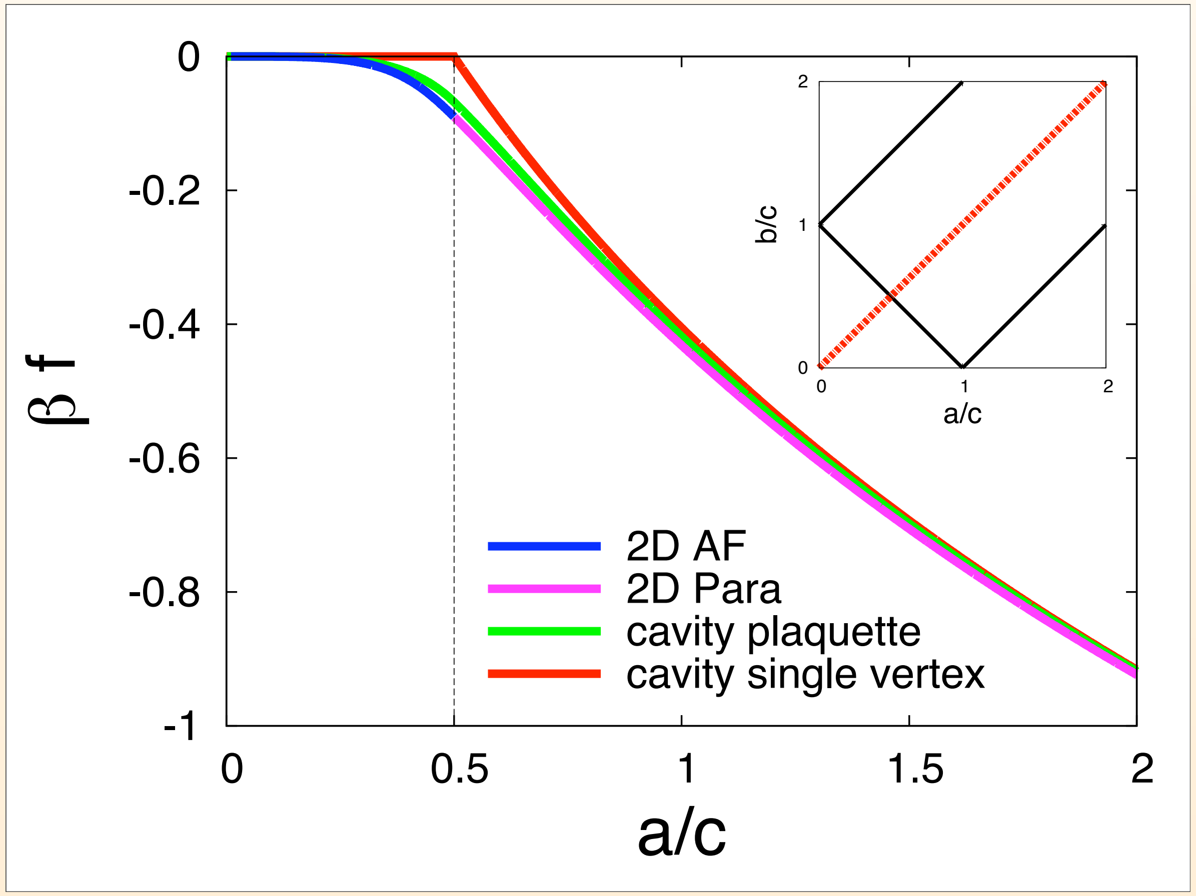
Foini, Levis, Tarzia & Cugliandolo 2012

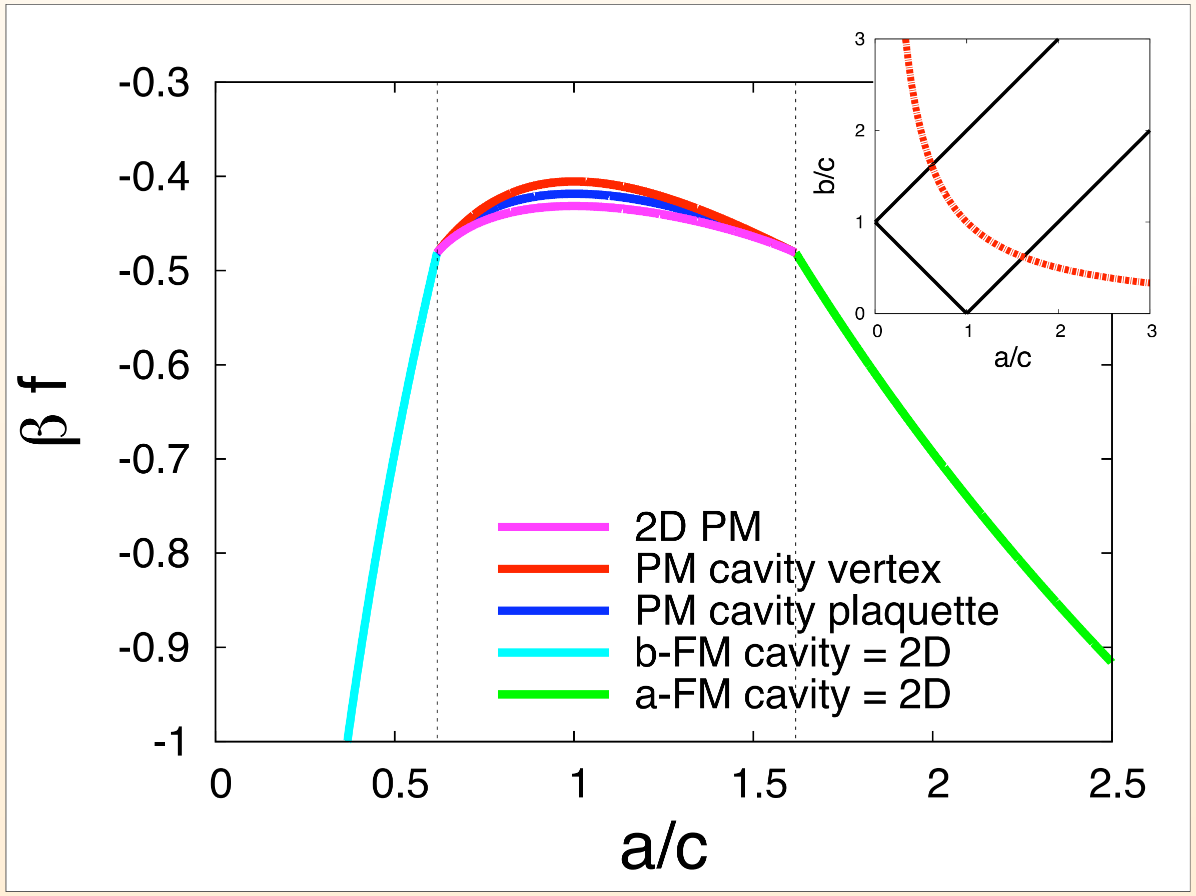
Spin-ice on a tree graph (cavity method)

Results for the six-vertex model:

1. **Exact** location of the transition lines with both the single vertex and the plaquette model.

2. Thermodynamic quantities in remarkable agreement with the 2D results.





Free energy of the 6V model on trees vs. exact results in 2D.

Foini, Levis, Tarzia & Cugliandolo 2012

Spin-ice on a tree graph (cavity method)

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	6V single vertices	6V plaquettes	6V 2D
PM-FMs	Frozen-to-PC	Frozen-to-PC	Frozen-to-crit
PM-AFs	Frozen-to-PC	Cont.-to-PC	KT

Foini, Levis, Tarzia & Cugliandolo 2012

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PM-AFs	Frozen-to-PC	Cont.-to-PC	KT

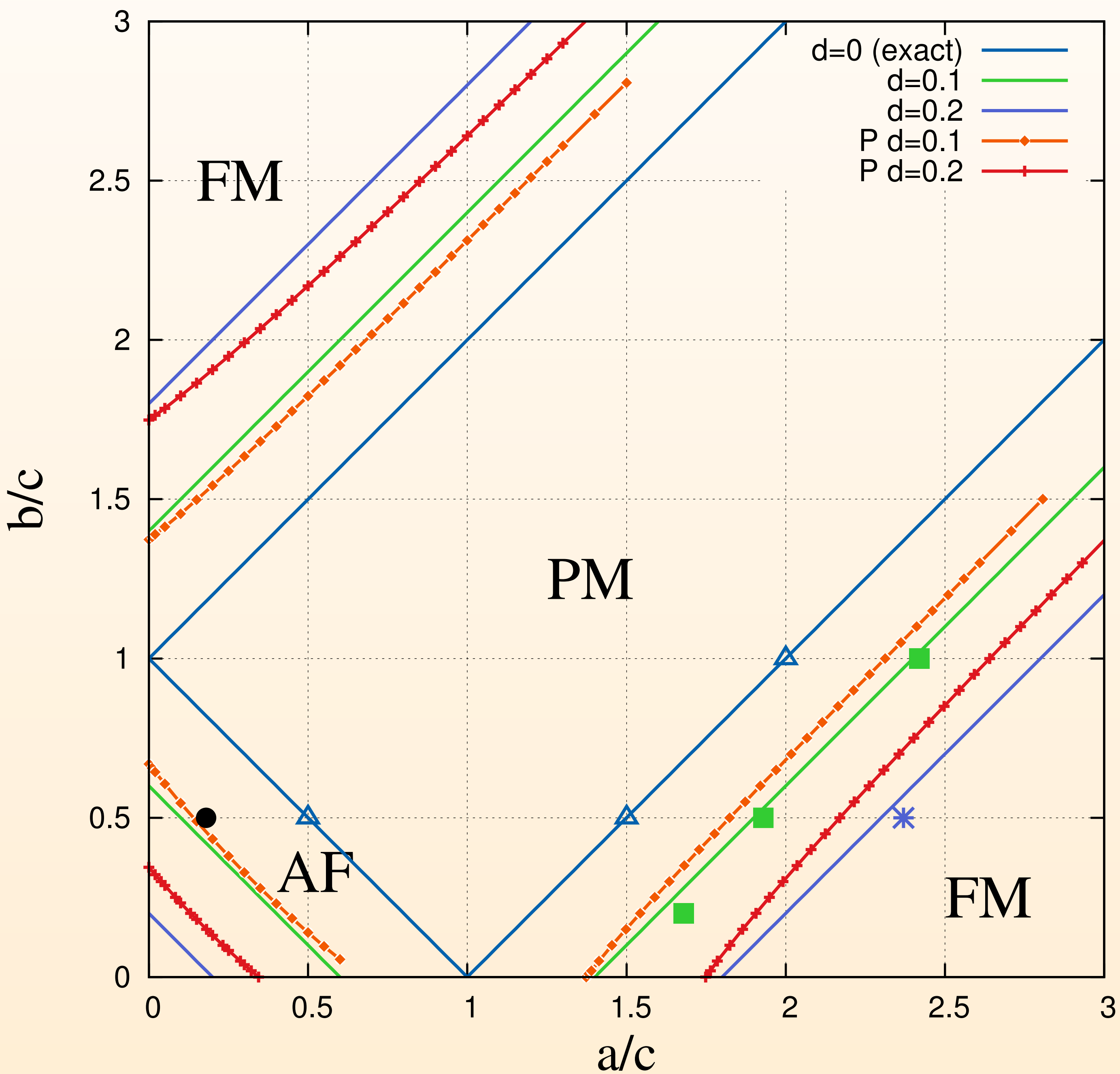
$\longrightarrow$ goes in the good direction

Foini, Levis, Tarzia & Cugliandolo 2012

Cavity method.

Spin-ice on a tree graph & CTMC

Results for the sixteen-vertex model:



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Cavity method.

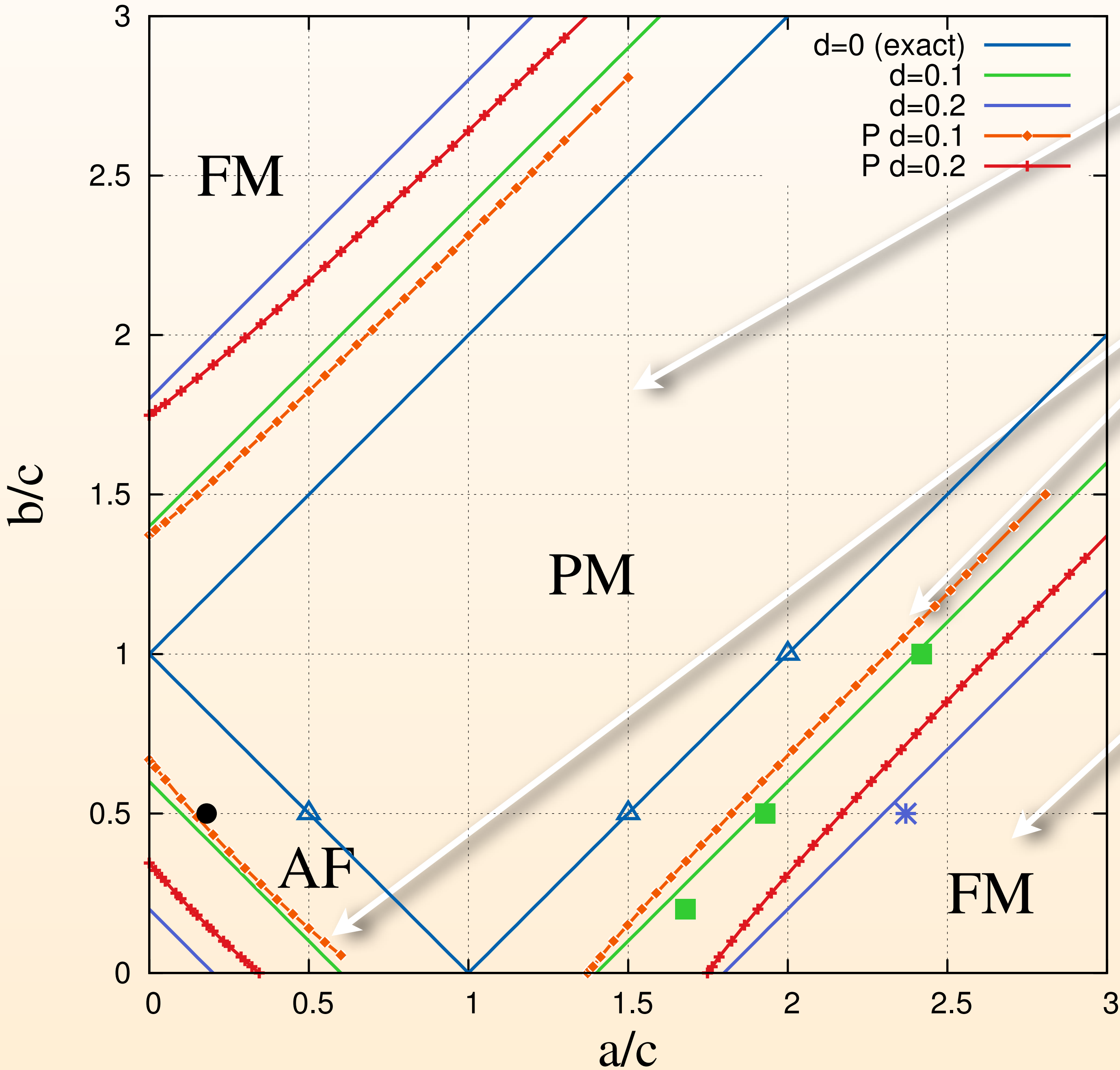
Spin-ice on a tree graph & CTMC

Results for the sixteen-vertex model:

Non-critical phase

continuous phase transitions

Non-frozen order

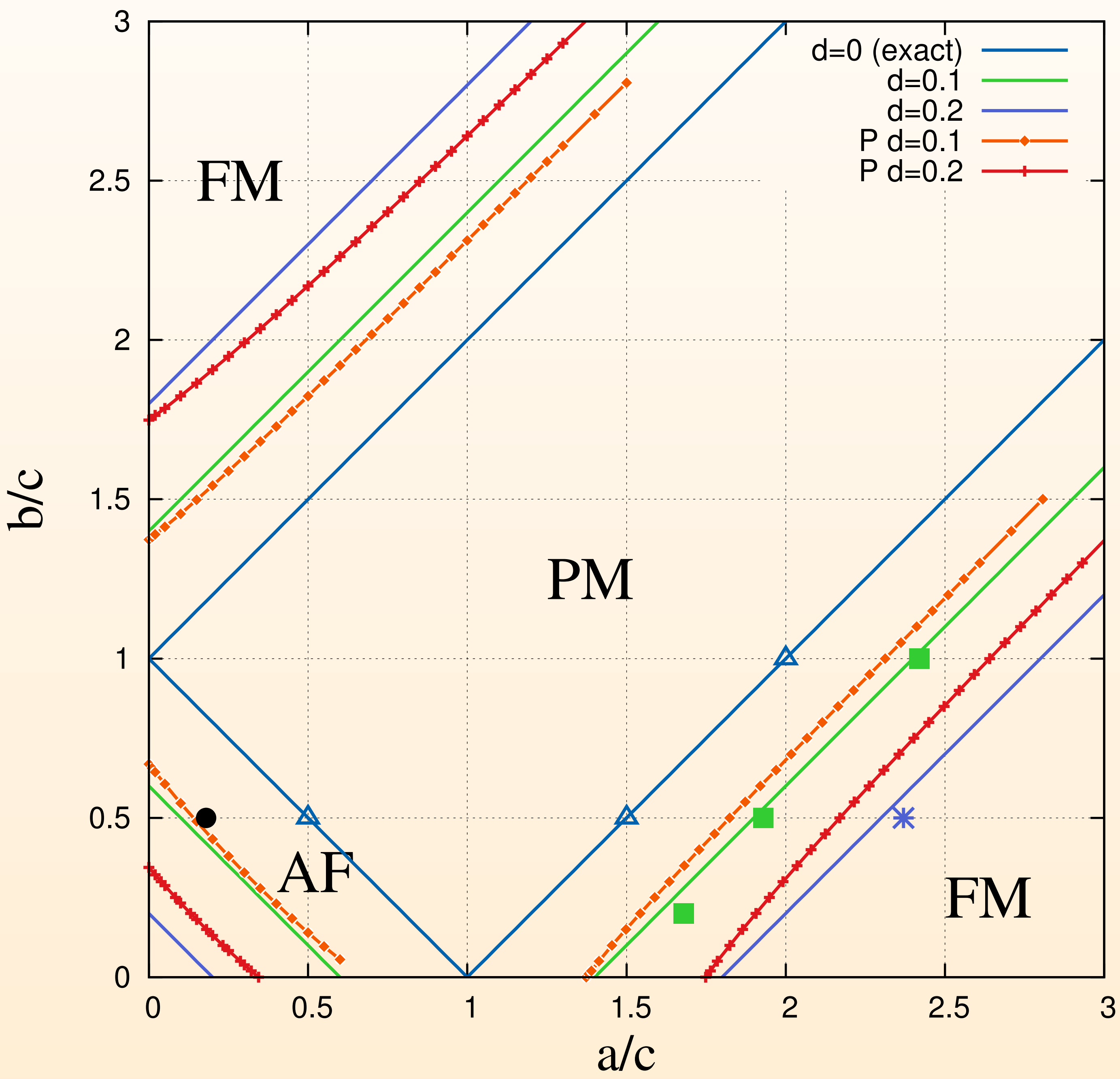


Foini, Levis, Tarzia & Cugliandolo 2012

Cavity method.

Spin-ice on a tree graph & CTMC

Results for the sixteen-vertex model:

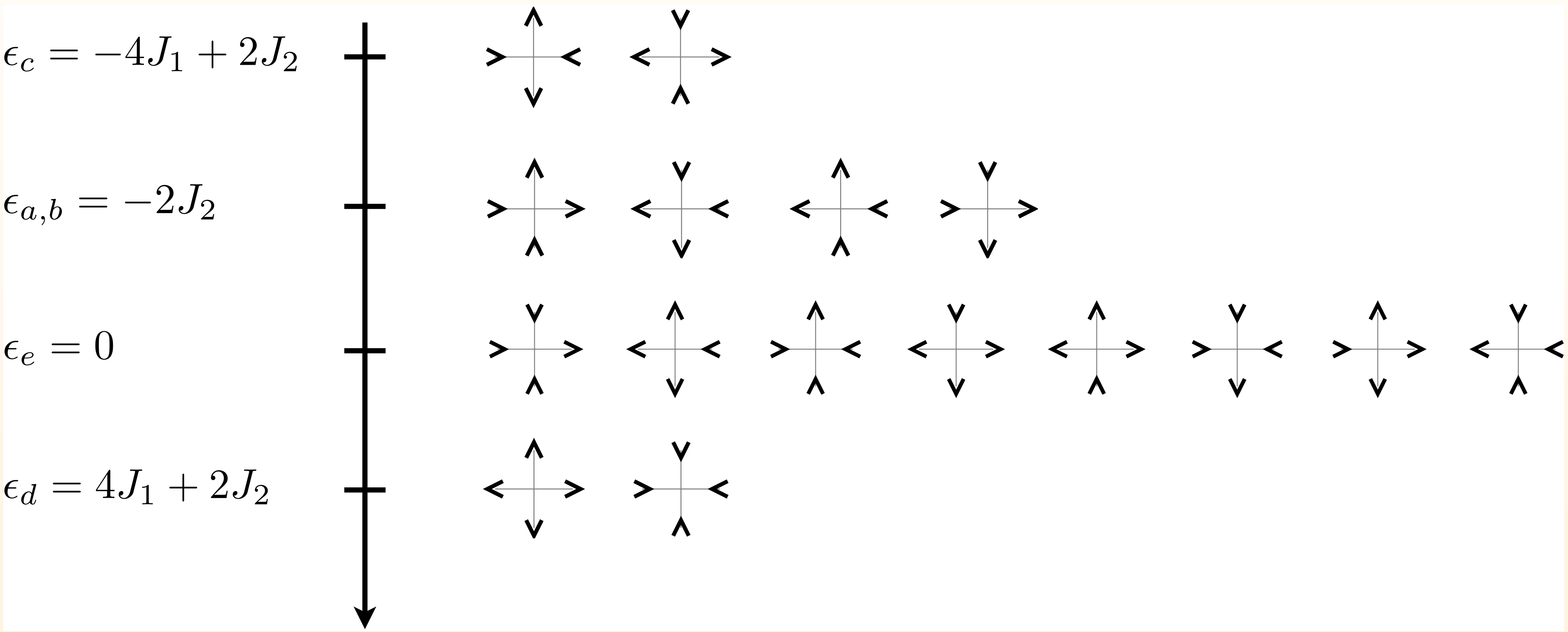


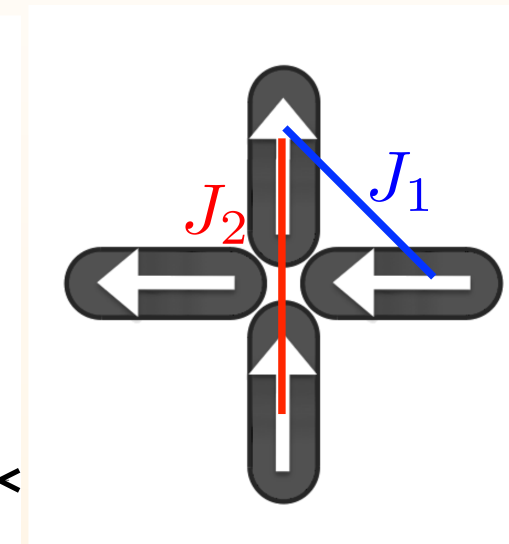
Apply this general picture to concrete experimental realisations

Foini, Levis, Tarzia & Cugliandolo 2012

Artificial spin ice

Fix the energy levels:





$$J_1 > J_2$$

Dipolar interactions in the square lattice

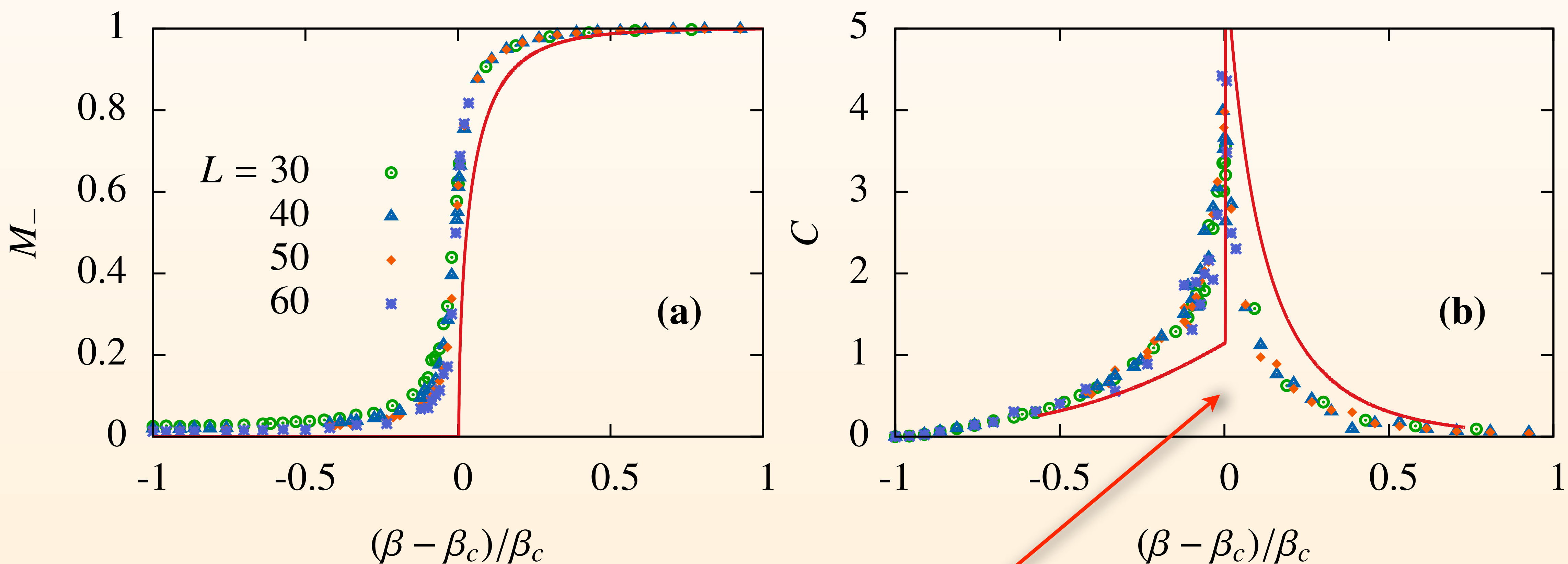
$\Longrightarrow$ *2 in - 2 out* vertices are not equivalent.

One external parameter, the canonical temperature: $\beta = -\dfrac{\ln a}{\epsilon_a}$

ASI.

‘Thermal’ Artificial spin ice

Numerics vs. Cavity calculation



Second order phase transition at $\beta_c \approx 2.65$

→ Compare with experiments

Levis, Foini, Tarzia & Cugliandolo 2012

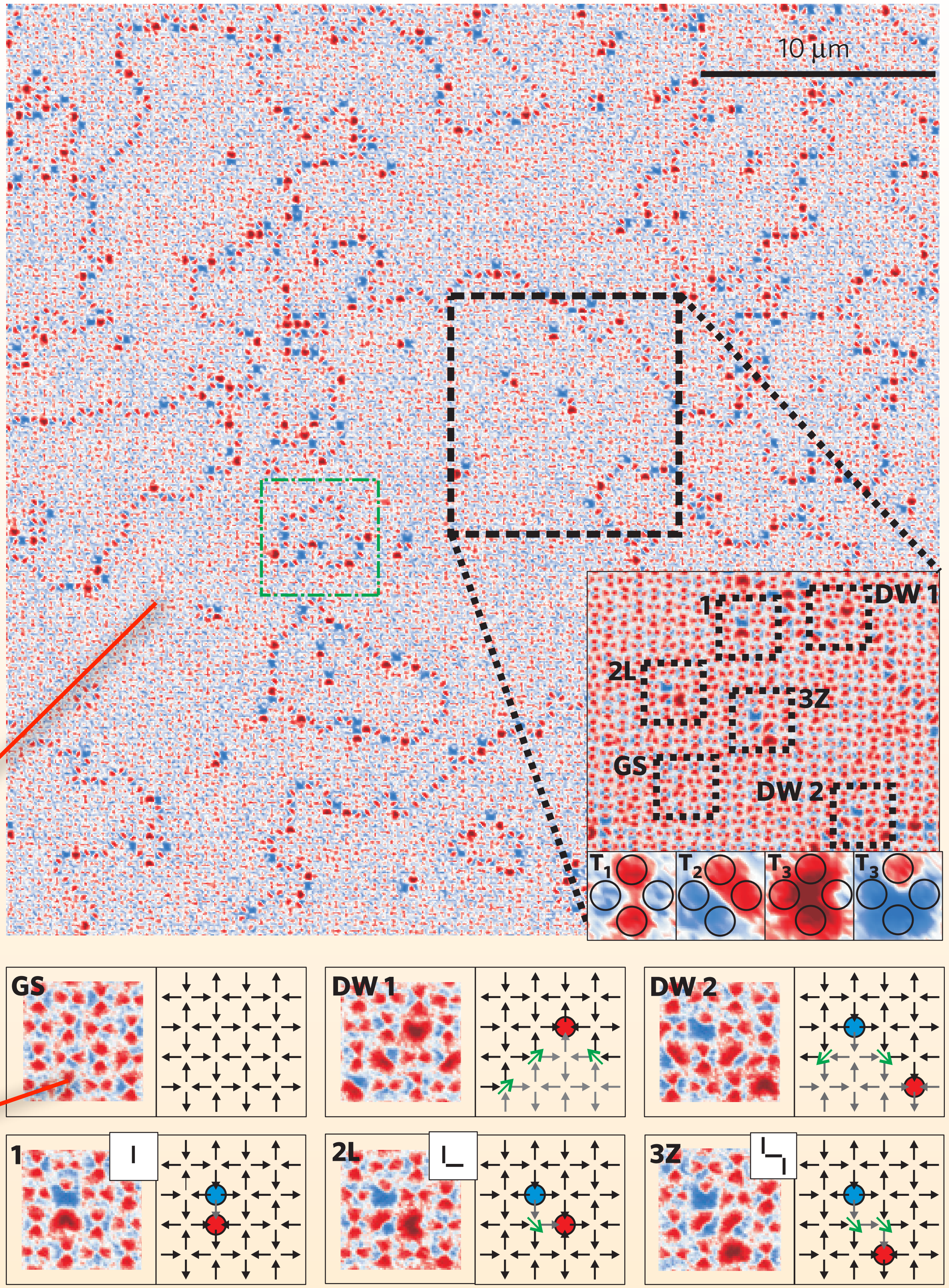
ASI.

‘Thermal’ Artificial spin ice

1. Thermal annealing during deposition.

2. Nano-islands feel thermal fluctuations during the growth process.

3. After the growing process the islands freeze.



Very close to the expected ground state

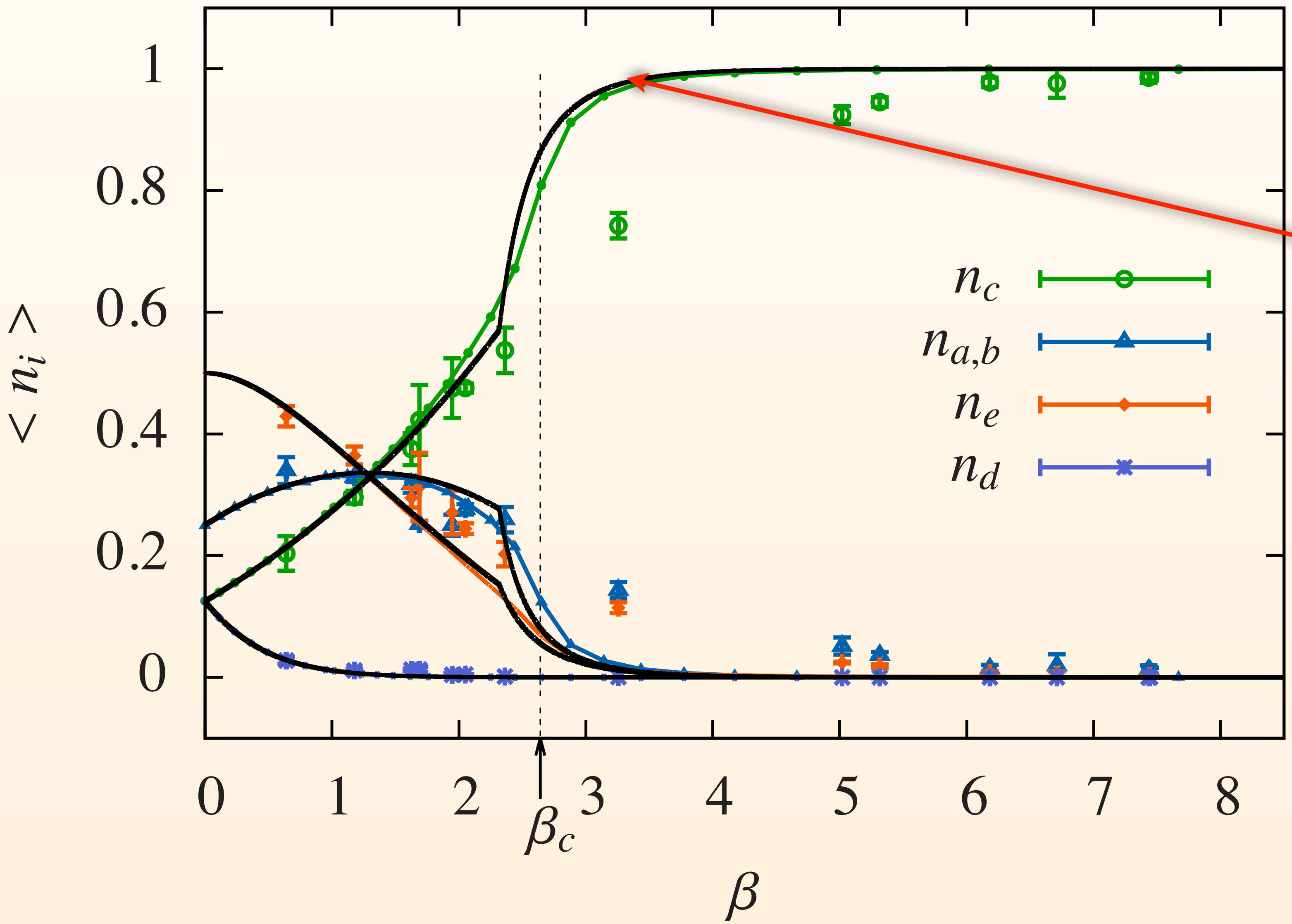
Excitations and defects can be visualised

From **Morgan** *et al.*, Nature Phys. 7 2011.

ASI.

'Thermal' Artificial spin ice

Unpublished data from **Morgan** *et al.* 2012



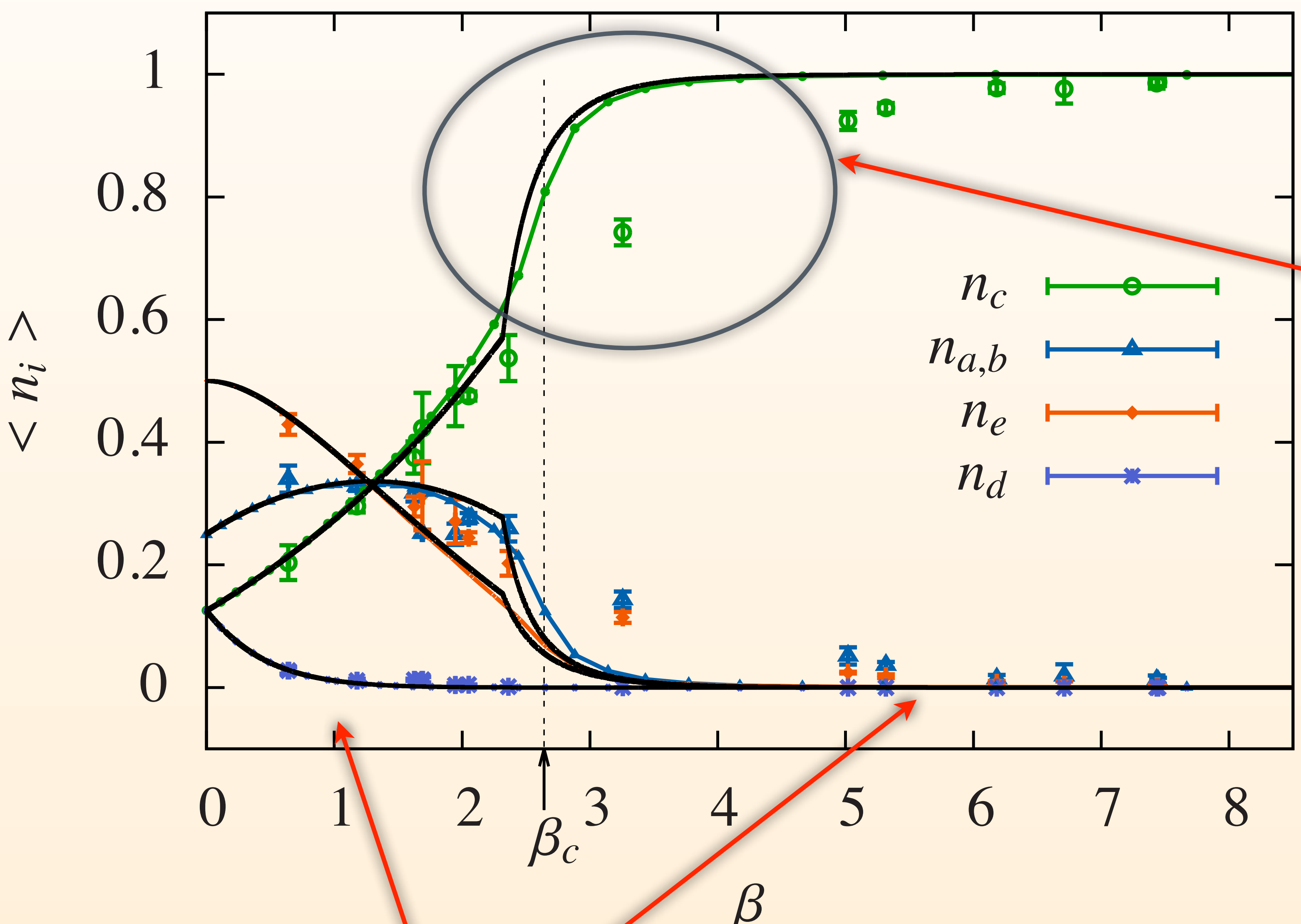
Numerics and analytics in agreement

Experimental data: $\beta_E = \ln\left(\frac{4n_c}{n_e}\right)(\epsilon_e - \epsilon_c)^{-1}$ vs. Canonical temperature

Levis, Foini, Tarzia & Cugliandolo 2012

'Thermal' Artificial spin ice

Unpublished data from **Morgan** *et al.* 2012



Critical slowing down

Equilibrium is not reached close to the critical point

$\beta \neq \beta_E$

Why ??

Our model reproduces quantitatively the experimental data

Levis, Foini, Tarzia & Cugliandolo 2012

Conclusions equilibrium.

1. Phase diagram of the sixteen-vertex model.
The inclusion of *defects* modifies dramatically the thermodynamics.

2. Continuously varying critical exponents verify 'weak universality'.

3. Approximating the system by a tree of square plaquettes leads to good agreement with 2D results.

4. The model reproduces experimental data away from the critical point

$\rightarrow$ Relationship between configurational temperature and canonical temperature in ASI?

$\rightarrow$ Out-of-equilibrium phenomena ?

Out-of-equilibrium.

Phase ordering dynamics after a quench.

Evolution of the system across a phase transition.

- Give updating rules: Non-conserved order parameter (single spin flips)

coupled to a thermal bath

Stochastic dynamics

- Prepare the system in a disordered equilibrium state and quench it into a known equilibrium ordered symmetry broken phase

→ *Competition between opposite orders*

Slow dynamics $L(t) \sim t^z$

→ Apply this procedure to the sixteen-vertex model

Out-of-equilibrium.

Quench into the PM phase.

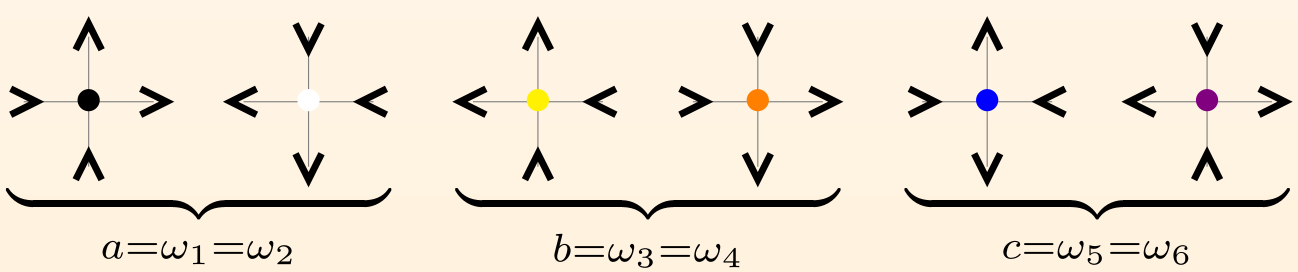
initial configuration:

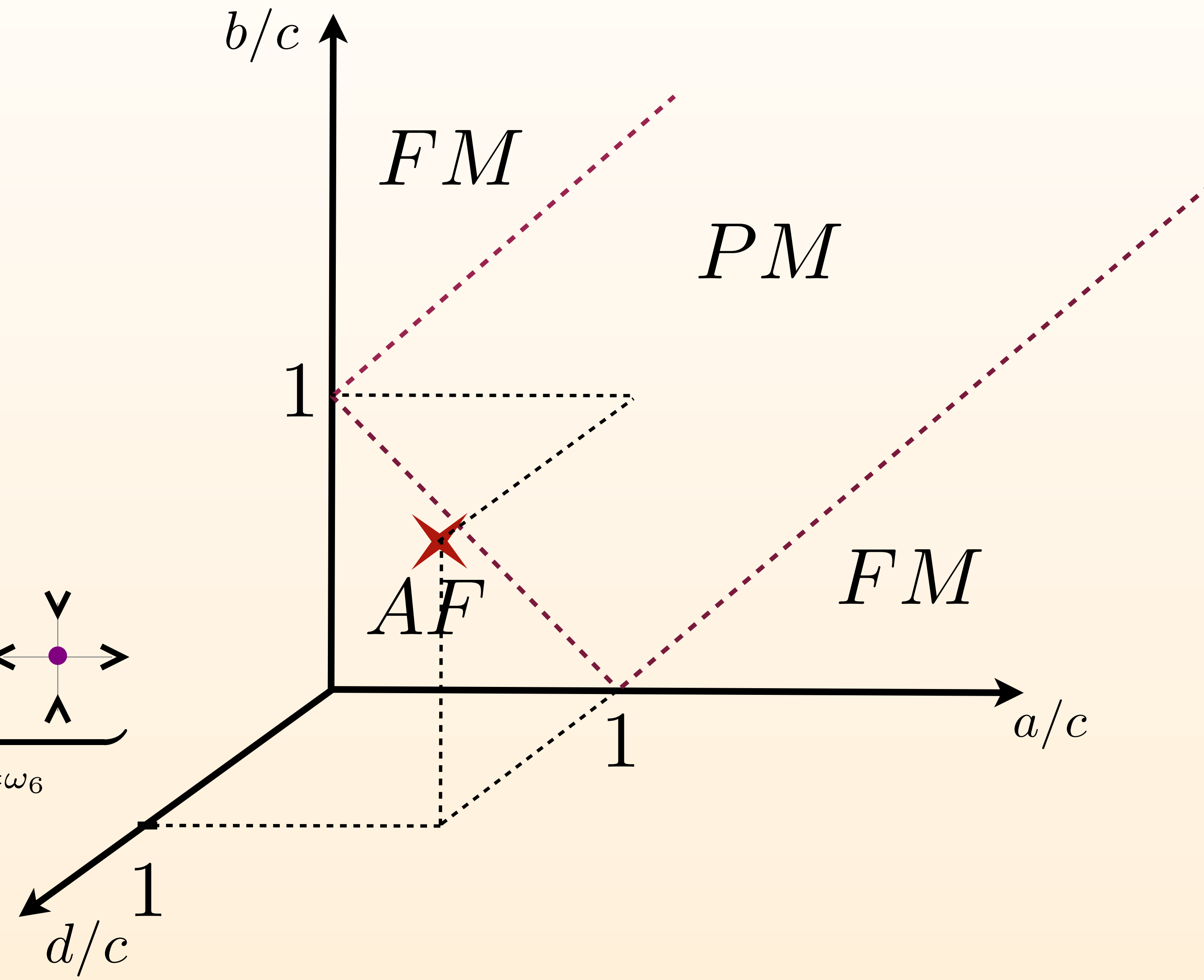
$a = b = c = d = e = 1 \ (T = \infty)$

$\big\downarrow$ at t=0 (instantaneous)

$a = b = c \gg d, e$

$\Longrightarrow$



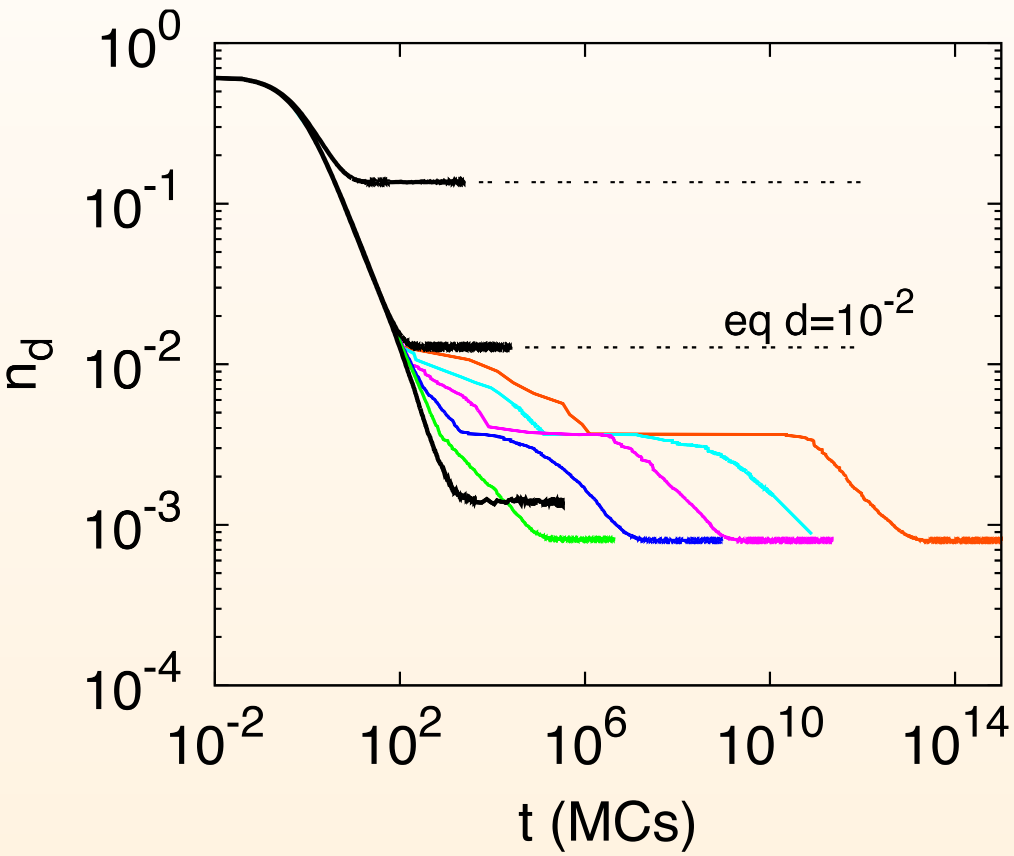


Magnetisation = 0 BUT ‘close’ to a QLRO phase $\Longrightarrow$ No order parameter to describe the relaxation process

Out-of-equilibrium.

Quench into the PM phase.

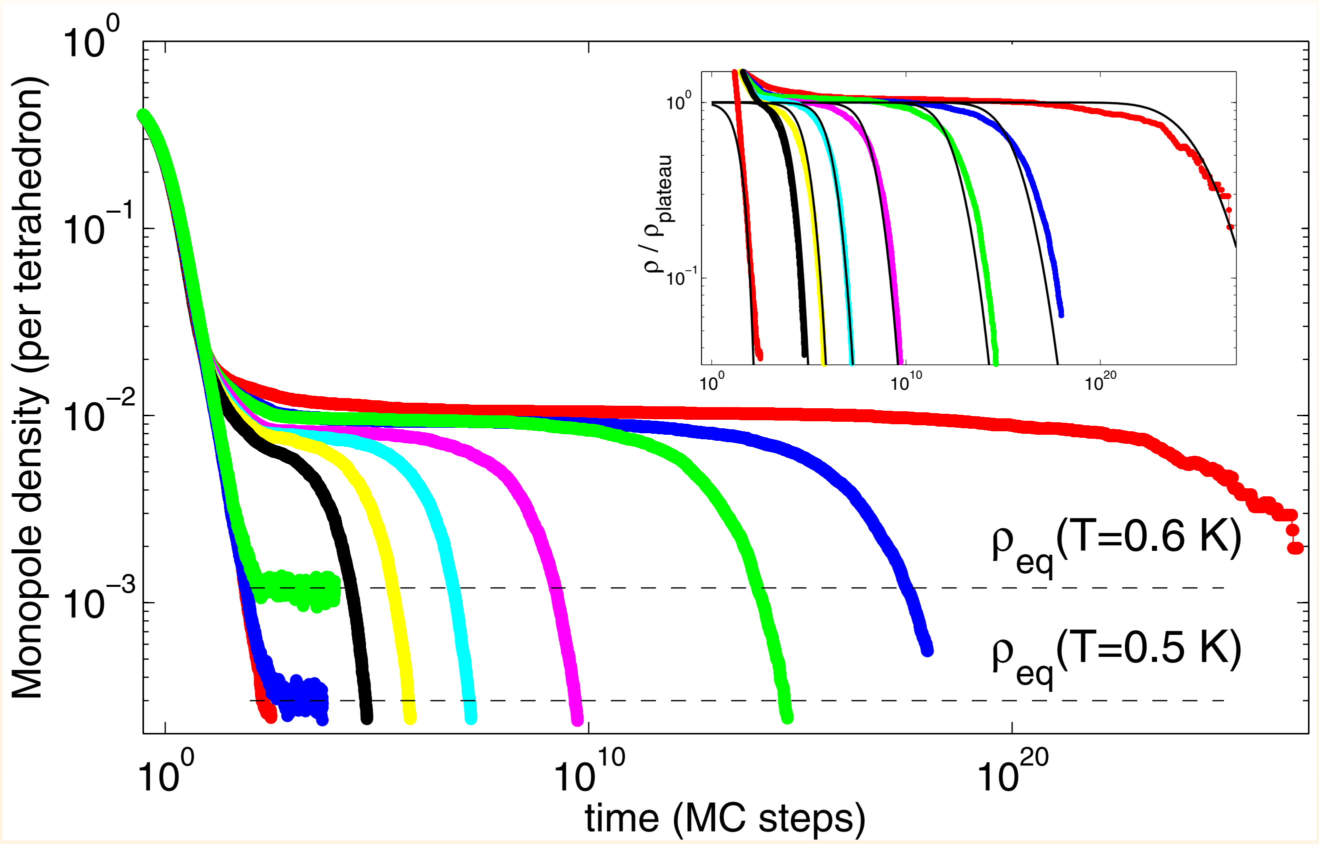
Density of defects



2D vertex model

$$d = e = 10^{-1}, ..., 10^{-4}, ..., 10^{-8}$$
$$N = 2L^2 = 5000$$

Levis & Cugliandolo 2011



3D pyroclhore spin ice with dipolar interactions

$$T = 0.6\,\mathrm{K}, ..., 0.025\,\mathrm{K}$$
$$N = 8192$$

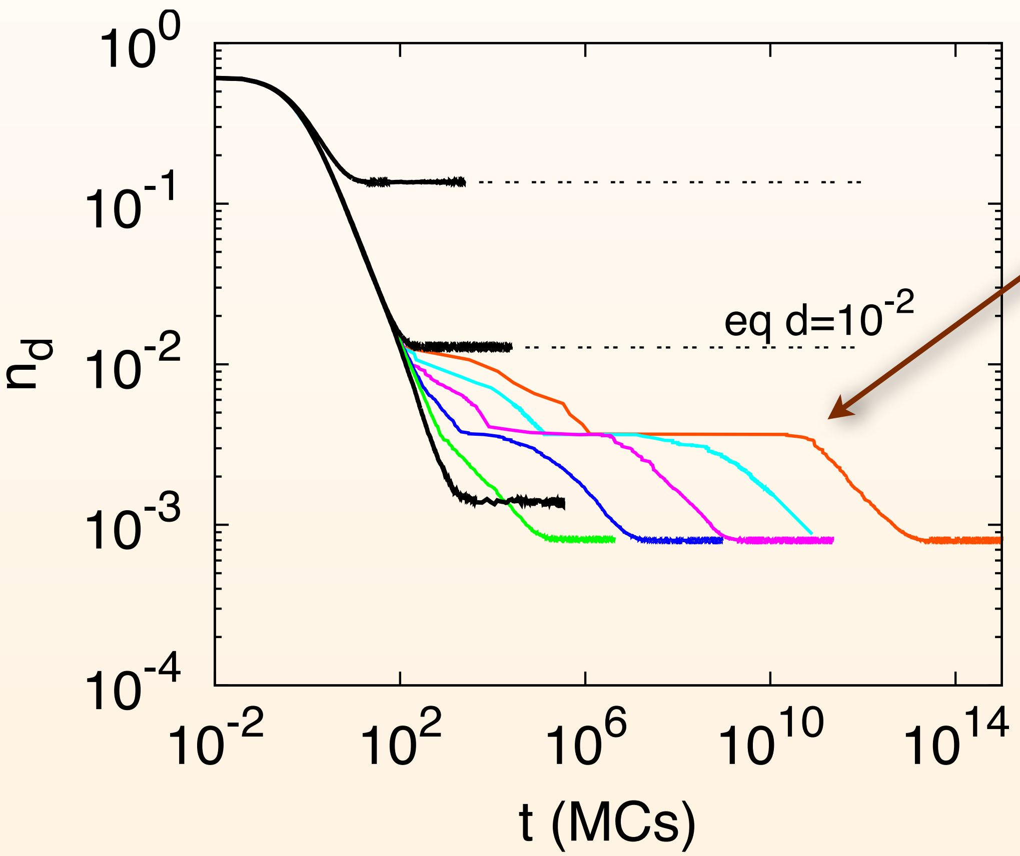
Castelnovo, Moessner & Sondhi 2012

Out-of-equilibrium.

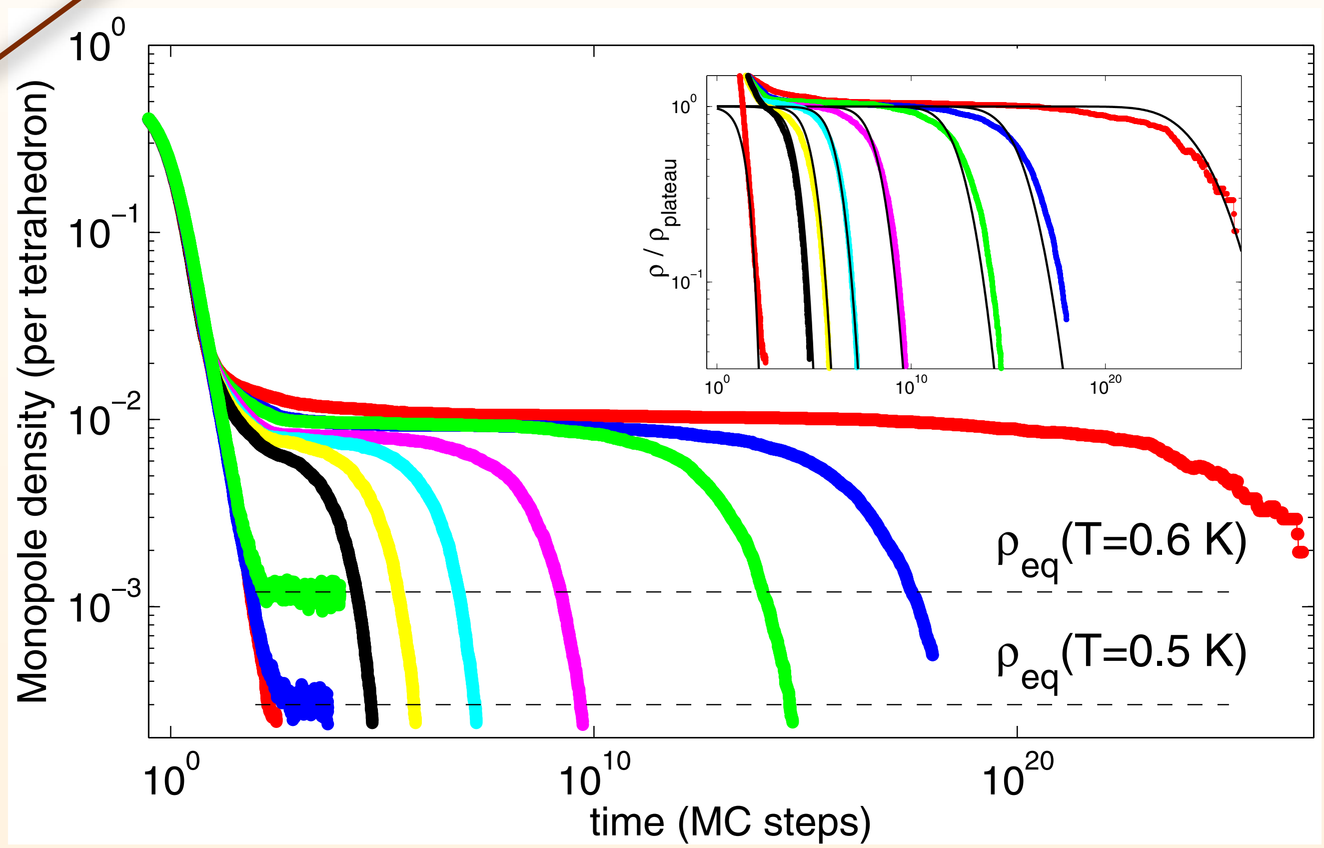
Quench into the PM phase.

Density of defects

Dynamical arrest for $d < 10^{-4}$



2D vertex model



3D pyroclhore spin ice with dipolar interactions

$$d = e = 10^{-1}, \ldots, 10^{-4}, \ldots, 10^{-8}$$
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Levis & Cugliandolo 2011

Castelnovo, Moessner & Sondhi 2012

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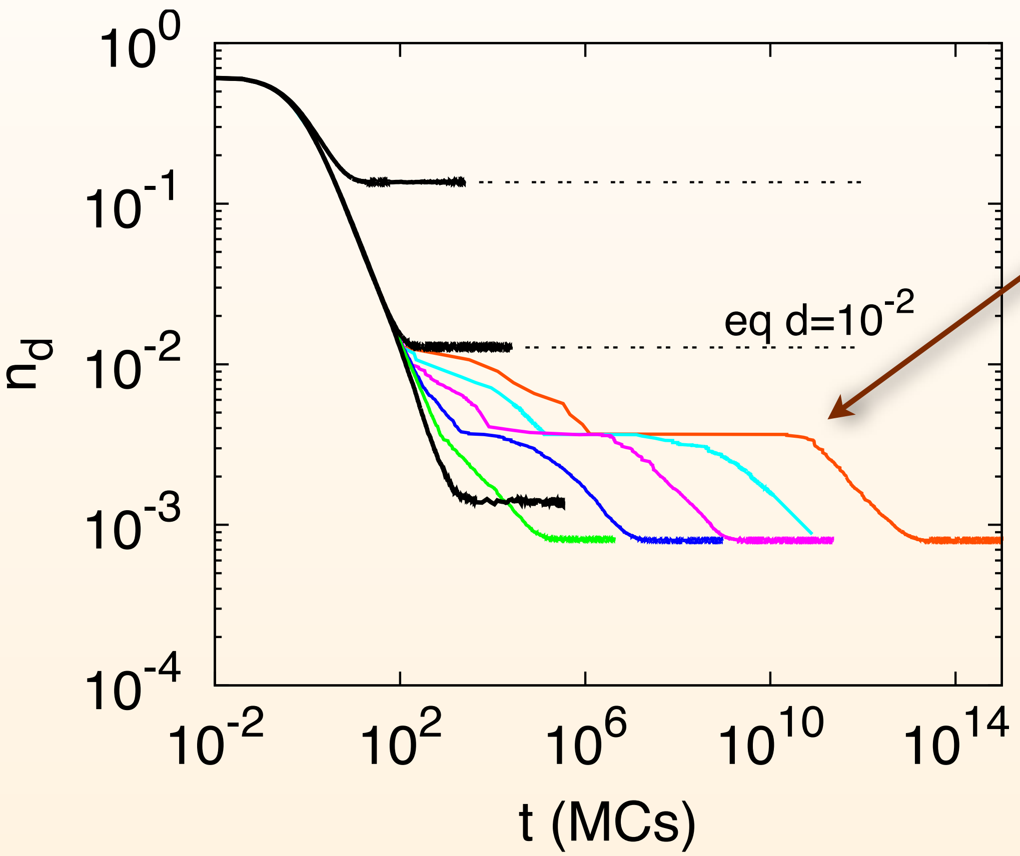

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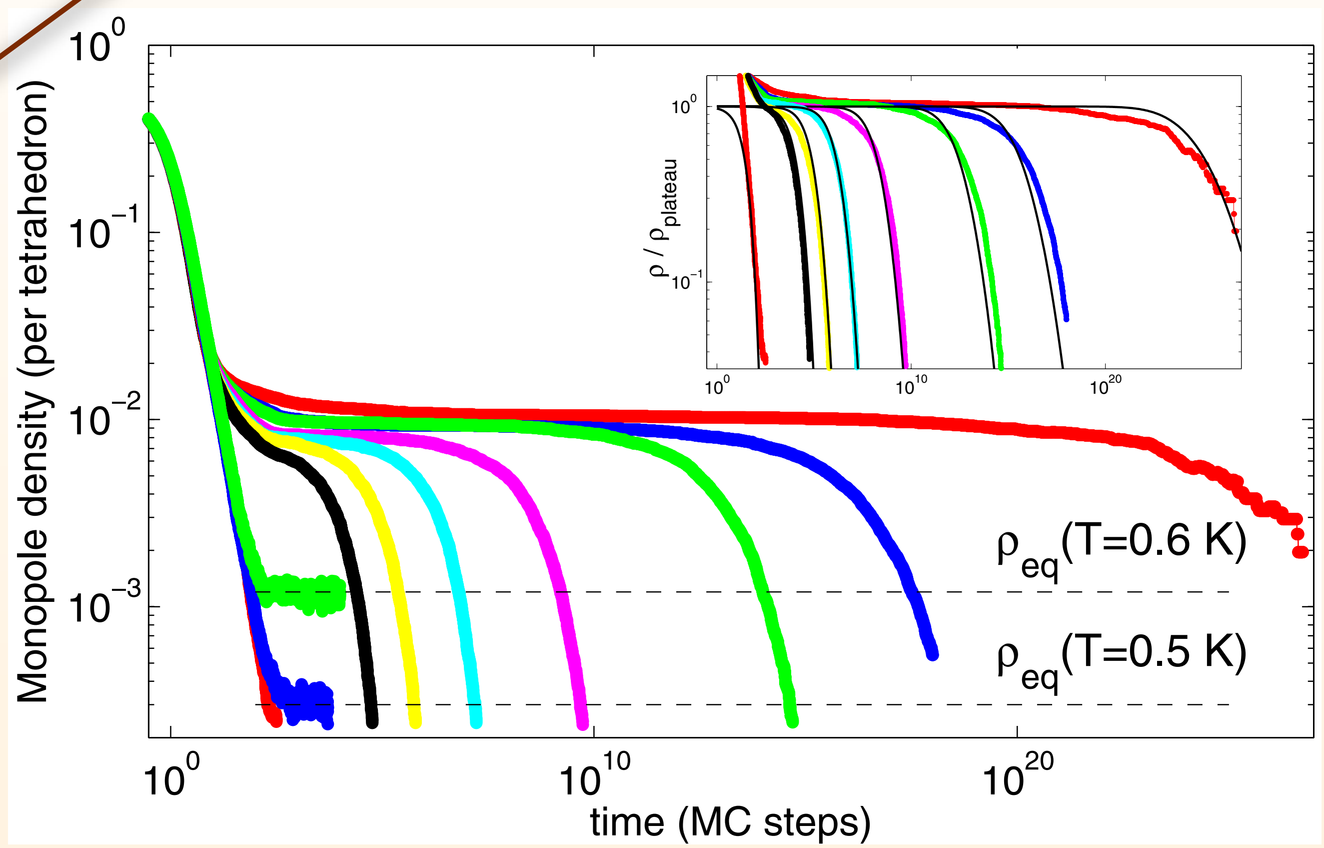

3D pyroclhore spin ice with dipolar interactions

$$d = e = 10^{-1}, \ldots, 10^{-4}, \ldots, 10^{-8}$$
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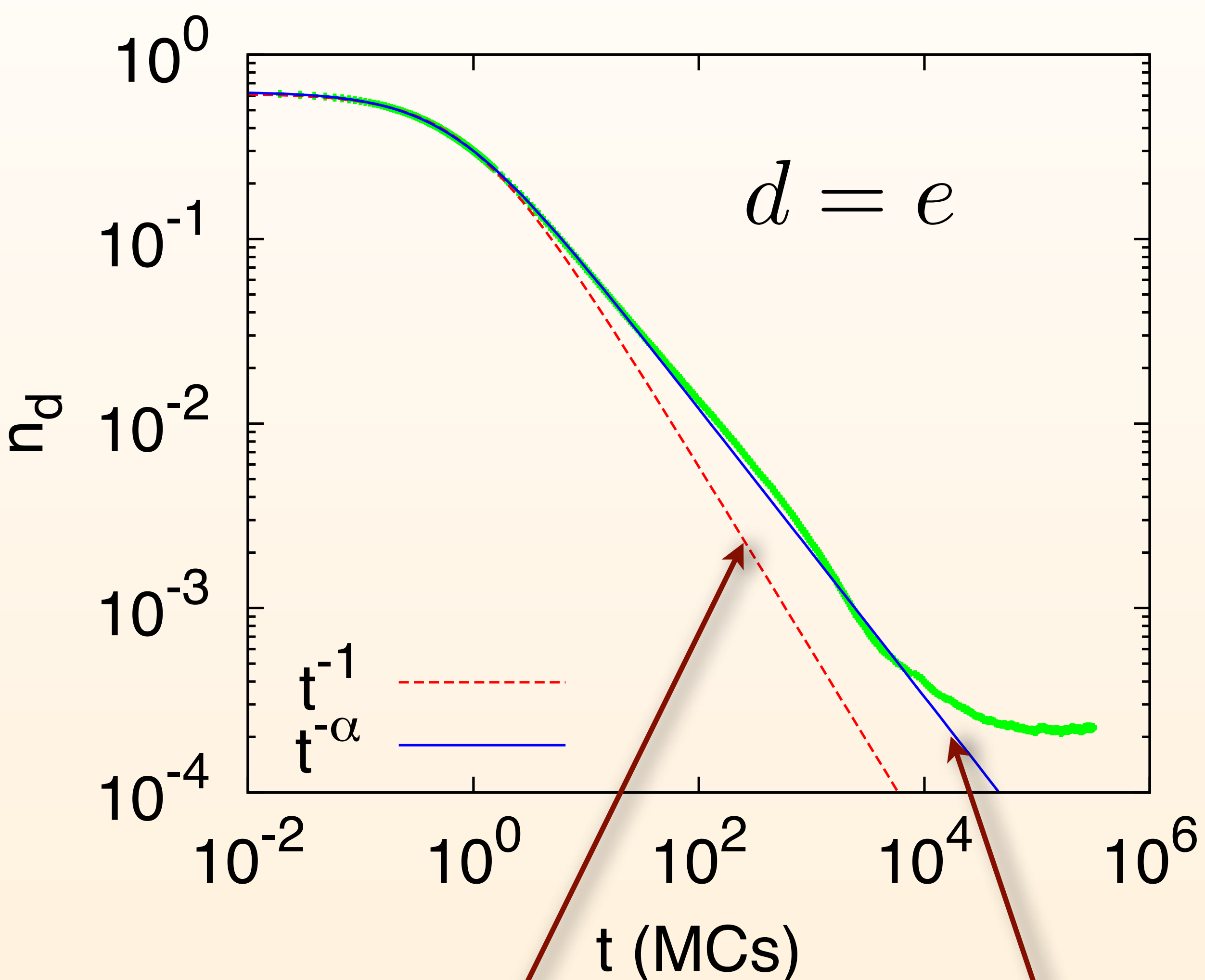
$$T = 0.6\,\mathrm{K}, \ldots, 0.025\,\mathrm{K}$$
$$N = 8192$$

Levis & Cugliandolo 2011

Castelnovo, Moessner & Sondhi 2012

$\Rightarrow$ Due to finite size ? $d = e$? are long range interactions needed ?

Quench into the PM phase.



NN spin ice model described by a mean field reaction-diffusion model

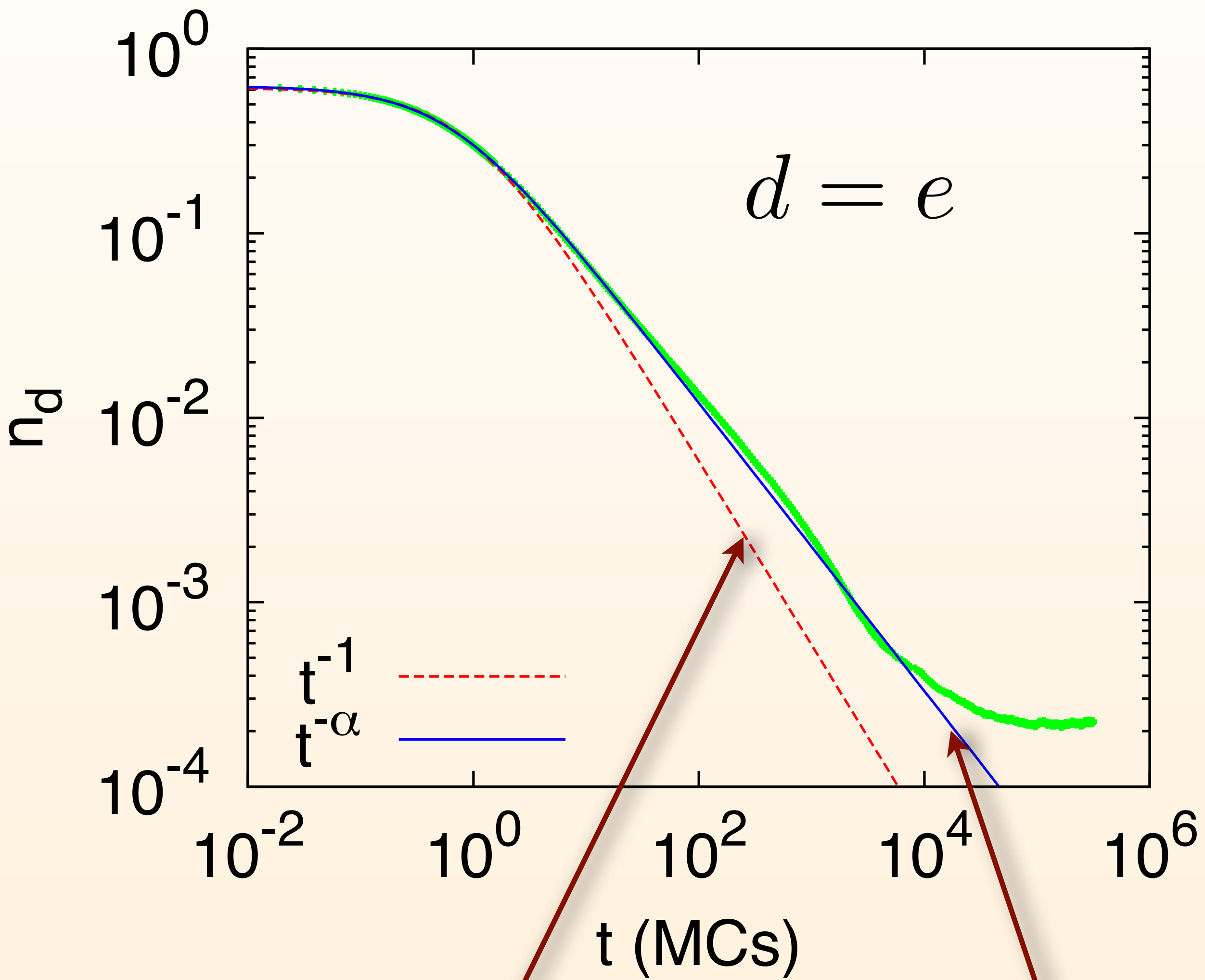
$\alpha \approx 0.78$

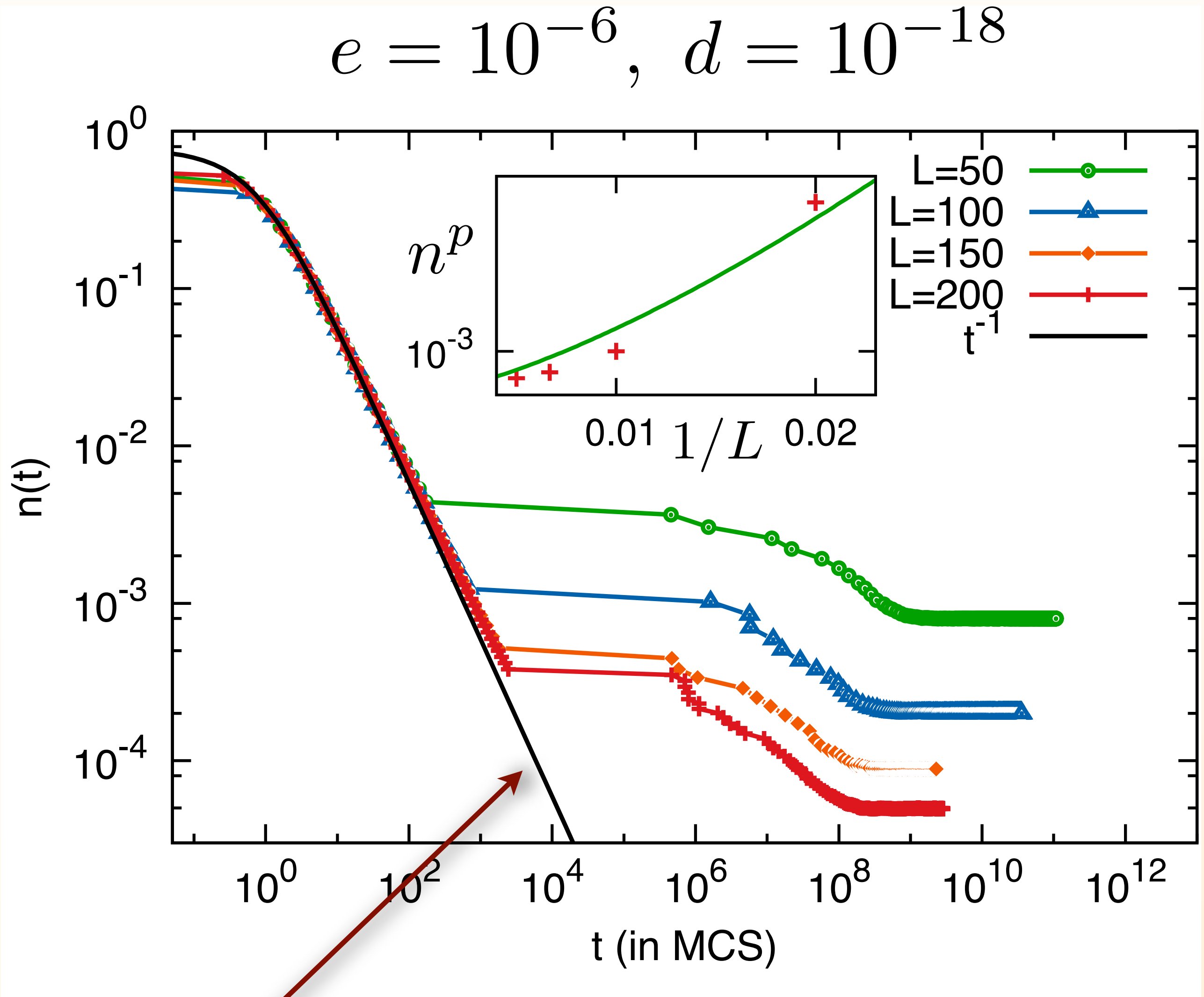
$$n(t) = \frac{\rho_0}{(1+\Omega t)^\alpha}$$

Levis & Cugliandolo 2011

Out-of-equilibrium.

Quench into the PM phase.





$\alpha \approx 0.78$

$\alpha \approx 1$

$d < e^2$

$\Rightarrow\ (2q) \to (q) + (q)$

spontaneously.

NN spin ice model described by a mean field reaction-diffusion model

$$n(t) = \frac{\rho_0}{(1+\Omega t)^\alpha}$$

Levis & Cugliandolo 2011

Out-of-equilibrium.

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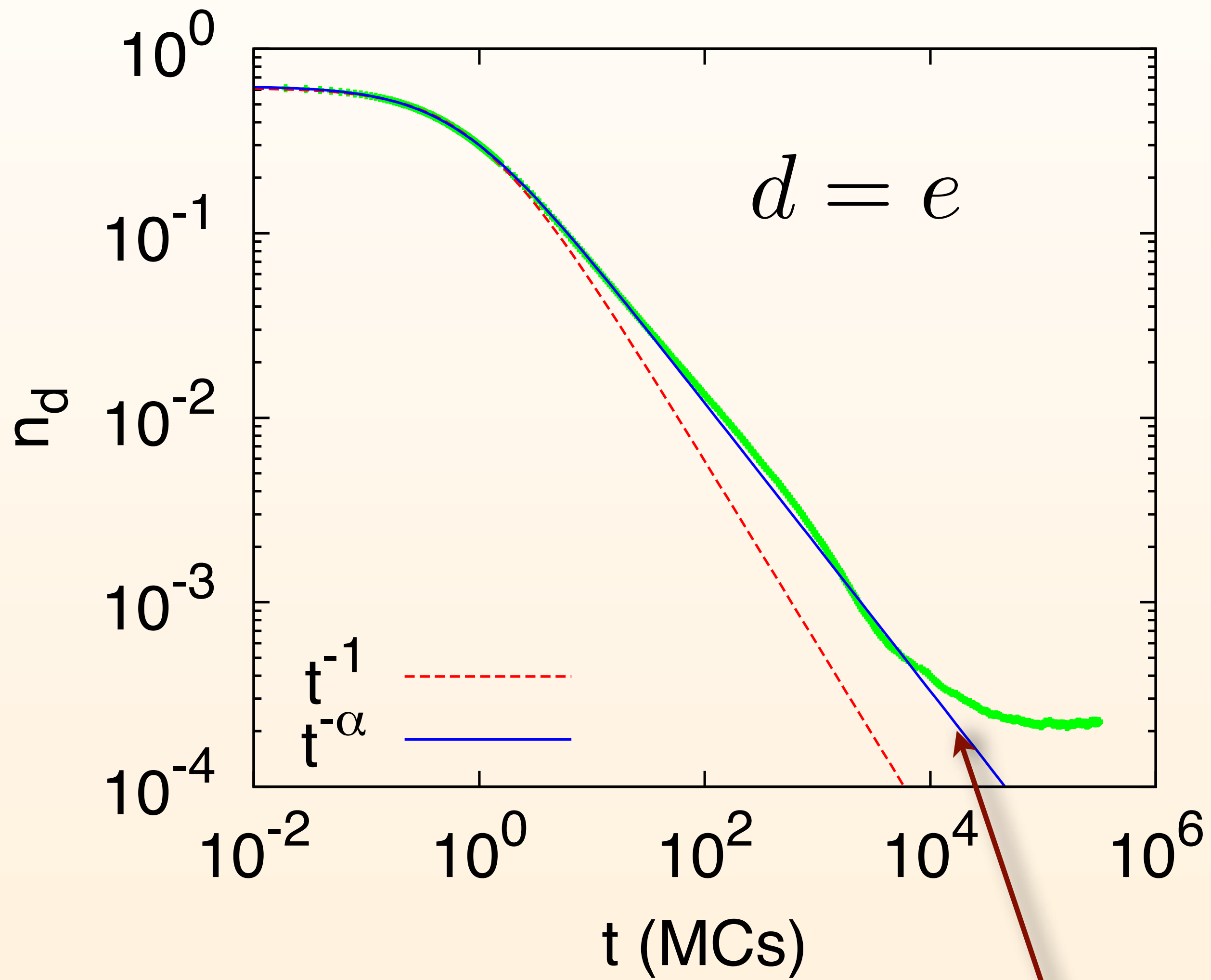
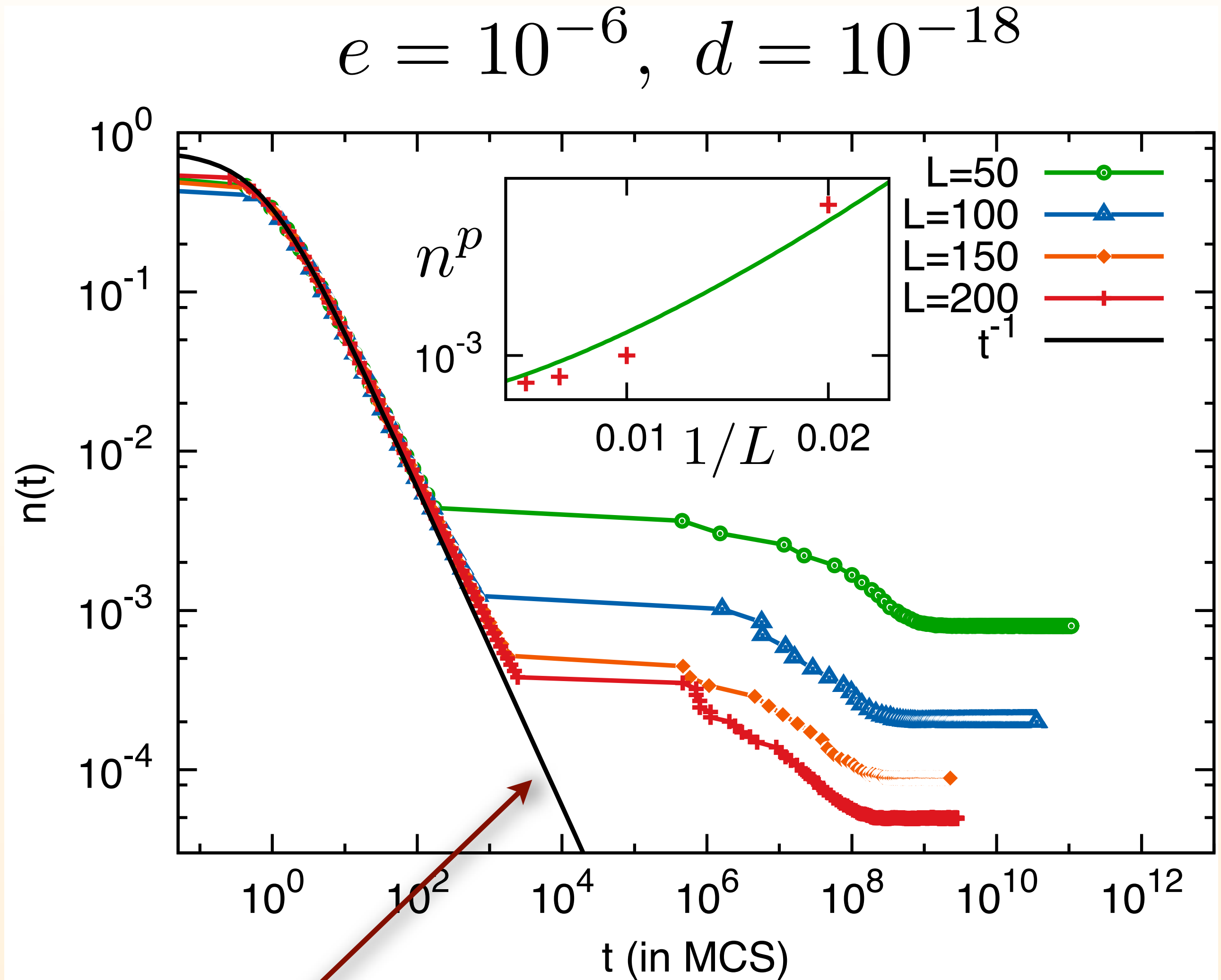




$\alpha \approx 0.78$

$\alpha \approx 1$

$$d < e^2$$

$$\Rightarrow \ (2q) \to (q) + (q)$$

spontaneously.

Same power law as in 3D spin-ice.

The decay power depends on the value of the parameters.

Levis & Cugliandolo 2011

Out-of-equilibrium.

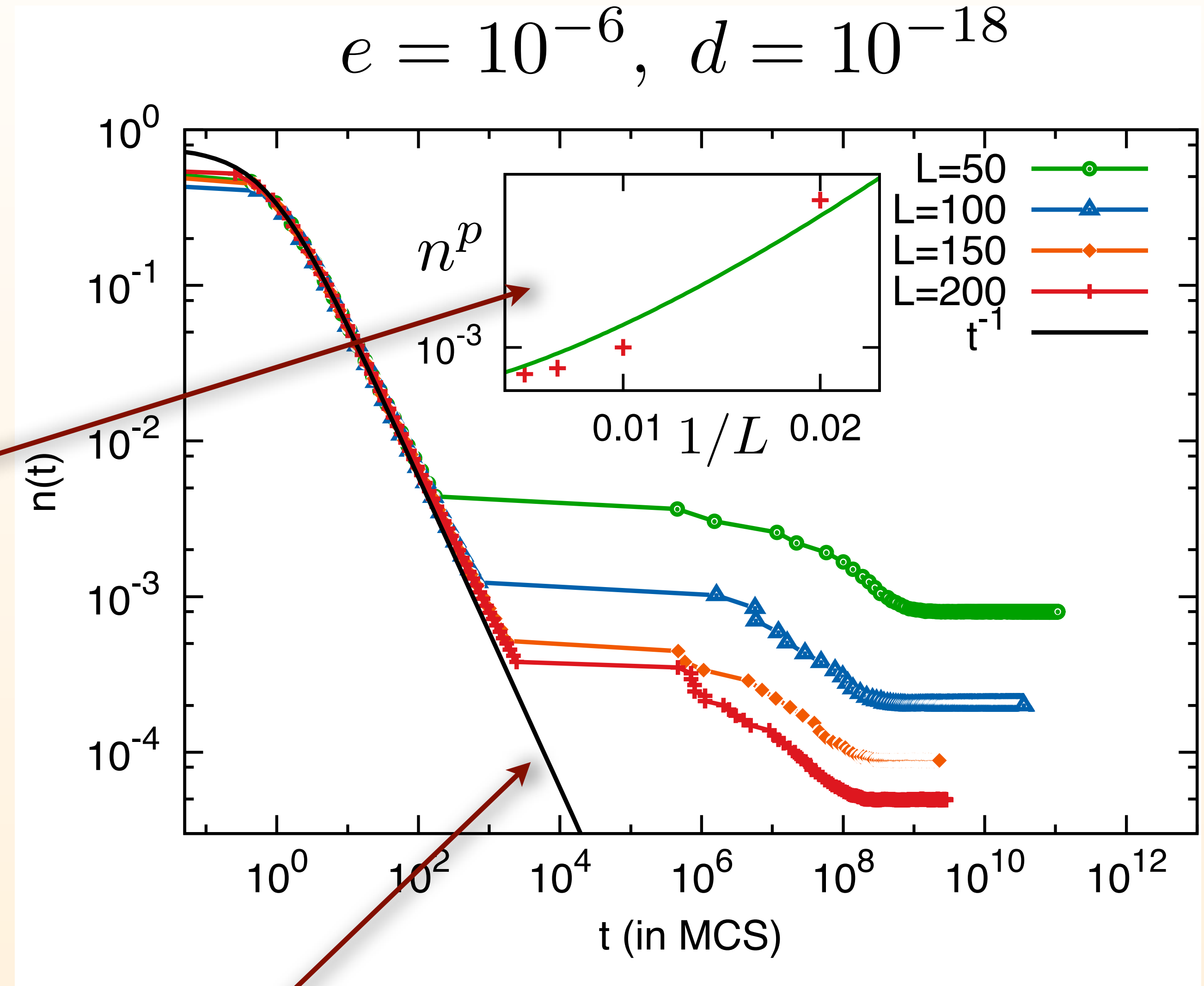
Quench into the PM phase.

Size dependence

The dynamical arrest is size dependent:

the height and the time spend in the plateau decrease for larger systems.

At the thermodynamic limit ?



$\alpha \approx 1$

$$d < e^2$$

$$\Rightarrow\ (2q) \to (q) + (q)$$

spontaneously.

Out-of-equilibrium.

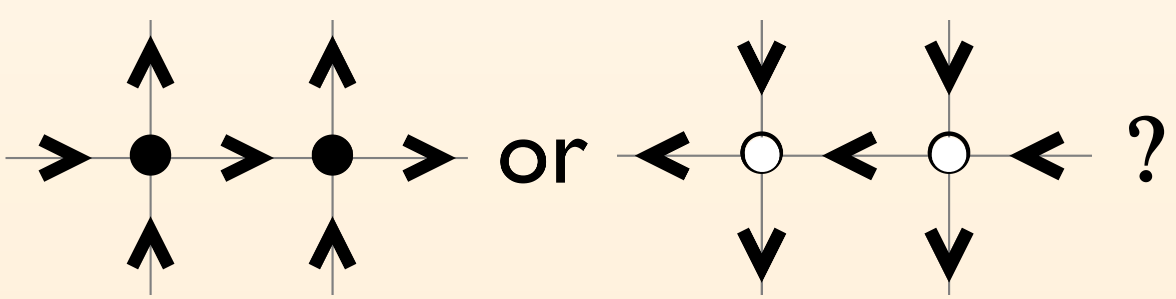
Quench into the FM phase.

initial configuration:

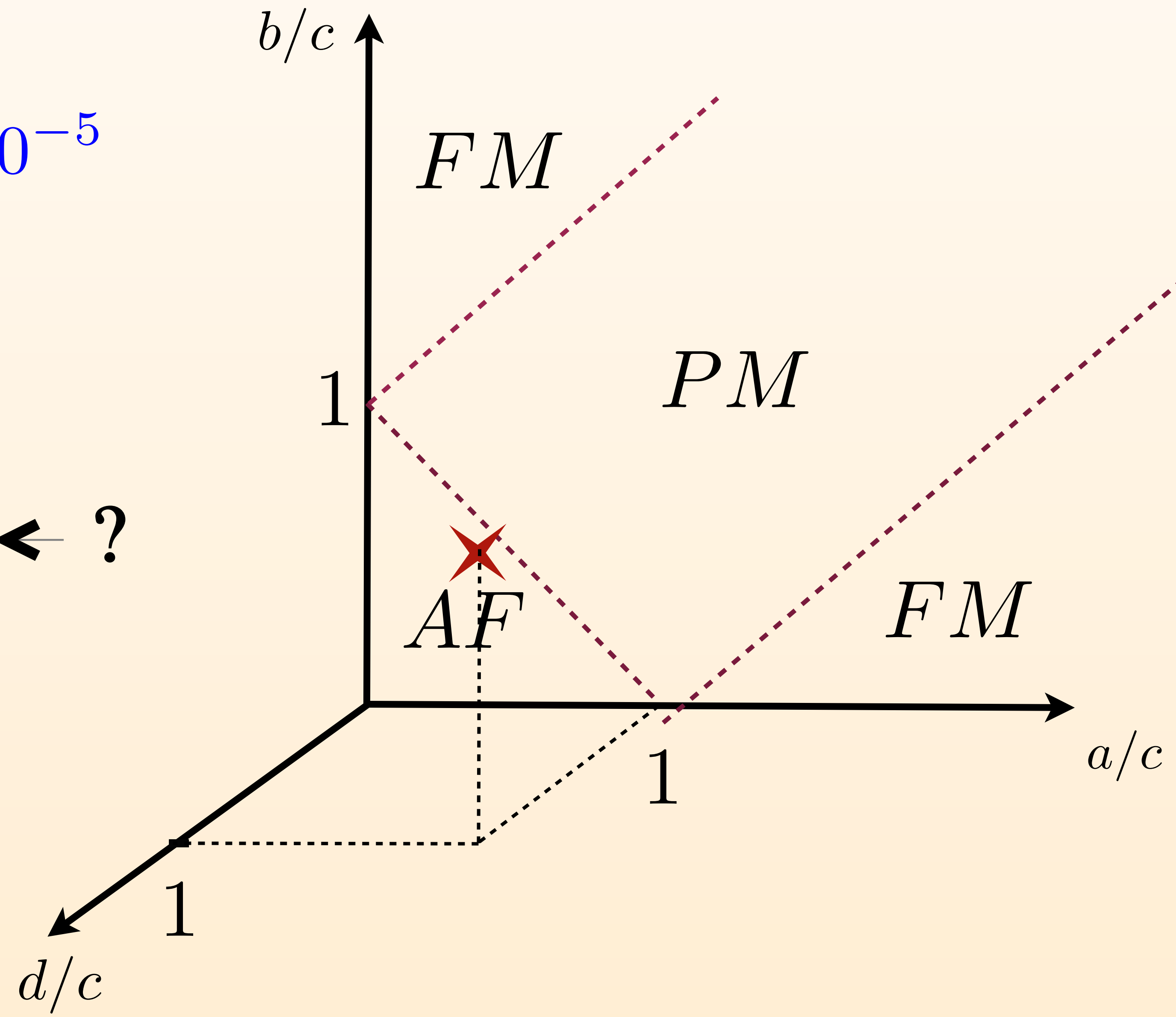
$$a = b = c = d = e = 1 \ (T = \infty)$$

$\Big\downarrow$ at t=0 (instantaneously)

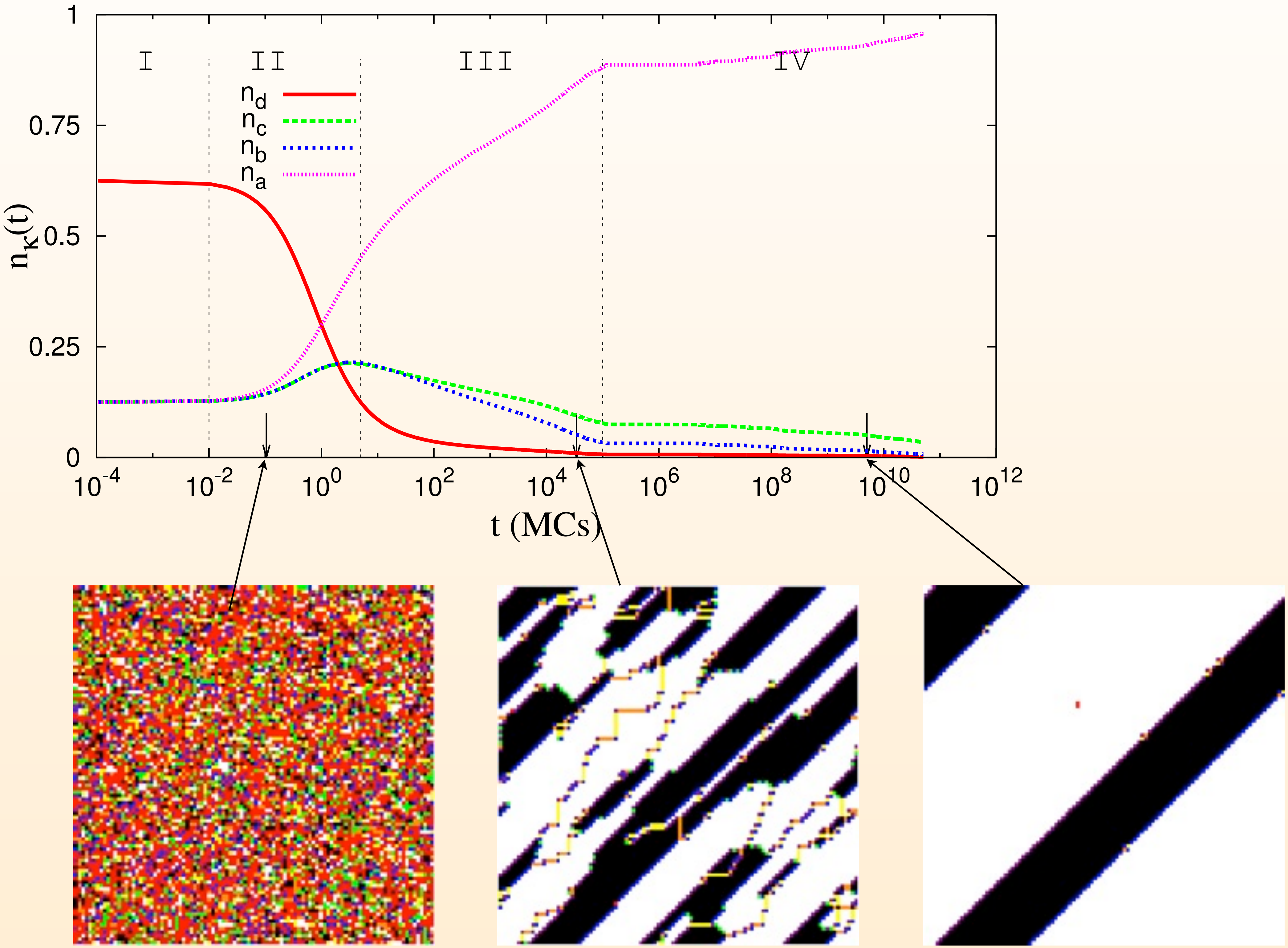
$$a = 5, \ b = c = 1, \ d = e = 10^{-5}$$
$$(T < T_c)$$



$\longrightarrow$ *slow relaxation*



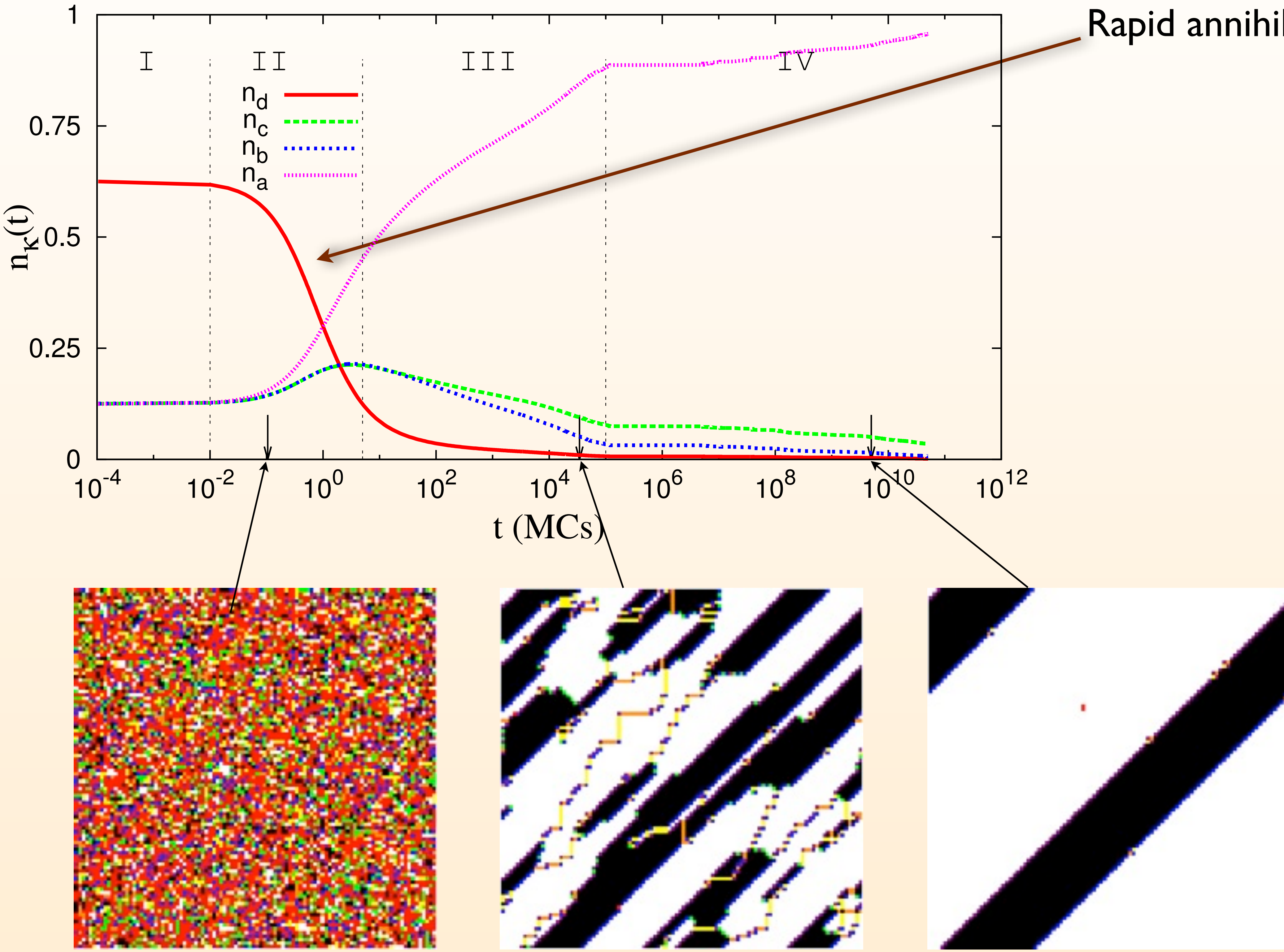
Quench into the FM phase.



Levis & Cugliandolo 2011

Out-of-equilibrium.

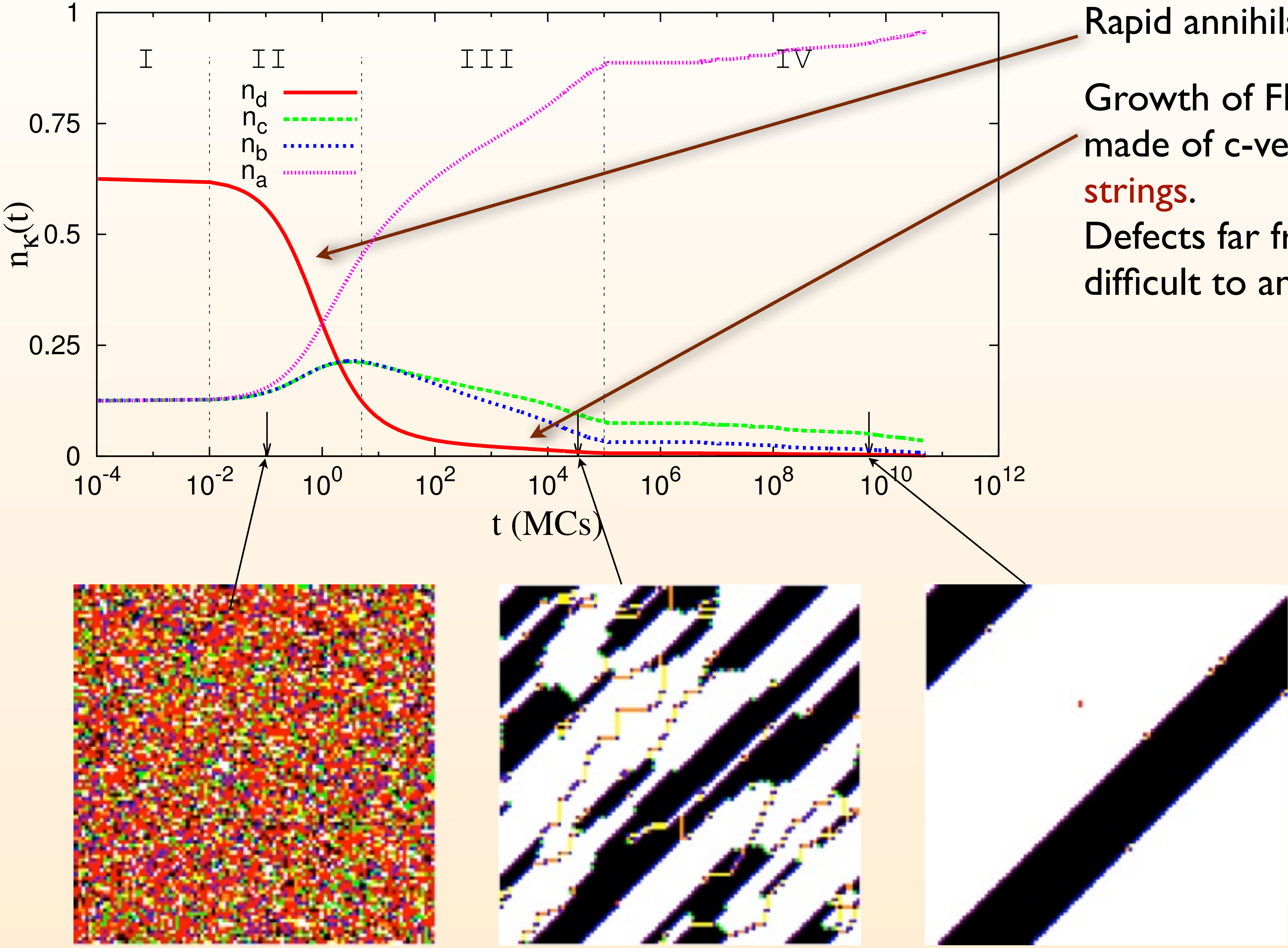
Quench into the FM phase.



Levis & Cugliandolo 2011

Out-of-equilibrium.

Quench into the FM phase.

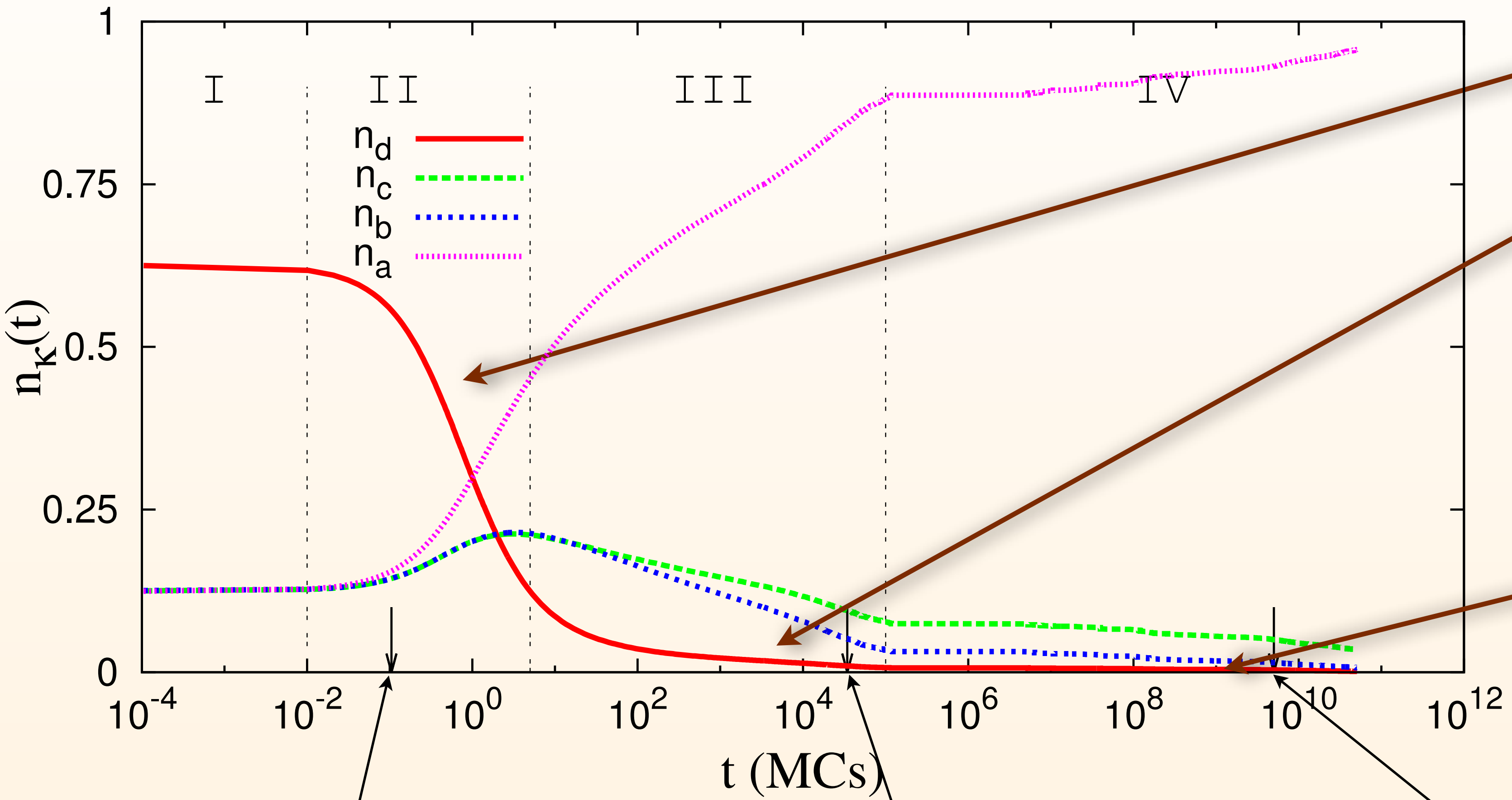


Rapid annihilation of defects

Growth of FM domains with walls made of c-vertices and linked by strings.
Defects far from each other, difficult to annihilate: slow dynamics.

Levis & Cugliandolo 2011

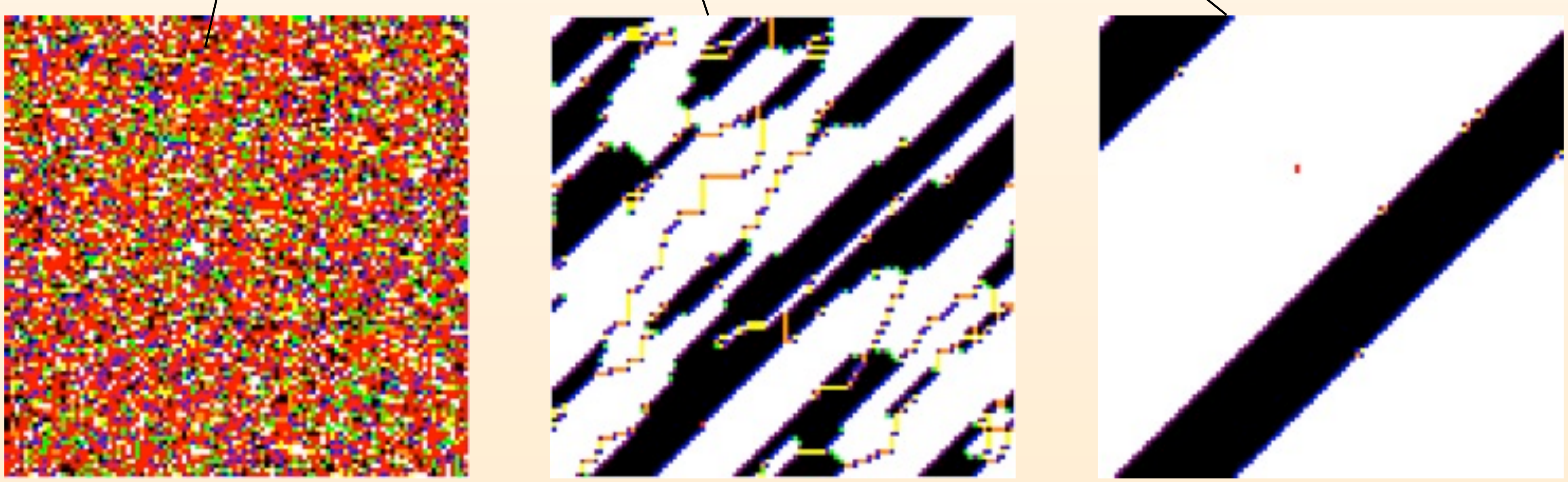
Quench into the FM phase.



Rapid annihilation of defects

Growth of FM domains with walls made of c-vertices and linked by strings.
Defects far from each other, difficult to annihilate: slow dynamics.

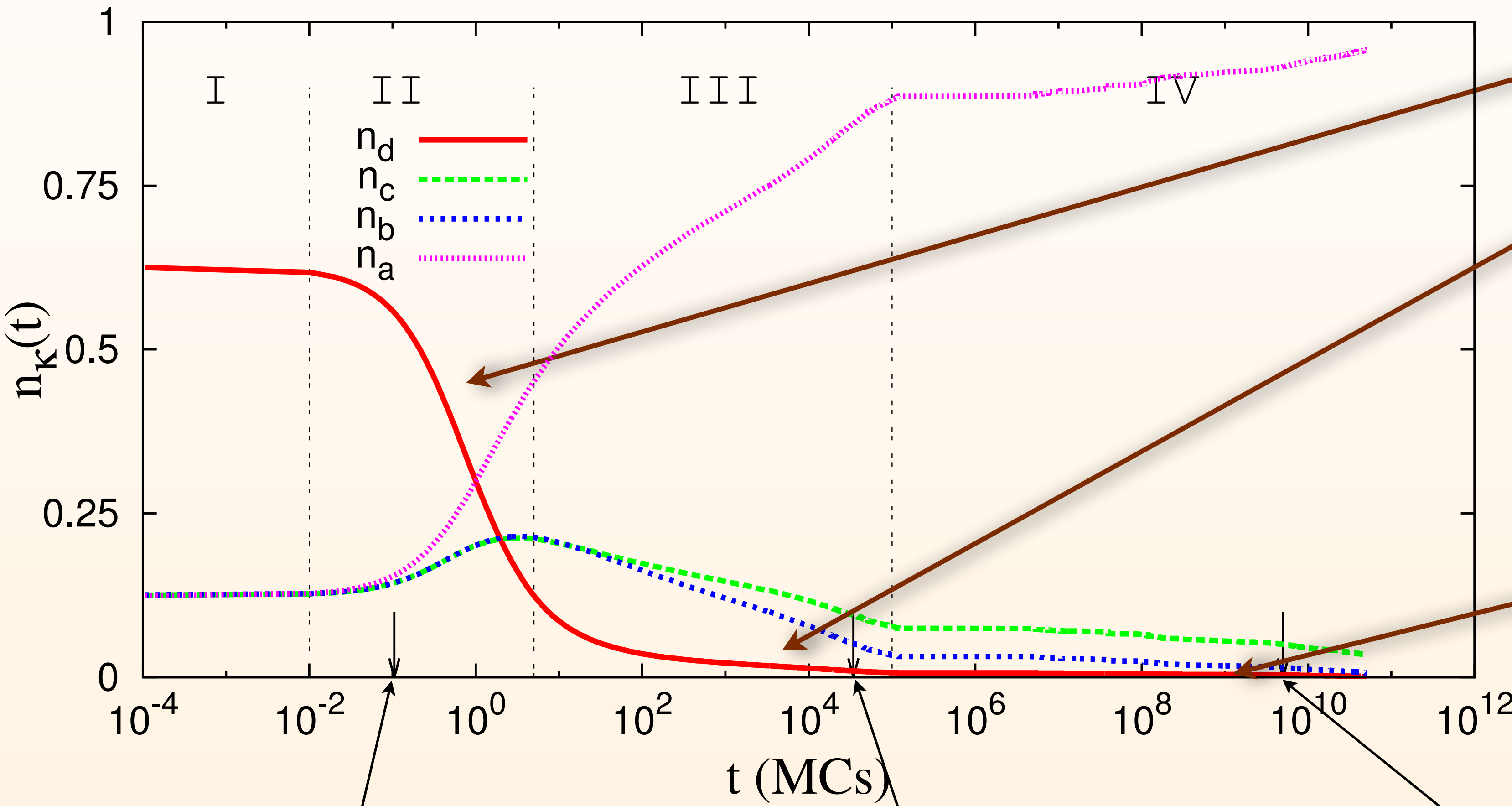
Very stable domains
= Parallel bands



Levis & Cugliandolo 2011

Out-of-equilibrium.

Quench into the FM phase.



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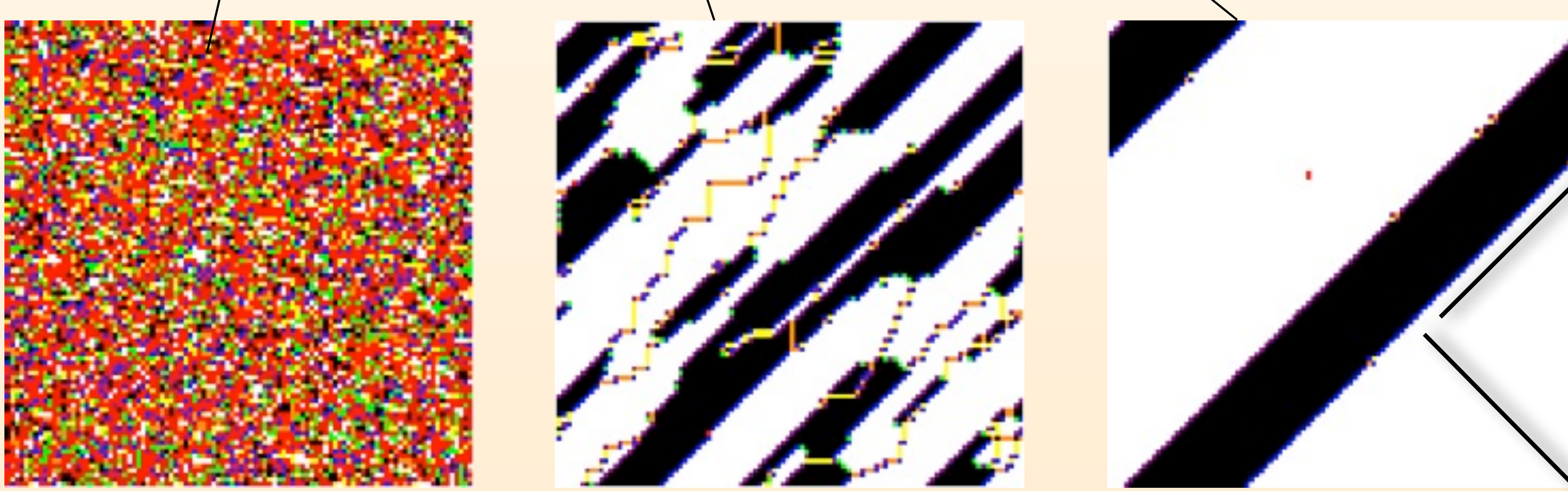


$L_{\parallel} \sim t^{1/2}$

$L_{\perp}$?

Levis & Cugliandolo 2011

Quench into the AF phase.

Same procedure

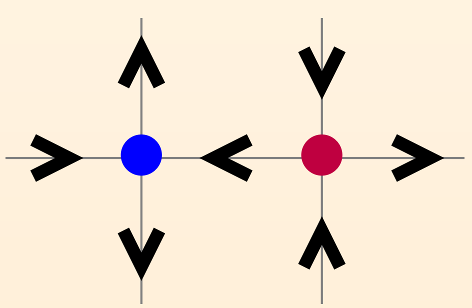
initial configuration:

$a = b = c = d = e = 1 \; (T = \infty)$

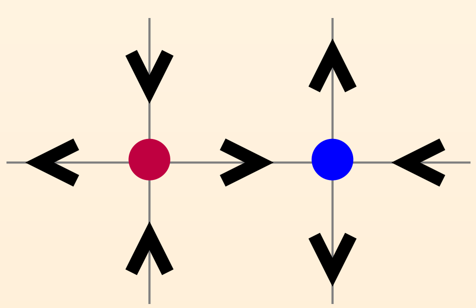
at t=0 (instantaneously)

$a = b = 0.1\,;\; c = 1,\;\; d = e = 10^{-5}$

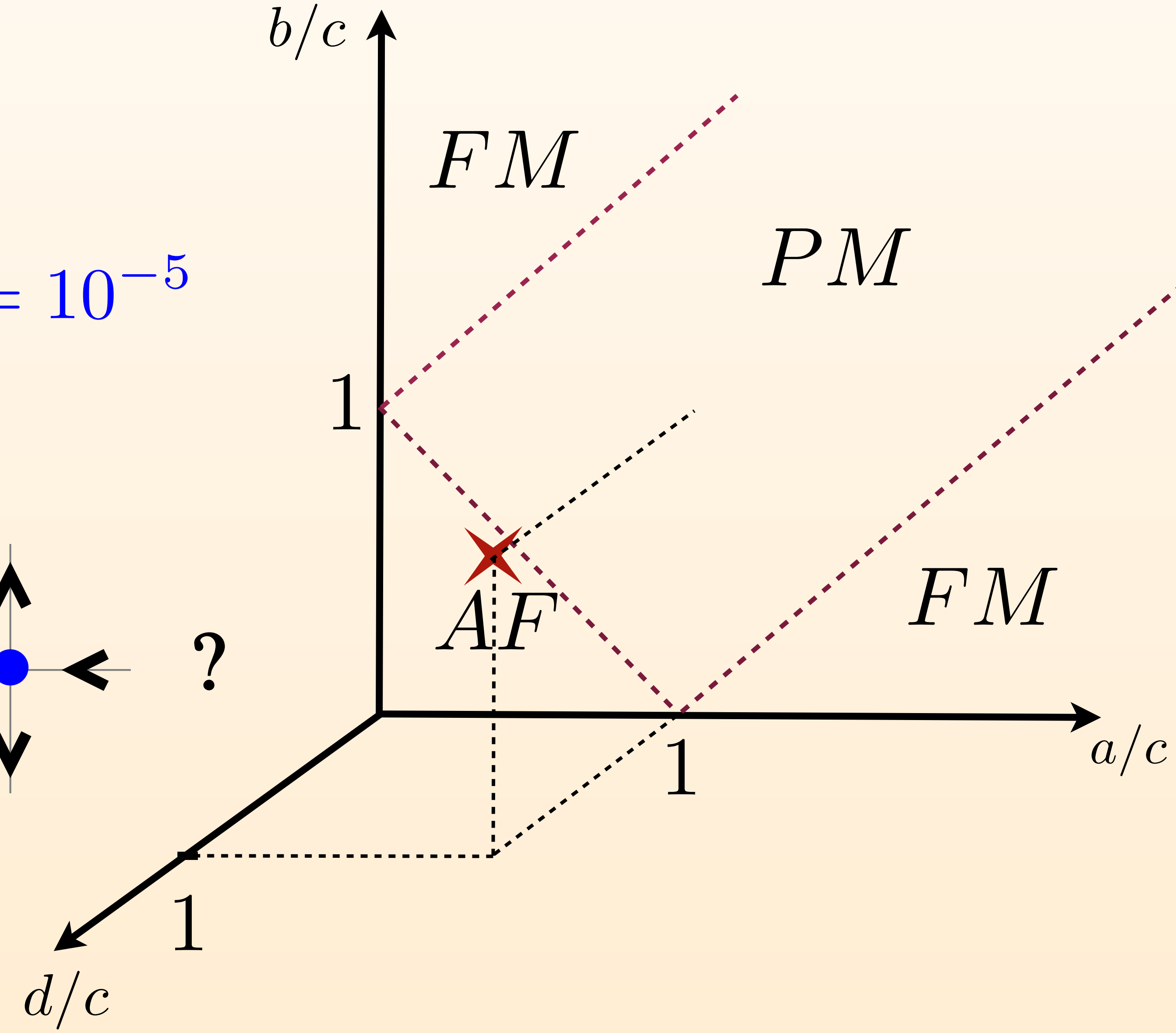
$(T < T_c)$



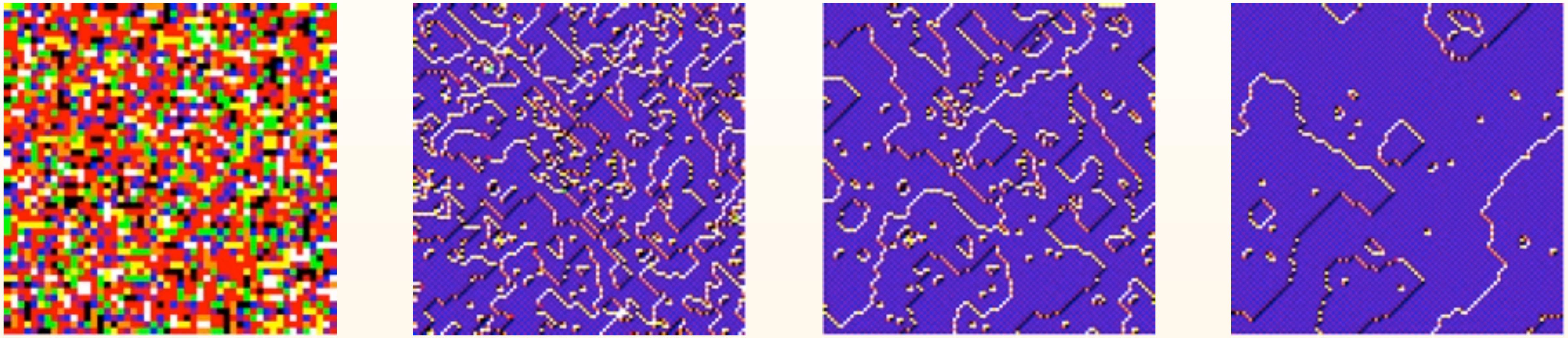
or



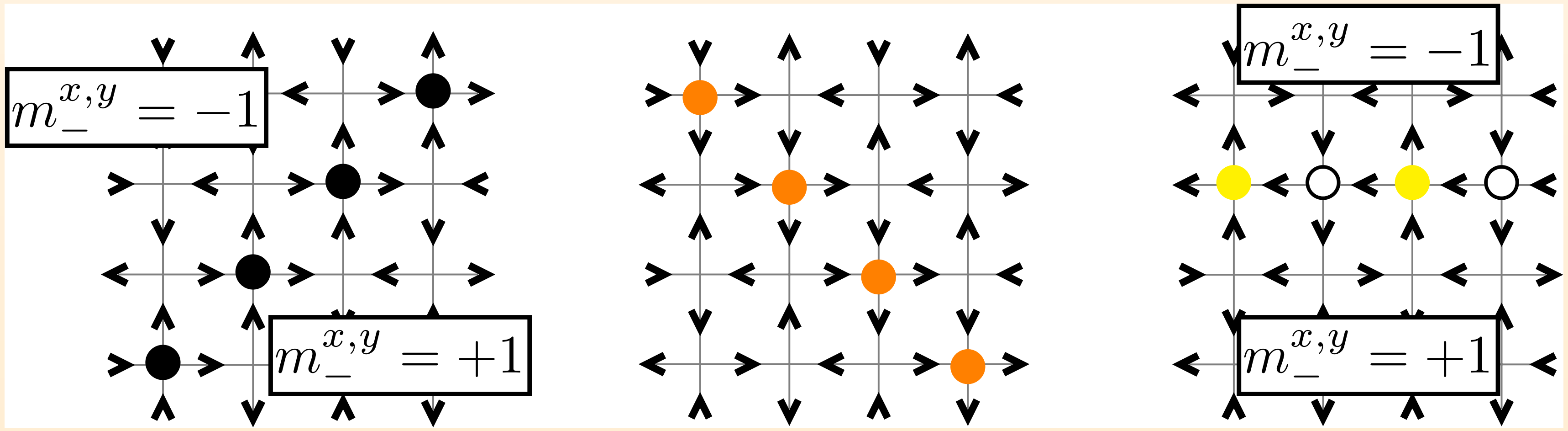
?



Quench into the AF phase.



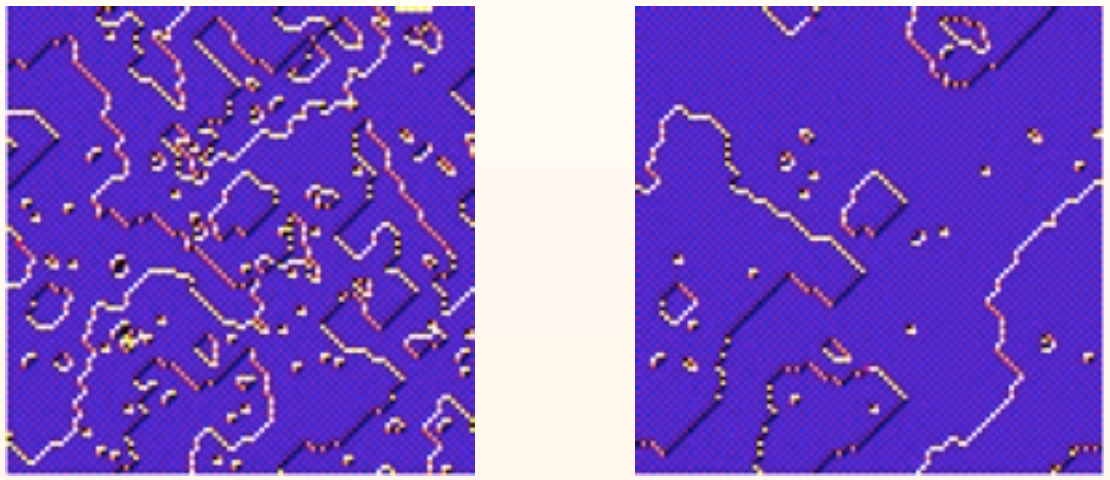
Growth of AF domains with walls
made of FM-vertices (a- b-type).
Defects on domain walls,
far from each other,
difficult to annihilate: slow dynamics.



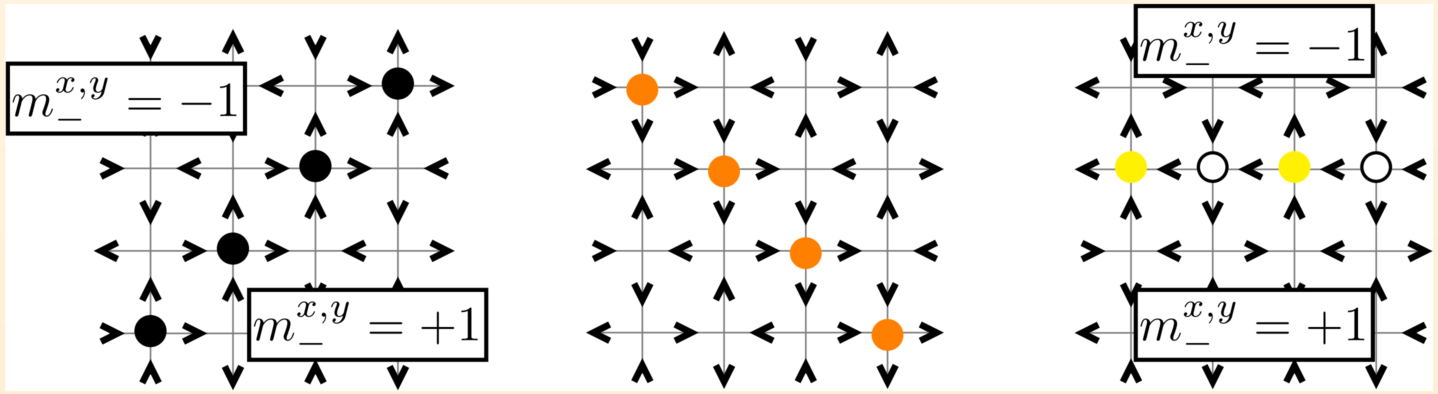
Levis & Cugliandolo 2011

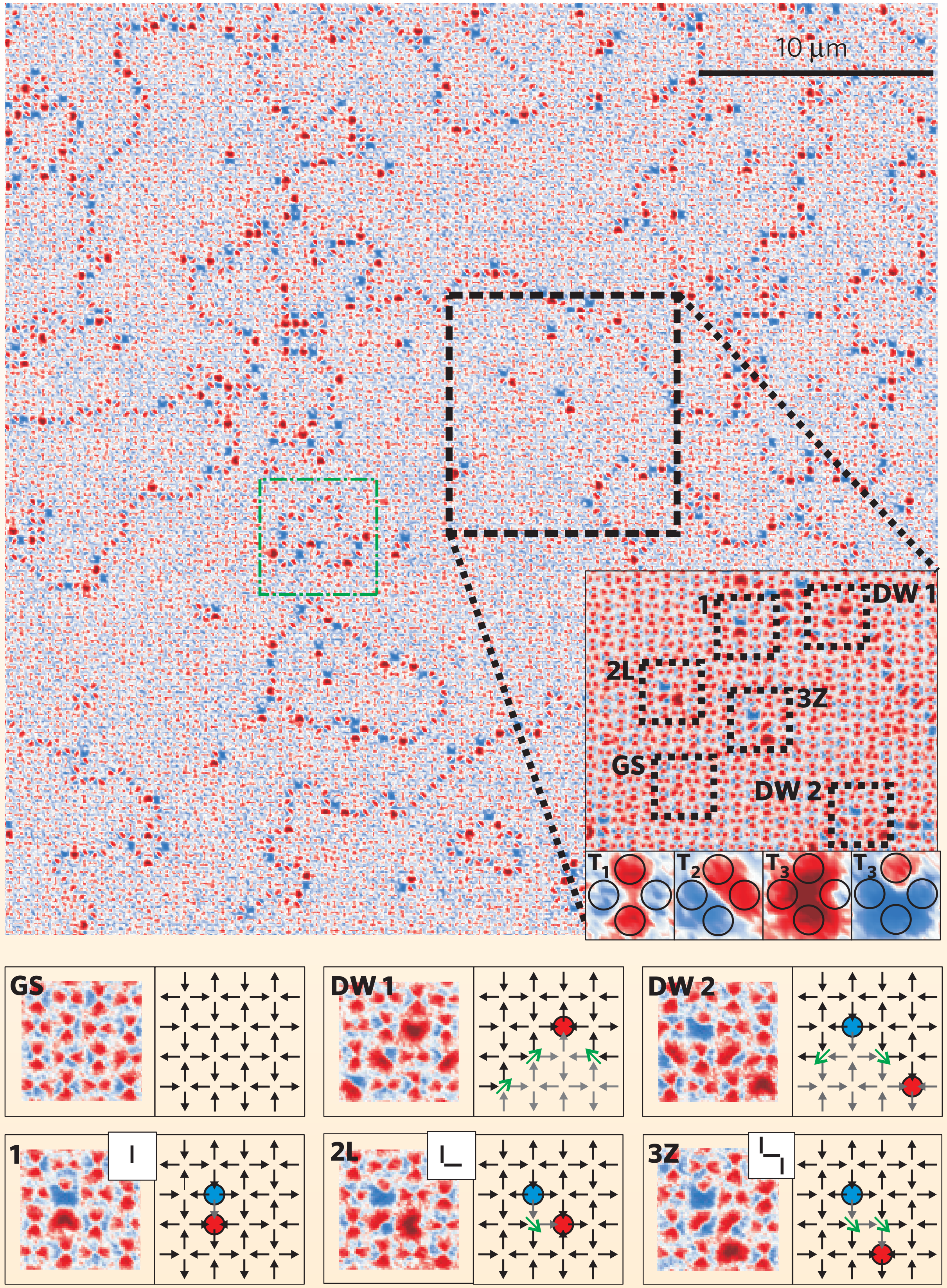
Out-of-equilibrium.

Quench into the AF phase.



Same kind of domain walls observed in as-grown ASI samples.





From **Morgan** *et al.*, Nature Phys. 7 2011.

Conclusion out-of-equilibrium.

1. Dynamical arrest appears for all type of quenches for $e < 10^{-4}$.
Why ? i) presence of 4in 4out vertices ? **NO**
ii) presence of winding loops ? **NO**
iii) finite-size effects? **YES** ? Arrhenius barrier $\tau \sim e^{-2}$

2. The evolution of the density of defects follows a power-law $n_d(t) \sim t^{-\alpha}$.
The exponent depends on the value of the parameters (how?).

3. Stripes of competing FM order with two different growing lenghts:
anisotropic coarsening.
Rich microscopic dynamical mechanisms.

4. Coarsening dynamics in the AF regime:
artificial spin-ice samples?
effective 'cooling rate' during deposition? vs. MC simulated annealing?
time-dependent correlations? geometric properties of the domain walls?

Conclusion & outlook.

Conclusion & outlook.

1. Investigate the variation of critical exponents in the 16V model by a real-space Renormalisation Group approach.

2. Extend the mappings between integrable vertex models (6V and 8V) into quantum spin chains (XXZ and XYZ resp.) to the unconstrained 16V model. Then, use our understanding on the classical model to predict the behaviour of the quantum system?

3. Link between configurational temperatures in ASI and the canonical temperature? In experiments:
   - properties of the domain walls?
   - thermal equilibration?
   - growth rate (link with a cooling rate)

4. Defects' motion in 2D (square or hexagonal).

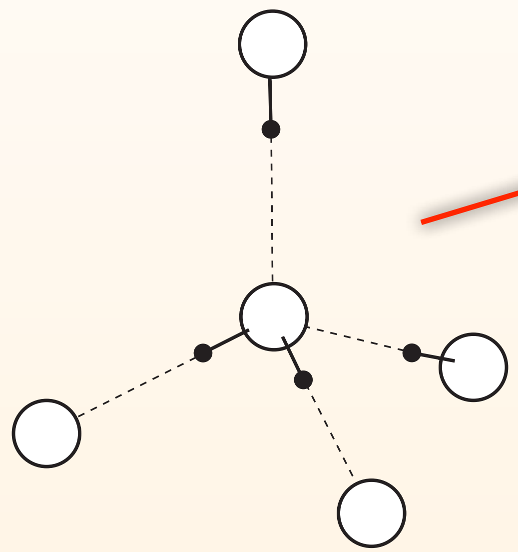
5. Effect of boundary conditions on the dynamics of hardly constrained models. Topological glass ?

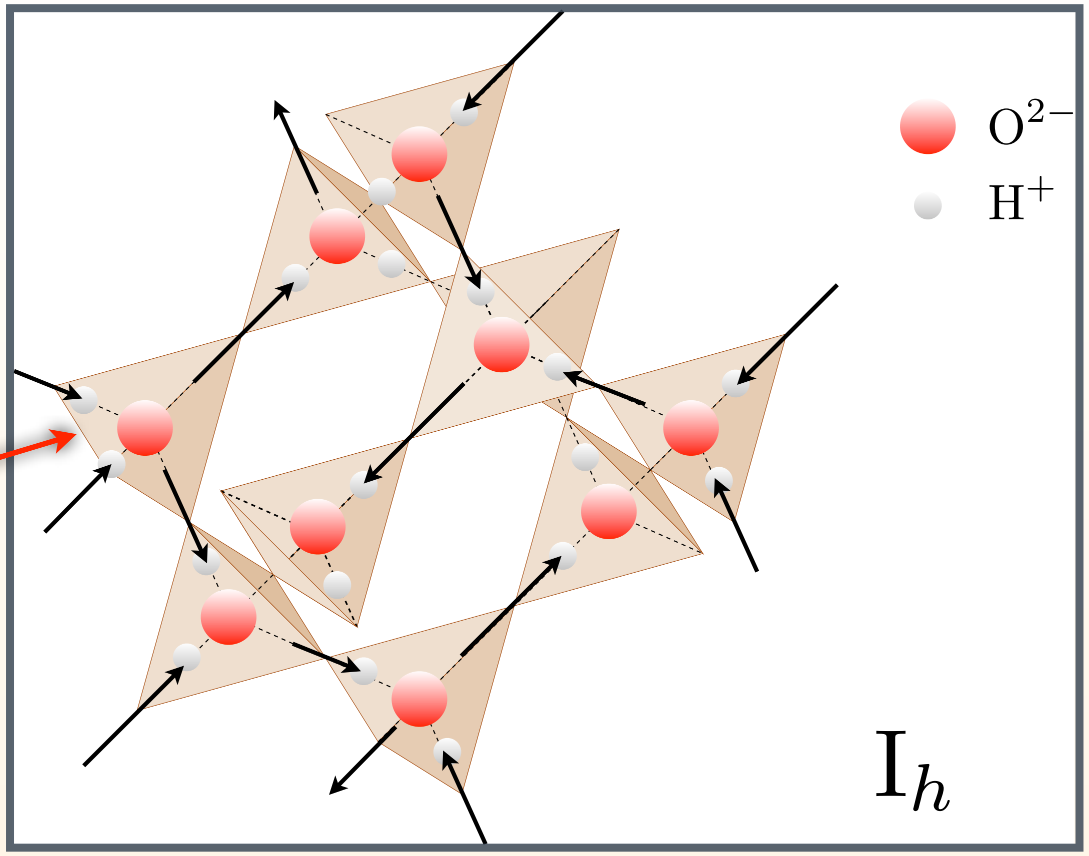
END OF EVERYTHING

Geometrical frustration

Crucial example: Water Ice

Single unit:





Experimental evidence of an extensive entropy at T=0 $\Longleftarrow$ Predicted by Pauling's model

Giauque & Stout 1936 **Pauling 1935**

$\longrightarrow$ each bond carries a dipolar moment $\vec{\mu}$ verifying the ***ice-rules*** $=$ at each ***vertex*** (oxygen atoms) *two in - two out*

$\Rightarrow$ **SIX** possible vertices

Ground state : $\vec{\nabla}.\vec{\mu} = 0$ Ice-rule vertices are favoured $\Rightarrow$ Geometrically frustrated

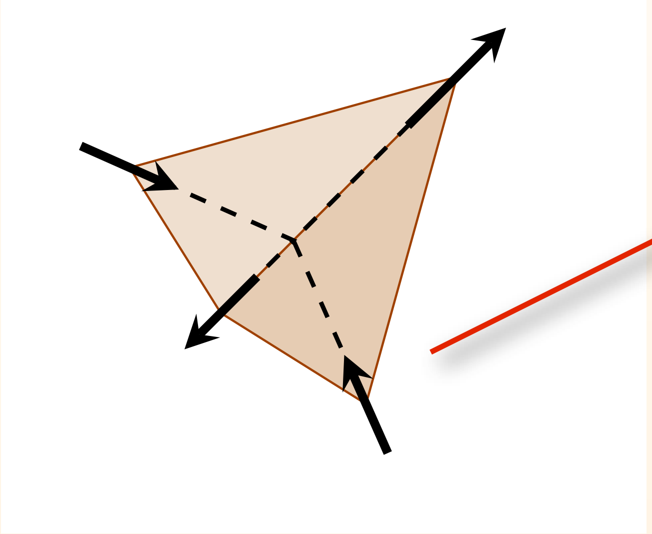
Introduction.

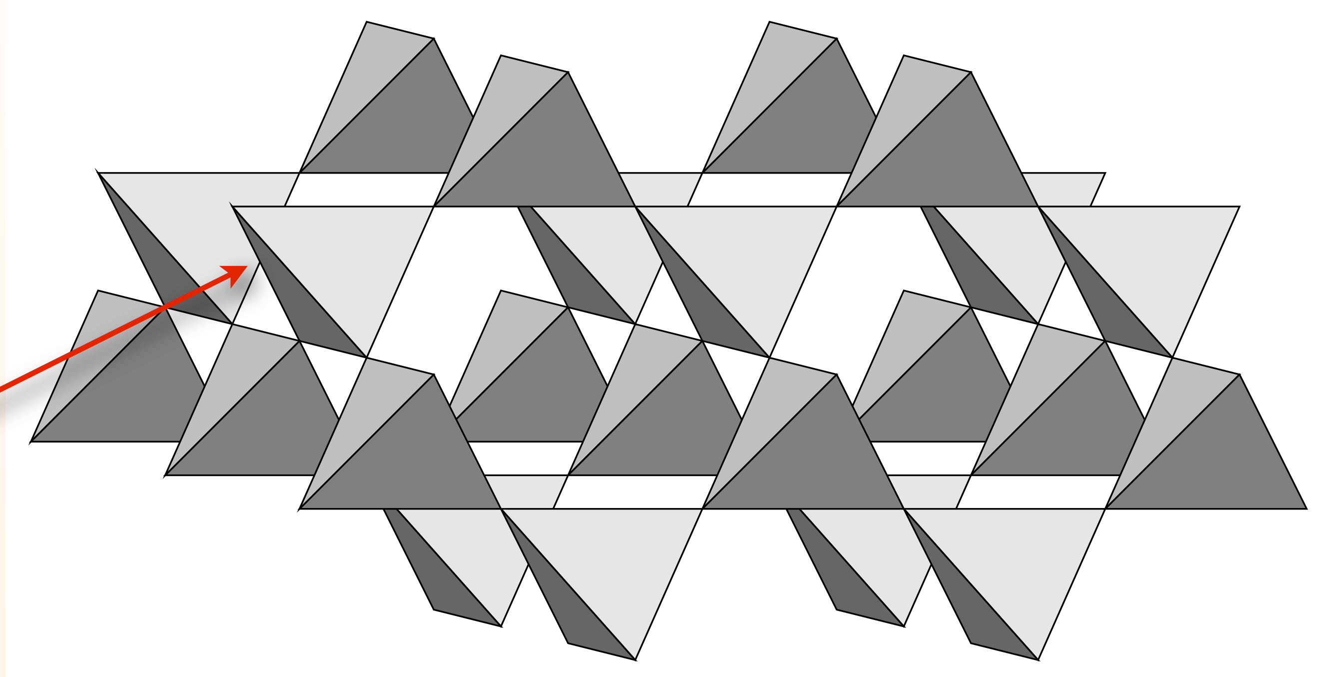
Geometrical frustration

in spin ice

Harris, Bramwell, McMorrow, Zeiske & Godfrey 1997

Pyroclhore lattice
+ magnetic moments





classical Ising spins pointing in the local direction connecting a site with the centre of its tetrahedron

DSI: nn FM + *dipolar interactions*

$$H = -J \sum_{<i,j>} \vec{S}_i.\vec{S}_j + Da^3 \sum_{i<j} \frac{\vec{S}_i.\vec{S}_j}{r_{ij}^3} - 3\frac{(\vec{S}_i.\vec{r}_{ij})(\vec{S}_j.\vec{r}_{ij})}{r_{ij}^5}$$

NN contribution: $H \approx \frac{J+5D}{3} \sum_{<i,j>} \sigma_i \sigma_j$ with: $\sigma_i = \pm 1$

AF

«in» or «out»

$\Rightarrow$ GS: 2 in-2 out (ice-rules)

Geometrically frustrated *Ferromagnet*

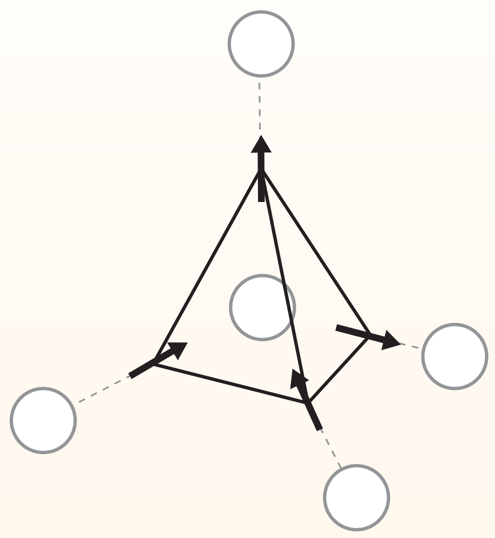
Zero-point entropy

Pauling's argument: N tetrahedra, $2N$ links (4 NN)

Pauling 1935 $\rightarrow$ $\Omega_0 = 2^{2N}$ (without ice-rules)

vertices indept. $\rightarrow \Omega_\infty = 2^{2N}(6/16)^N$

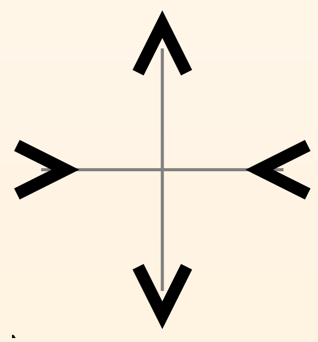
$\rightarrow S_\infty \approx 0.40R$



2D Lieb's **exact** result (Bethe *Ansatz*):

Lieb 1967

$$S_{2D} = Nk_B \frac{3}{2} \ln\left(\frac{4}{3}\right) \approx 0.43R$$



3D **Experiments** (both Spin and Water Ice):

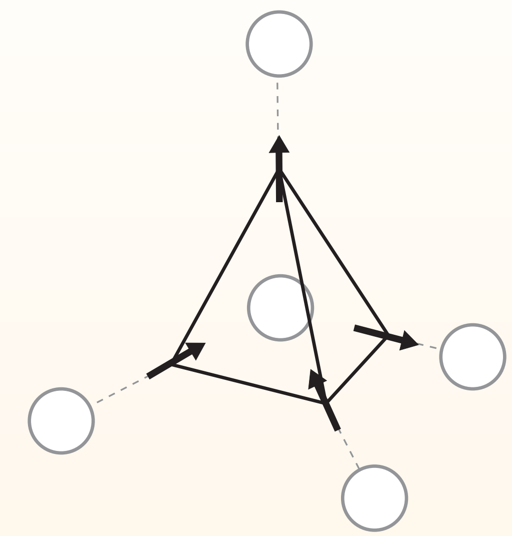
$S_w \approx 0.41R$

Giauque & Stout 1936

$S_{si} \approx 0.46R$

Ramirez, Hayashi, Cava, Siddharthan & Shastry 1999

Zero-point entropy



Pauling's argument: N tetrahedra, $2N$ links (4 NN)

Pauling 1935 $\rightarrow \quad \Omega_0 = 2^{2N}$ (without ice-rules)

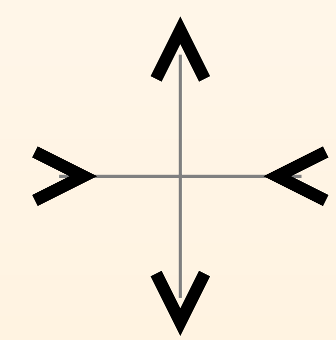
vertices indept. $\rightarrow \Omega_\infty = 2^{2N}(6/16)^N$

$\rightarrow S_\infty \approx 0.40R$

2D Lieb's **exact** result (Bethe *Ansatz*):

Lieb 1967

$$S_{2D} = Nk_B \frac{3}{2} \ln\left(\frac{4}{3}\right) \approx 0.43R$$



3D **Experiments** (both Spin and Water Ice):

$S_w \approx 0.41R$

Giauque & Stout 1936

$S_{si} \approx 0.46R$

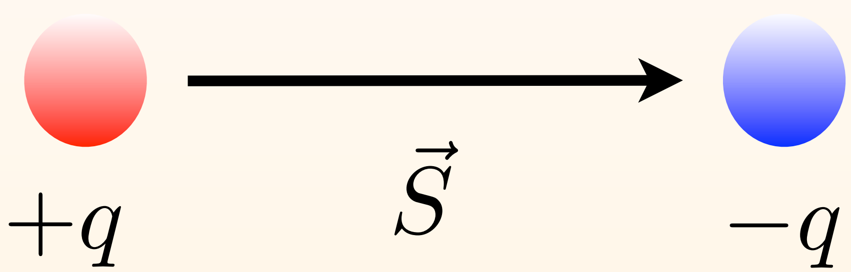
Ramirez, Hayashi, Cava, Siddharthan & Shastry 1999

Thermal excitations in spin-ice

$$H = -J \sum_{<i,j>} \vec{S}_i.\vec{S}_j + Da^3 \sum_{i<j} \frac{\vec{S}_i.\vec{S}_j}{r_{ij}^3} - 3\frac{(\vec{S}_i.\vec{r}_{ij})(\vec{S}_j.\vec{r}_{ij})}{r_{ij}^5} \approx \sum_{(\alpha,\beta)} V_{\alpha\beta}$$

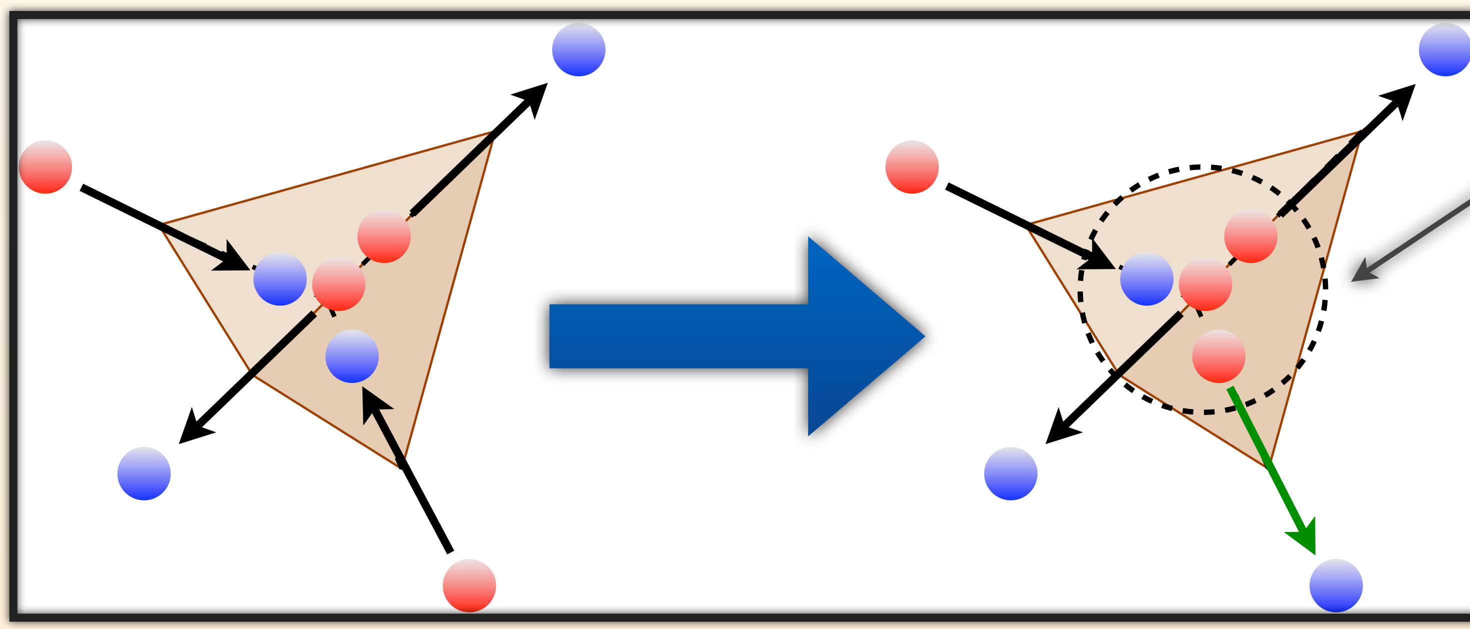
Pyroclhore lattice
= dipoles

Diamond lattice
= monopoles



Thermal excitations violating the ice rules =

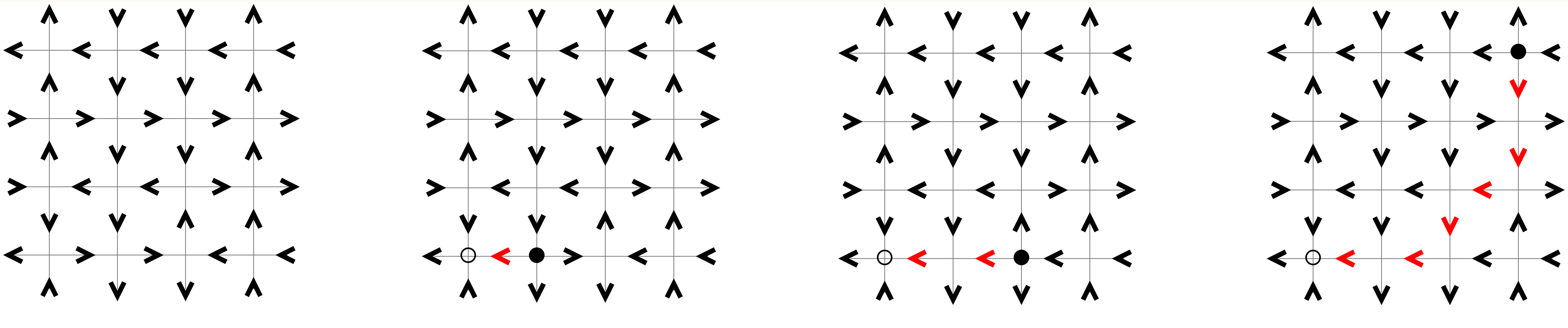
magnetic monopoles



$\Rightarrow$ Ice configurations $\equiv \{q_\alpha = 0\}$

Castelnovo, Moessner & Sondhi 2008

Magnetic monopoles in spin-ice



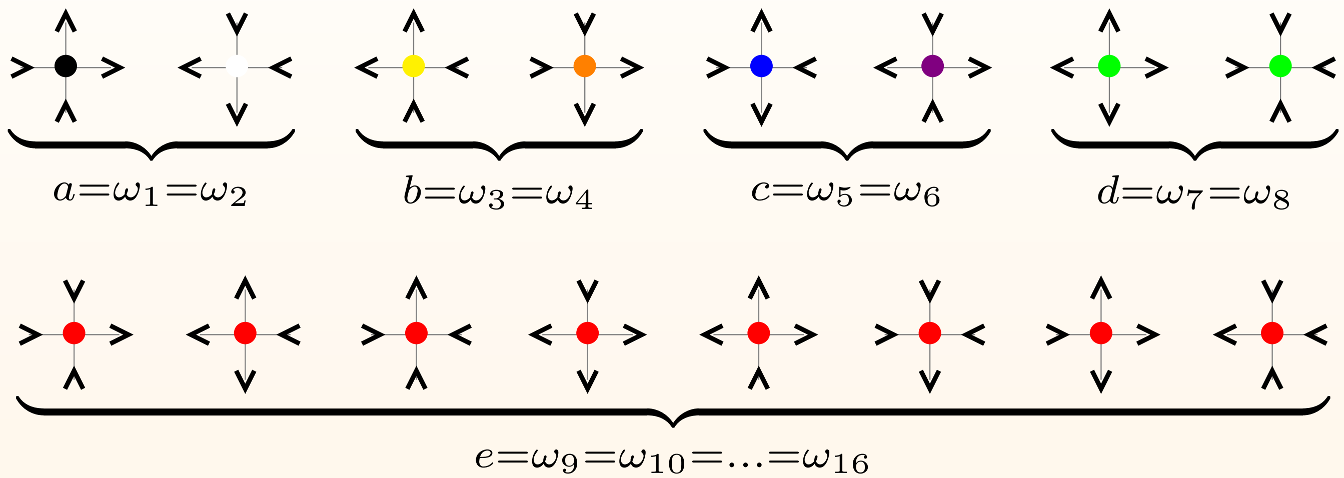
(a) (b) (c) (d)

- If all the 2 in - 2out vertices are equivalent, the only energy cost comes from the $1/r$ Coulomb interaction.

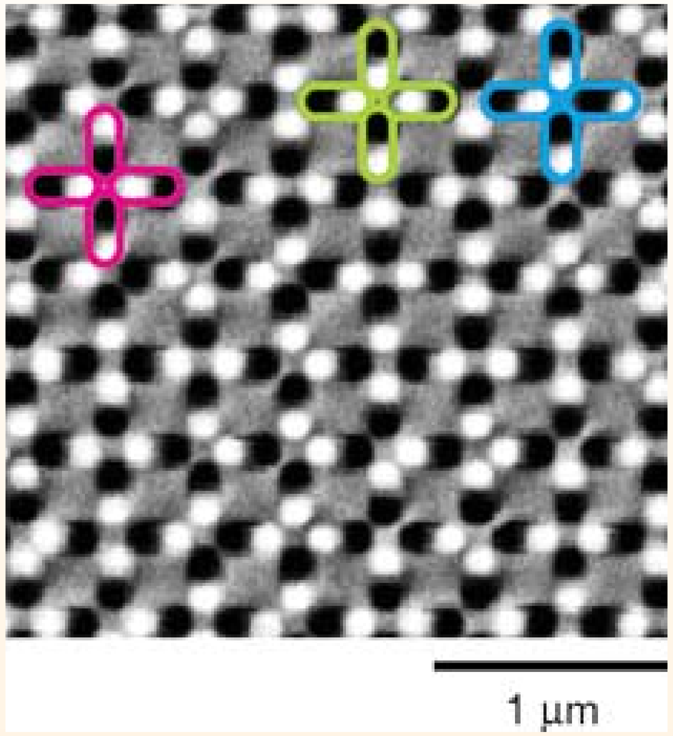
- Finite energy to separate two monopoles to infinity = *de-confined*

$\longrightarrow$ *Fractionalisation* in 3D

2D experiments: Artificial Spin-Ice



from **Wang et al., Nature** 2008



Experimental motivation:

1. direct visualisation of defects' dynamics
Mengotti, Heyderman, Rodríguez, Nolting, Hügli, Braun 2011.

2. vertex weights can be tuned $\longrightarrow a = b = c$
Moller & Moessner 2006

3. A-thermal system: energy barrier $\sim 10^5 K$;

$\longrightarrow$ effective thermodynamics can be recovered
Nisoli, Li, Ke, Garand, Schiffer, Crespi 2010;
Morgan, Stein, Langridge, Morrows 2011.

Equilibrium.

Sixteen-vertex model (numerics)

Allow defects $\Rightarrow \quad d = e \neq 0$

Conjecture: (numerics)

$$\boxed{\Delta_{16} = \frac{a^2 + b^2 - c^2 - (3e+d)^2}{2(ab + c(3e+d))}}$$

$$\xrightarrow{e=0} \quad \Delta_8 = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

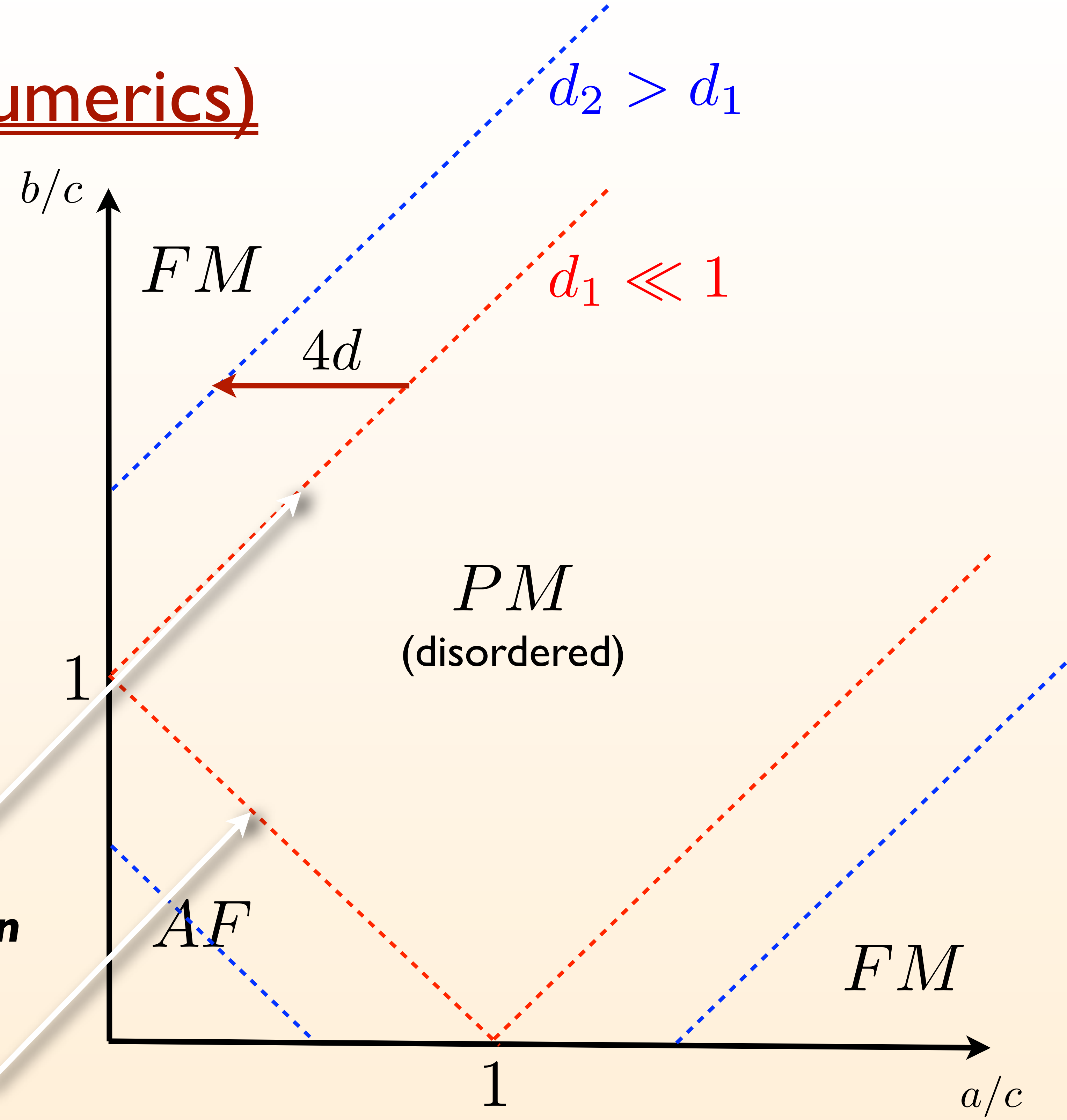
$\Delta_{16} > 1$ *ordered* FM phase

continuous phase transition

$1 > \Delta_{16} > -1$ *disordered* PM phase

continuous phase transition

$\Delta_{16} < -1$ *ordered* AF phase



Levis & Cugliandolo 2011

Phase ordering dynamics after a quench.

Evolution of the system across a phase transition.

- Give updating rules: Non-conserved order parameter (single spin flips)

  coupled to a thermal bath → *Stochastic dynamics*

- Prepare the system in a disordered equilibrium state and quench it into a known equilibrium ordered symmetry broken phase

  → *Competition between opposite orders* *Slow dynamics* $L(t) \sim t^z$

- Dynamical scaling theory: at late times, there is a single length scale $L(t)$ such that:

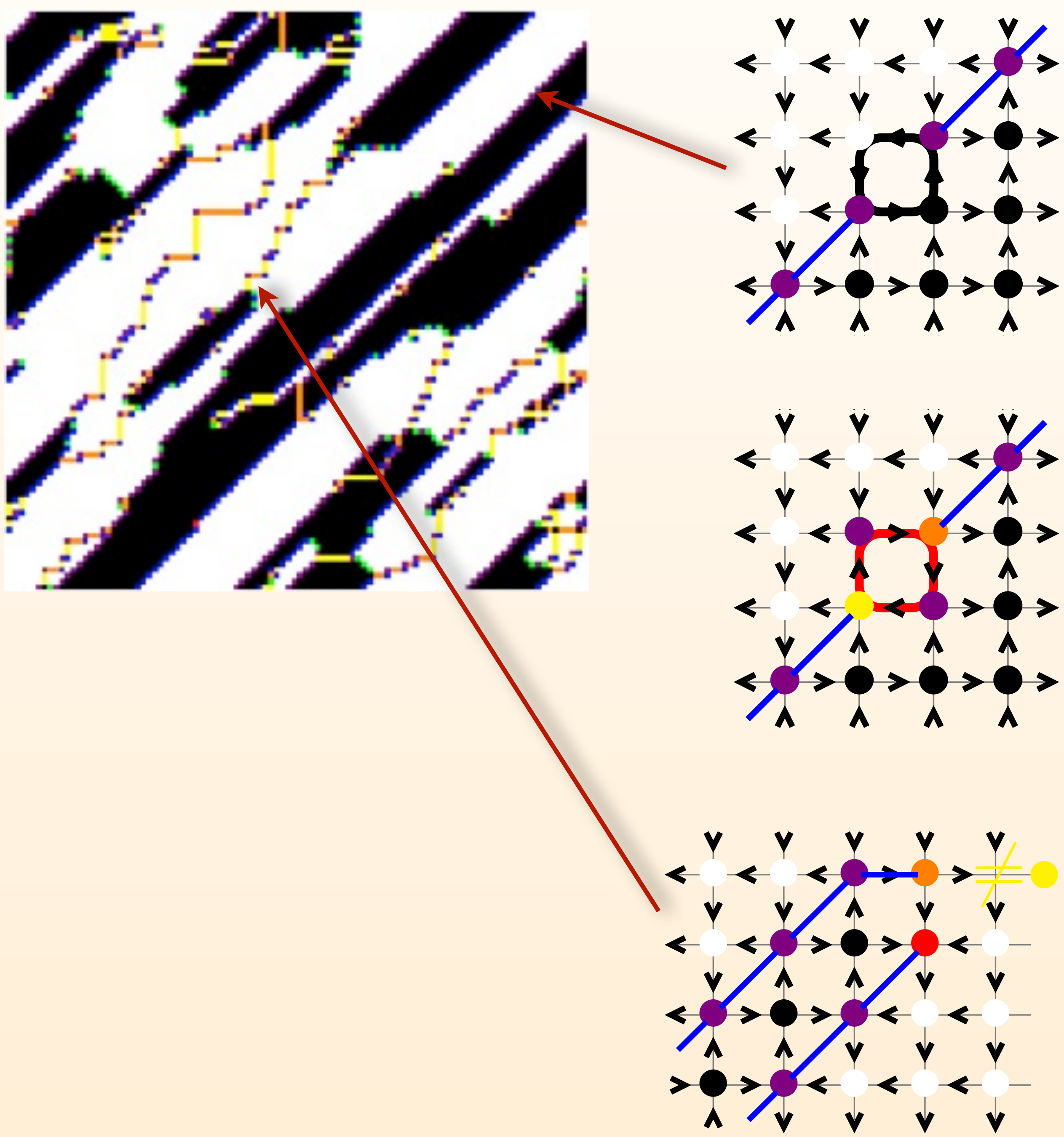
$$C(t, t_w) = \langle s_i(t) s_i(t_w) \rangle \simeq M_{eq}^2 \, F_C \left(\frac{L(t)}{L(t_w)} \right)$$

$$G(r, t) = \langle s_i(t) s_j(t) \rangle \simeq M_{eq}^2 \, F_G \left(\frac{r}{L(t)} \right)$$

cf. review **Bray** 1994

Quench into the FM phase.

Dynamic mechanisms



- anisotropy $a \neq b$ tends to create diagonal domain walls made of AF vertices.

- loop fluctuations are the elementary moves that do not break the ice-rules.

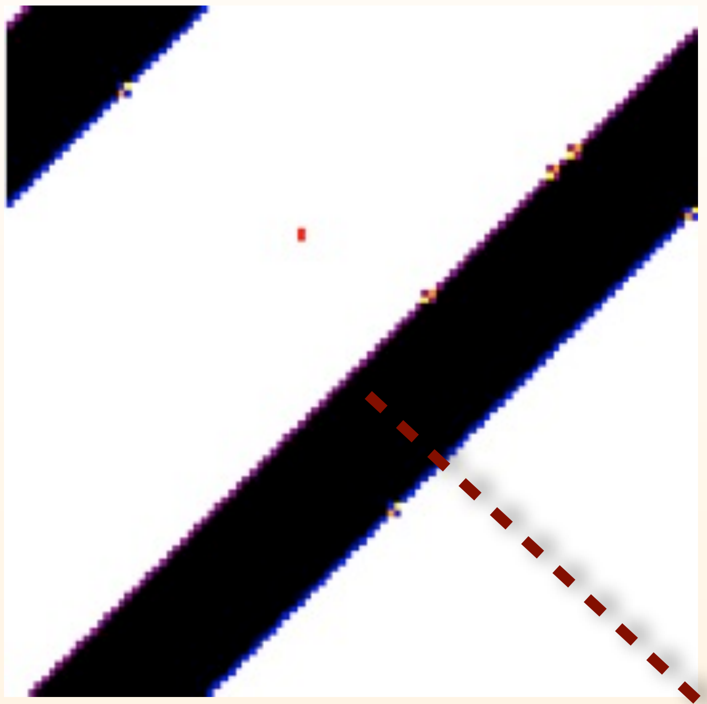
- 'corners' of domains cannot have a neighboring a-vertex. Avoiding defects, this explains the presence of strings.

- Strings connect two domains and mediate their growth.

Levis & Cugliandolo 2011

Quench into the FM phase.

Dynamic mechanisms

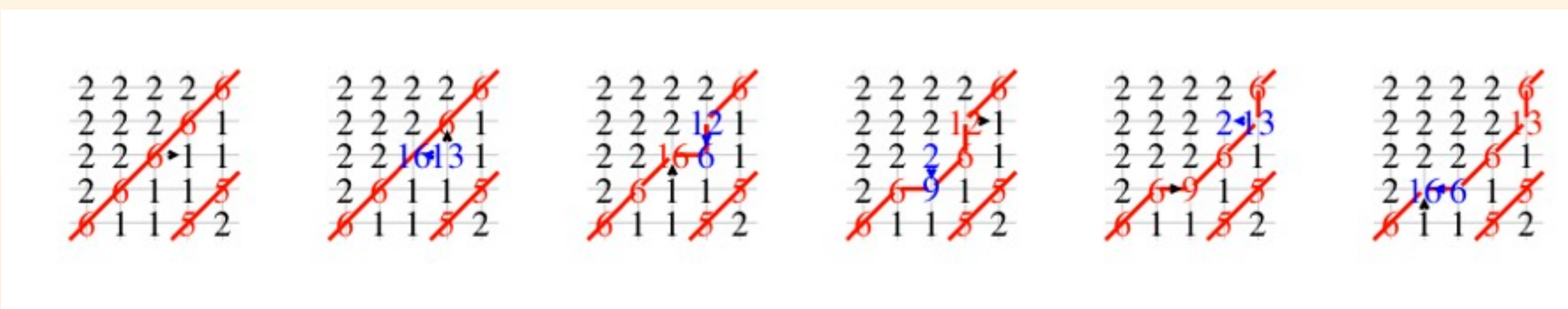


- once the bands are created we must create a pair of defects and made them move along the walls to restore the equilibrium configuration.

$\longrightarrow$ Extremely slow process $\quad T \sim L$

(diverging at the thermodynamic limit)

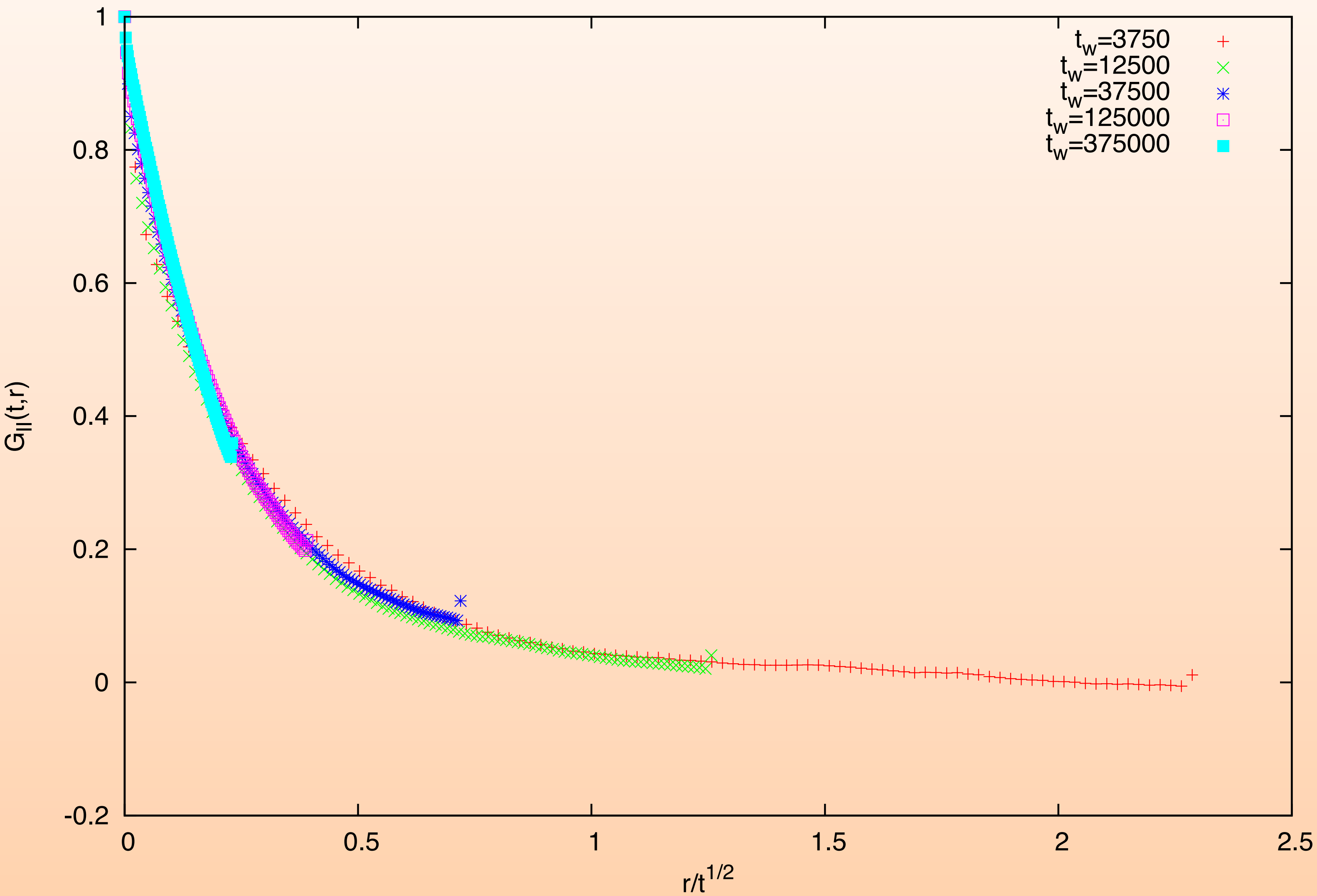
Equilibrium is reached when magnetic order percolates in the $\perp$ direction.

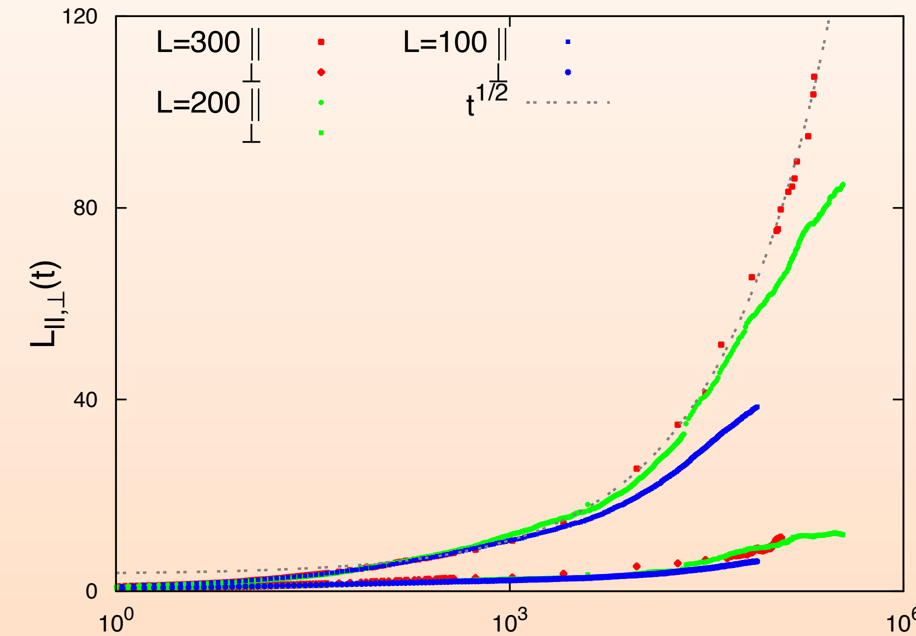


Levis & Cugliandolo 2011

Out-of-equilibrium.

Quench into the FM phase.





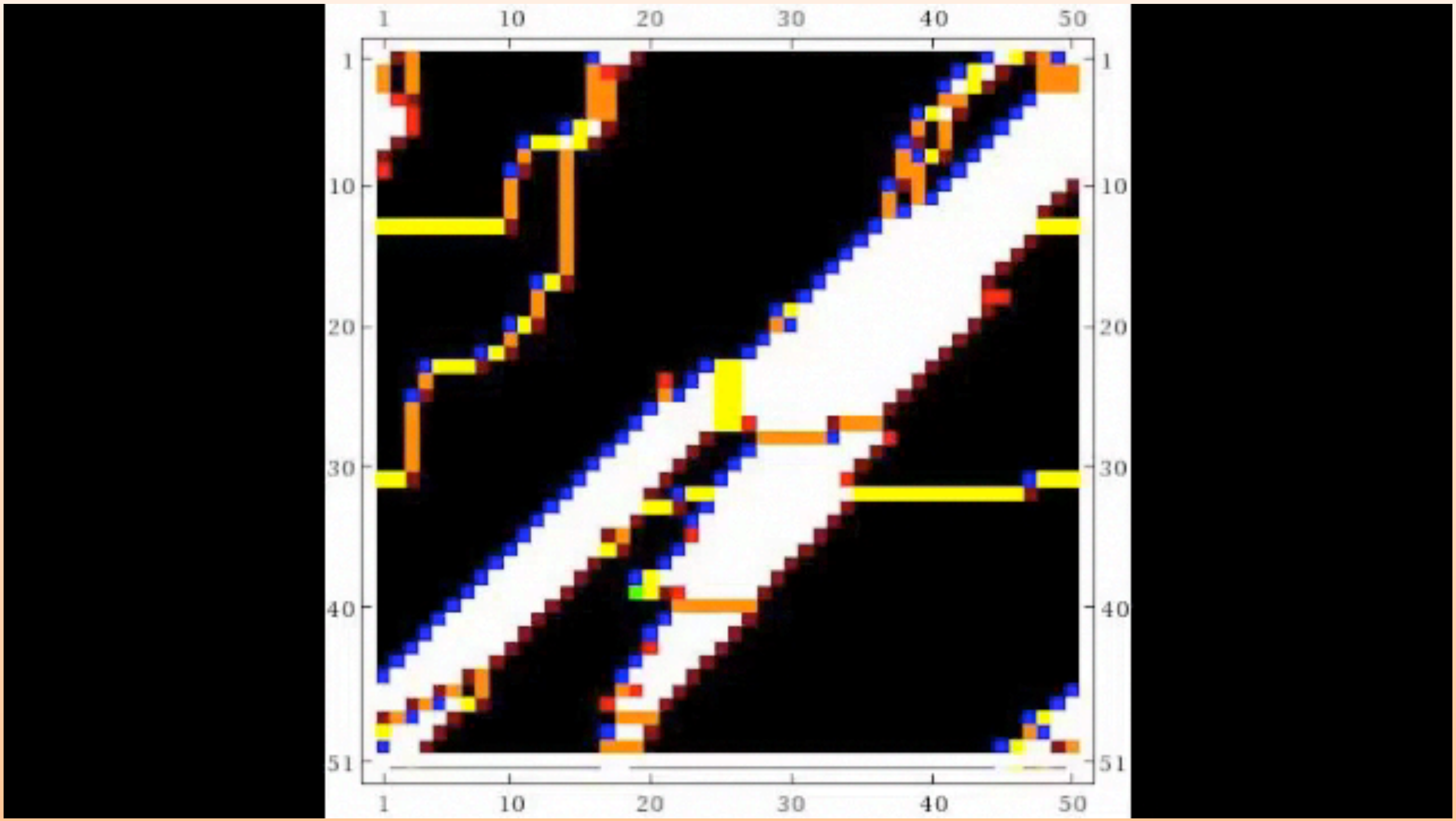
L=200

Two-point self correlation function in the $\|$ direction:

$$G_{\|}(t,r) = \sum_{i,j} \langle S_{i,j}(t) S_{i+\frac{r}{\sqrt{2}},j+\frac{r}{\sqrt{2}}}(t) \rangle$$

Quench into the FM phase.

Evolution from a D i.c. to $a = 5,\ b = 1,\ e = 10^{-7},\ d = e^2$

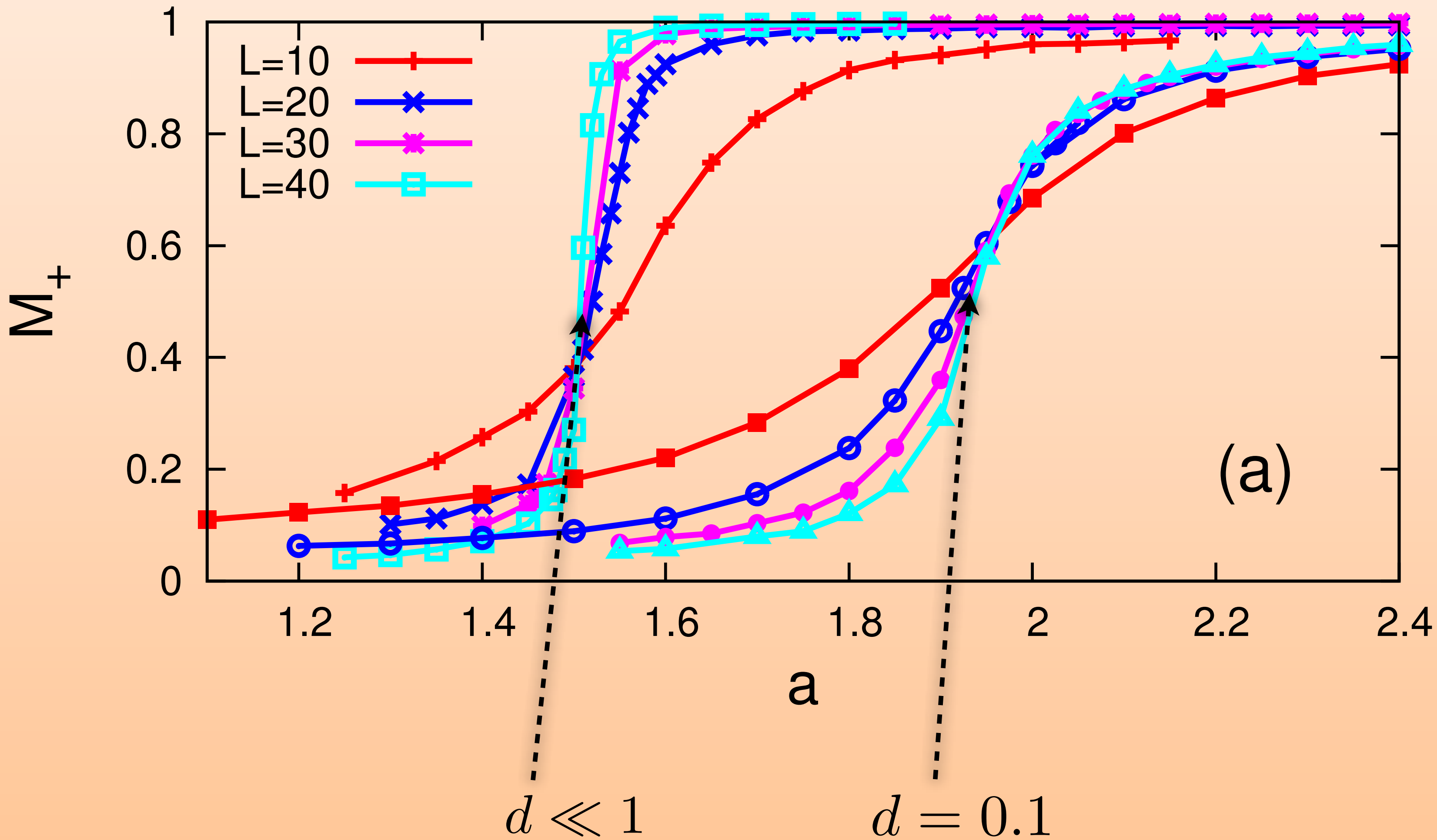


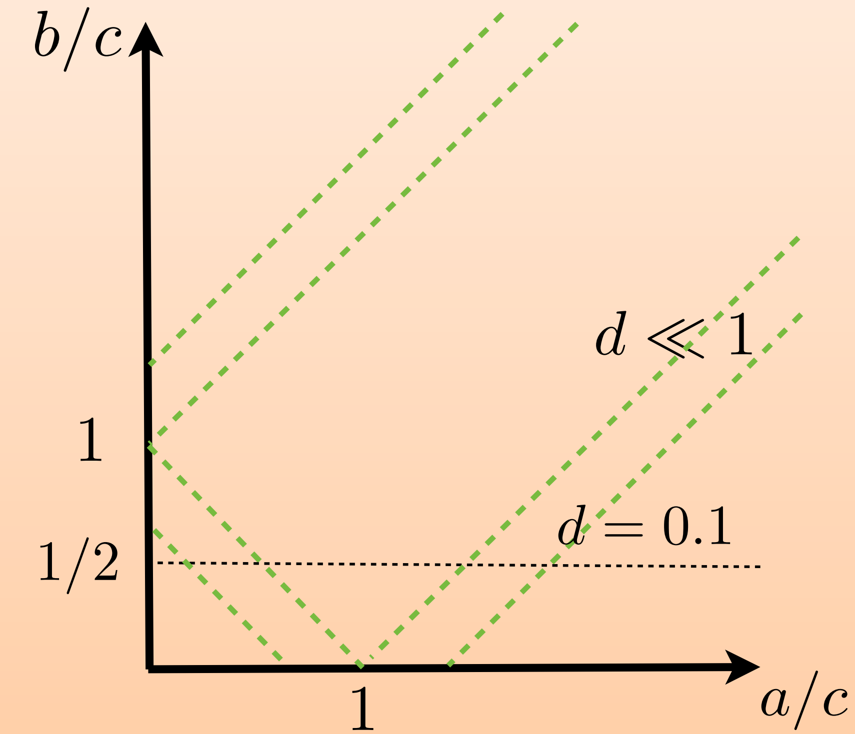
Equilibrium.

Sixteen-vertex model

Allow defects $\Rightarrow \quad d = e \neq 0$

Example: $\quad b = 1/2\,, c = 1$





$\rightarrow$ The FM transition gets "smoother" by increasing the weight of the defects

Equilibrium.

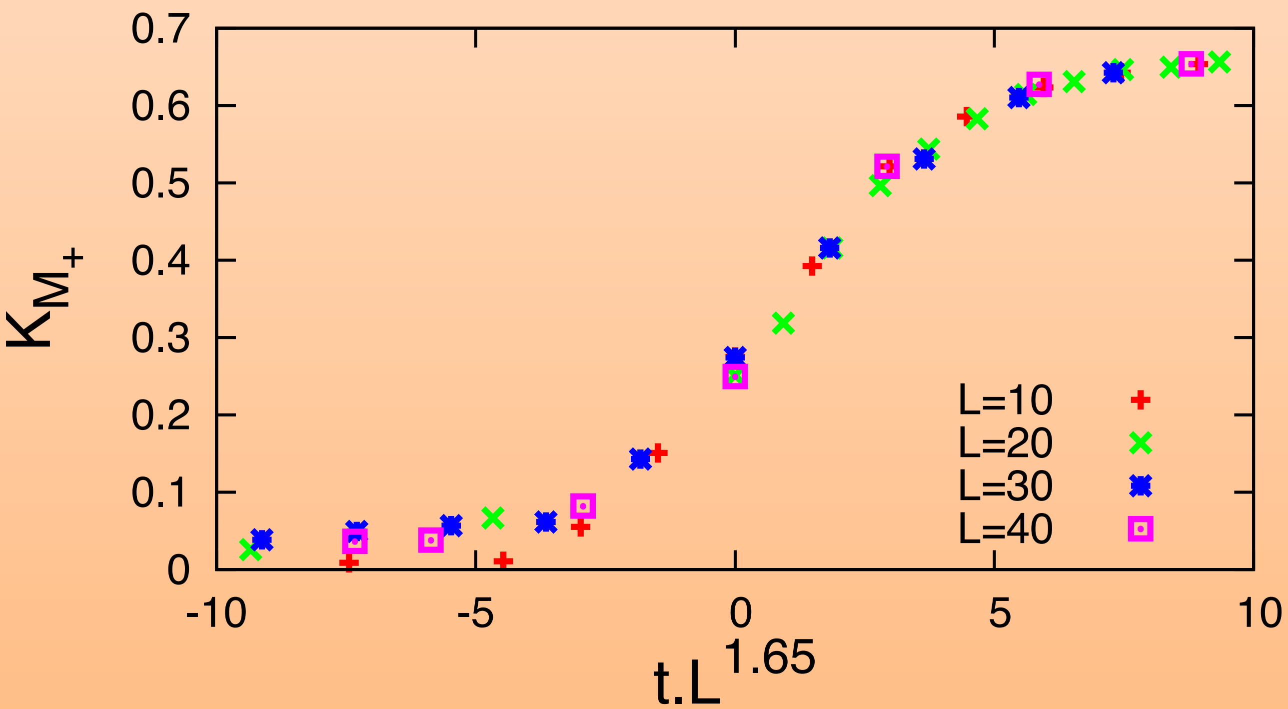
Sixteen-vertex model

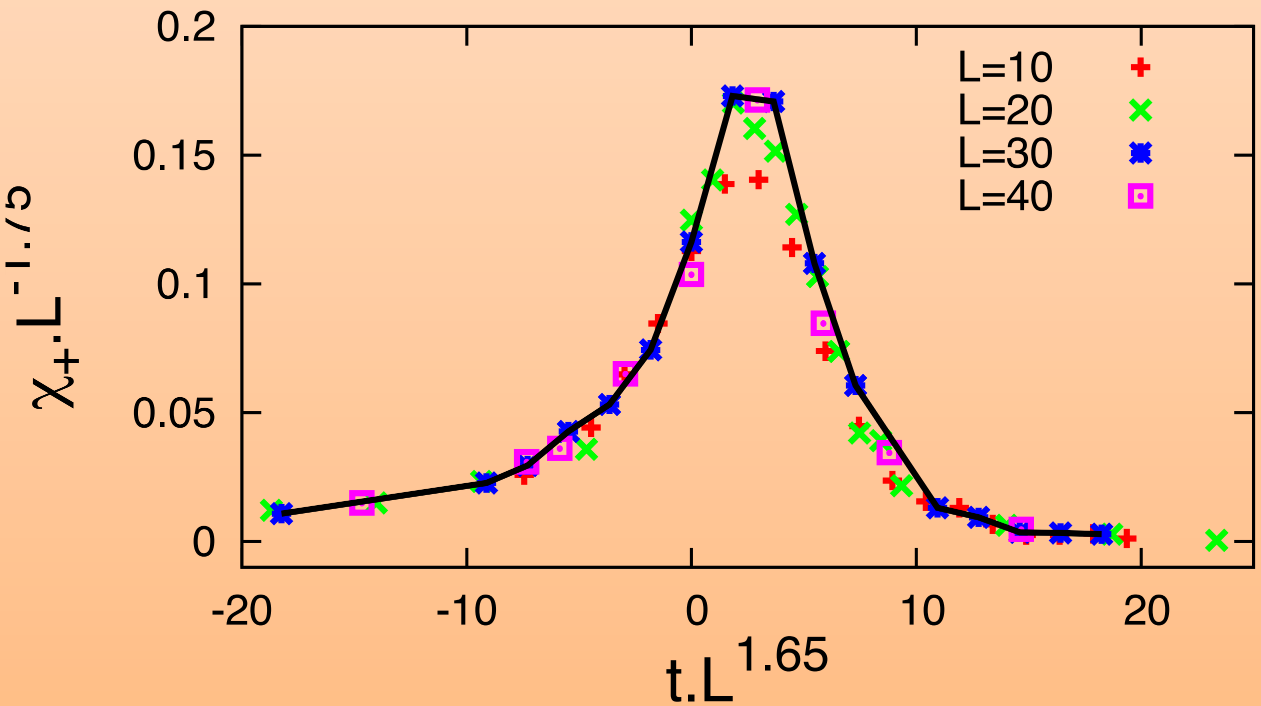
Finite size scaling:

$$K_{M_+} = 1 - \frac{\langle M_+^4 \rangle}{3\langle M_+^2 \rangle^2} \sim \Phi_K(tL^{1/\nu})$$

$$\chi_+ \sim \Phi_{chi_+}(tL^{\gamma/\nu})$$

Distance from the transition: $t = \dfrac{a - a_c}{a}$





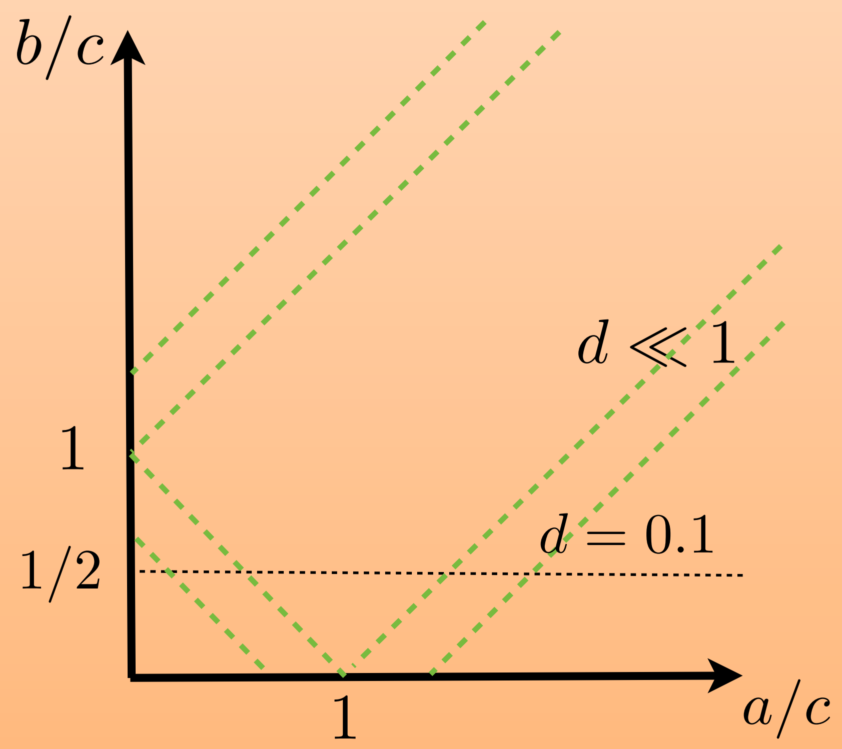
Sixteen-vertex model

Ex: critical exponents for the FM transition

	six-vertex	$2d$ Ising	SI $(d=e=10^{-5})$	SI $(d=e=0.1)$
$\hat{\gamma}=\gamma/\nu$	$7/4$	$7/4$	$=1.75\pm0.02$	$=1.75\pm0.02$
$\hat{\beta}=\beta/\nu$	$1/8$	$1/8$	$\approx 0.125\pm0.05$	≈ 0.125
$\hat{\alpha}=(2-\alpha)/\nu$	2	2	$=2.00\pm0.15$	≈ 2
γ	$7/8$	$7/4$	$=1.06\pm0.03$	$=1.75\pm0.18$
β	$1/16$	$1/8$	$=0.050\pm0.014$	≈ 0.125
α	1	0	$=0.84\pm0.23$	≈ 0
ν	$1/2$	1	$=0.60\pm0.02$	$=1.0\pm0.1$

Define new exponents : $M_+ \sim \xi^{-\hat{\beta}}$

Consistent with scaling relations



Sixteen-vertex model

	six-vertex	$2d$ Ising	SI $(d = e = 10^{-5})$	SI $(d = e = 0.1)$
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Exponents depend on the vertex weights

Equilibrium.

Sixteen-vertex model

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Exponents depend on the vertex weights

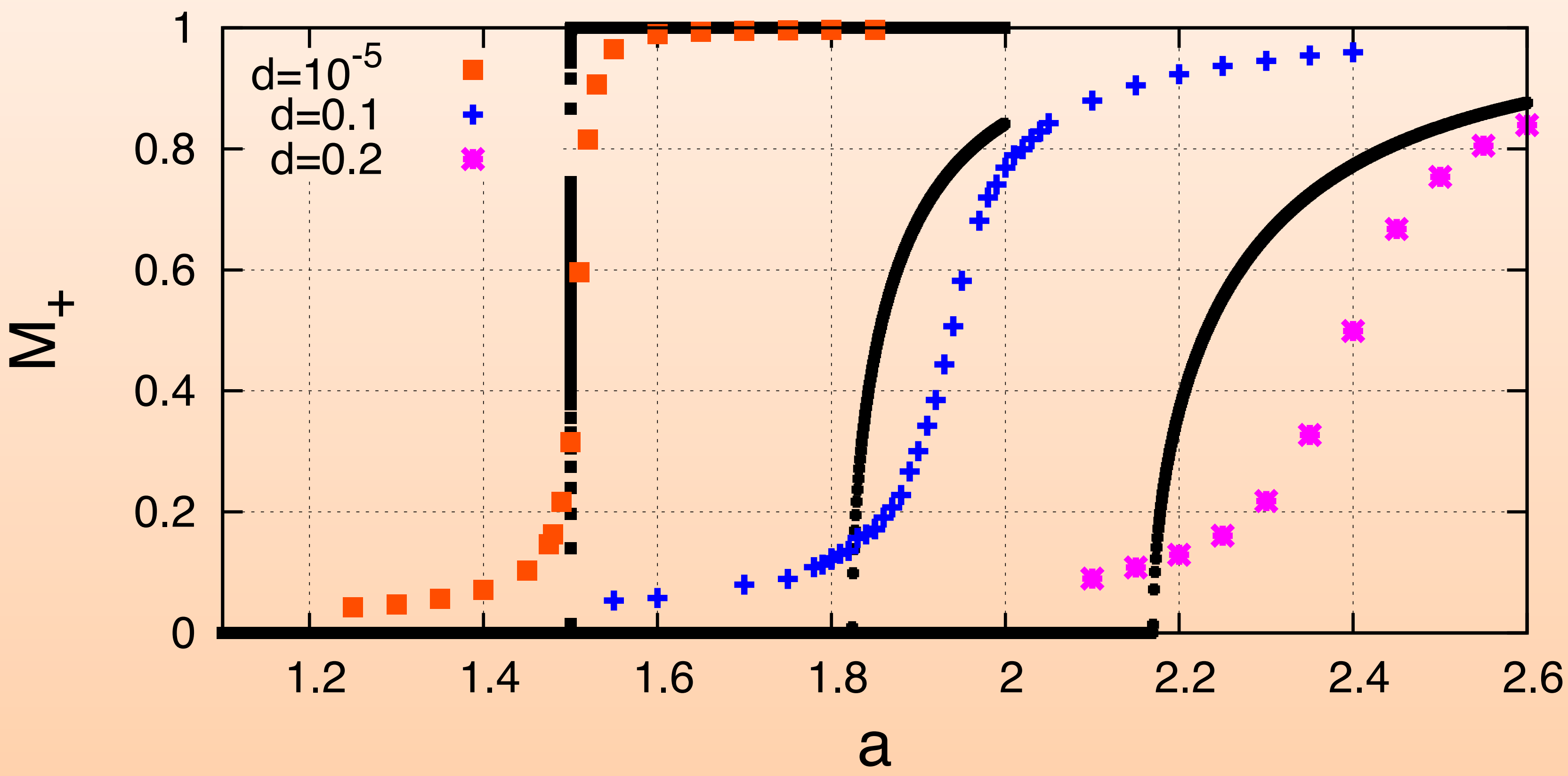
$\hat{\gamma}\,, \hat{\alpha}\,, \hat{\beta}$ does not!

$\Rightarrow$ ***Weak universality*** *(Suzuki 1974)*

Cavity method vs. Monte Carlo.

Spin-ice on a tree graph (cavity method)

Cavity vs. numerical results (sixteen-vertex model)



1. The approx. becomes worse when increasing the weight of the defects and the transition gets "softer".

2. The location of critical surfaces obtained by the cavity calculation are almost parallel to the six-vertex ones (as predicted by MC)

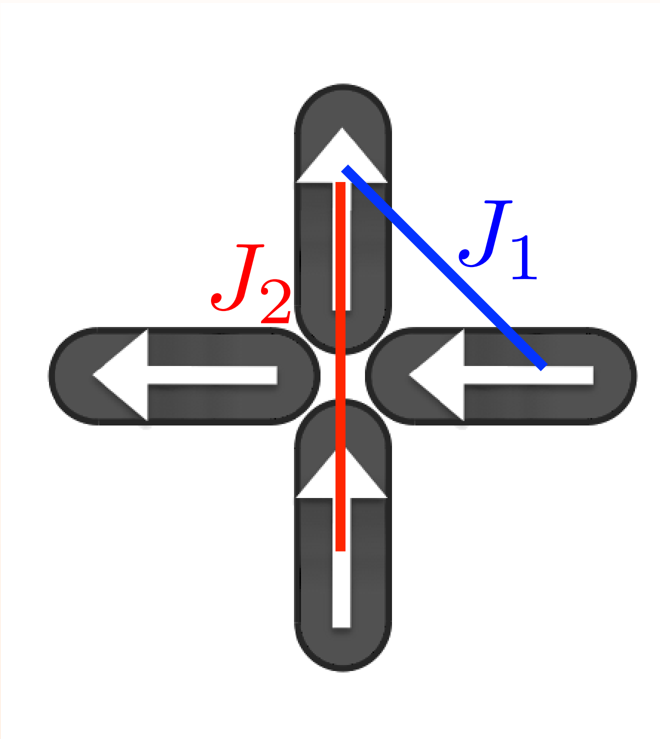
Foini, Levis, Tarzia, Cugliandolo In preparation (2012)

ASI.

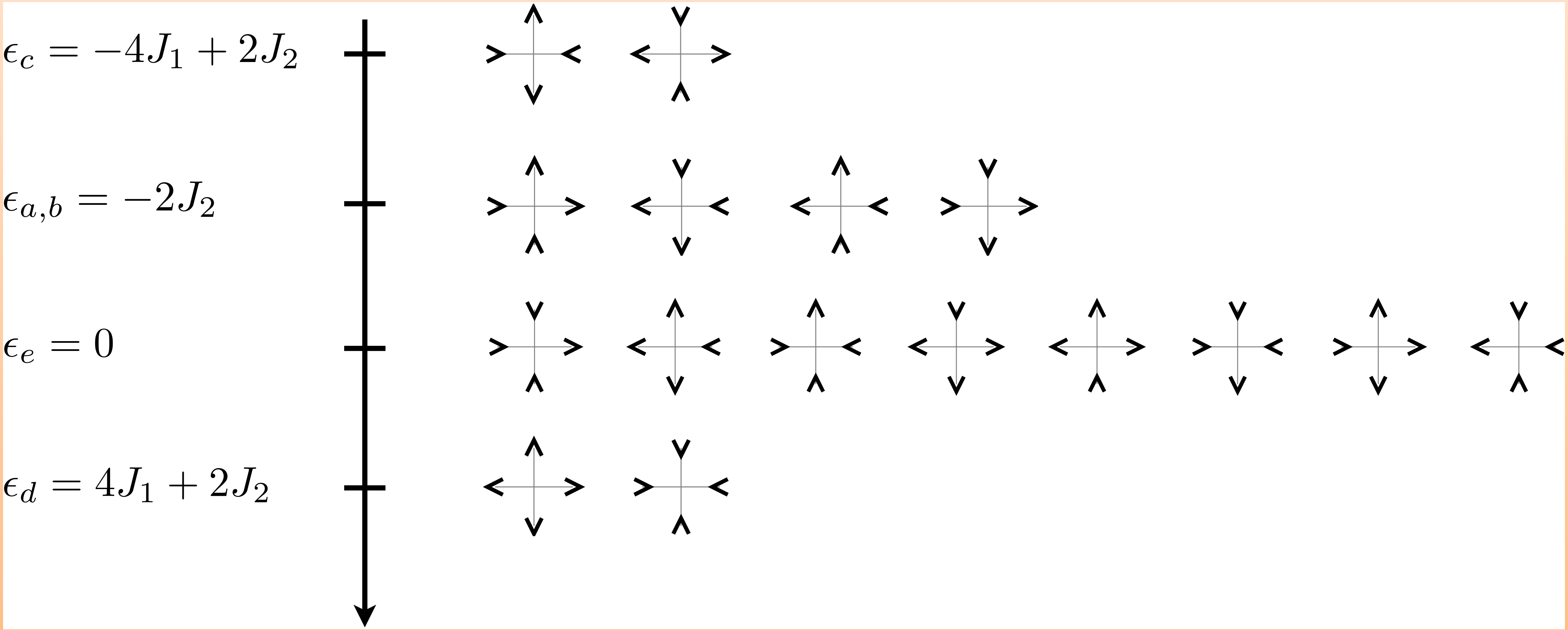
Artificial spin ice

Wang et al., Nature 439 (2006)

$$H = Dr_0^3 \sum_{i<j\in\mathcal{P}} \left(\frac{\mathbf{S}_i.\mathbf{S}_j}{||\vec{r}_{ij}||^3} - 3\frac{(\mathbf{S}_i.\vec{r}_{ij})\,(\mathbf{S}_j.\vec{r}_{ij})}{||\vec{r}_{ij}||^5} \right)$$



Energy levels

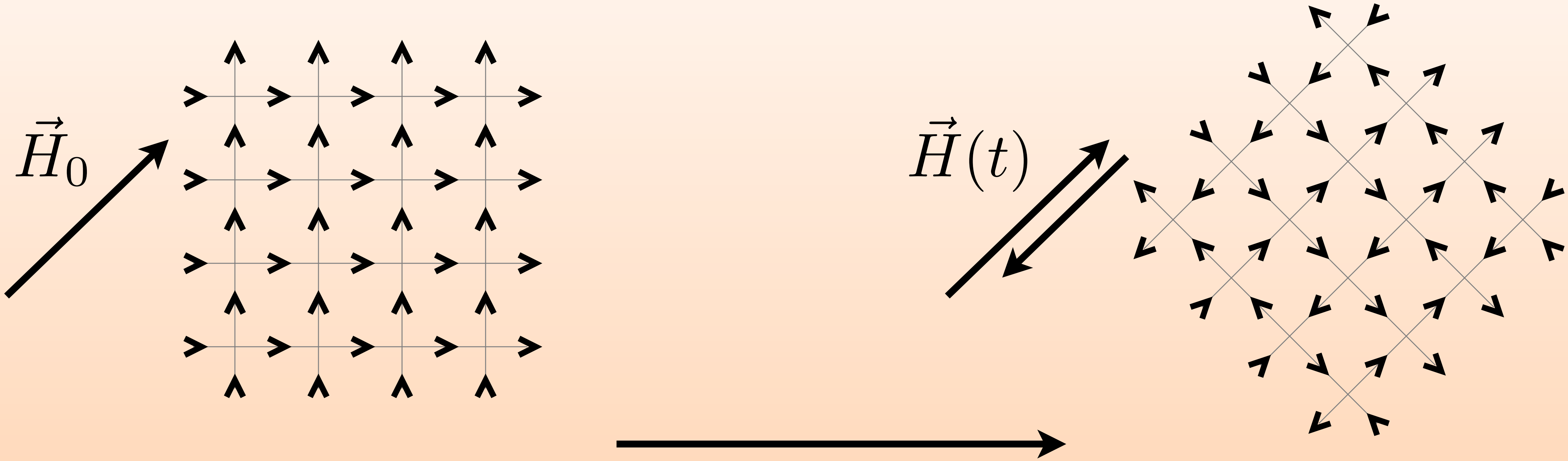


$\Longrightarrow$ *2 in - 2 out* vertices are not equivalent in the square lattice

ASI.

‘A-thermal’ Artificial spin ice

(*Nisoli et al.* 2007-2008)



$t = 0$ $\qquad\qquad$ $t > 0$

Demagnetisation protocol: $\quad H(t = n\Delta t) = (-1)^n (H_0 - nH_s)$

- polarised initial configuration
- rotate the sample
- decrease magnetic field
- change polarity of the field after a cycle

Shearing and shaking in granular matter ??

'A-thermal' Artificial spin ice

(*Nisoli et al. 2007-2010*)

Assume: (i) $E \approx \mathrm{ct}$

(ii) vertices can be chosen independently (mean-field)

→ Construct an Edwards' measure

1. Count configurations $S(E,N) = \ln \Omega(E,N)$

2. Maximise the entropy under the constraint $E \approx \mathrm{ct}$

Gives population of vertices created during the demagnetisation :

$$n_\alpha = \frac{\exp(-\beta_E \epsilon_\alpha)}{Z(\beta_E)}$$

With $$\frac{1}{T_E} = \frac{\partial S}{\partial E}$$ configurational 'temperature'

ASI.

‘A-thermal’ Artificial spin ice

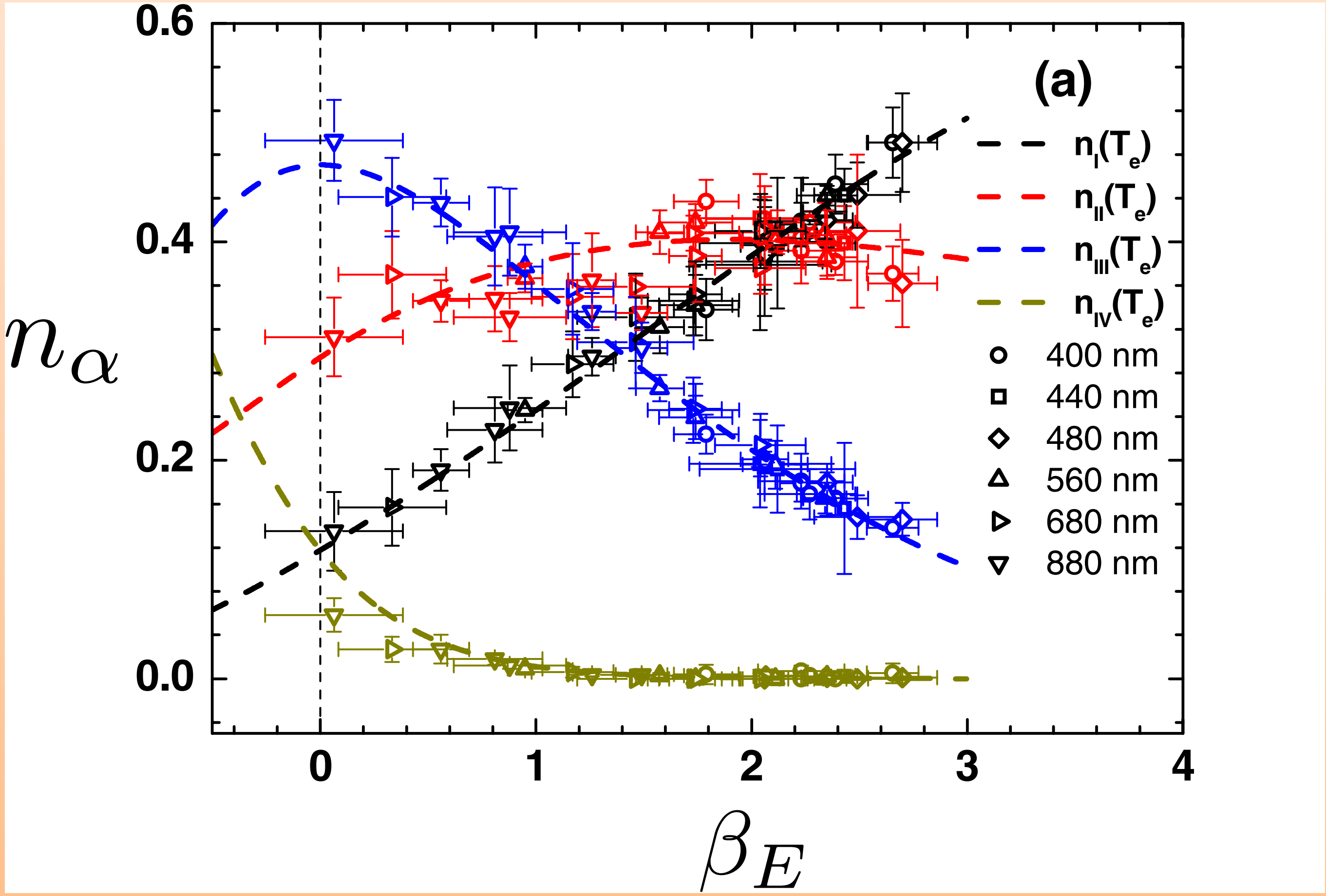
(*Nisoli et al. 2007-2010*)

$$n_\alpha = \frac{\exp(-\beta_E \epsilon_\alpha)}{Z(\beta_E)}$$

Count vertices in the samples

$$\beta_E = \ln\left(\frac{4n_c}{n_e}\right)(\epsilon_e - \epsilon_c)^{-1}$$

From: *Nisoli et al.*, PRL 105 *(2010)*

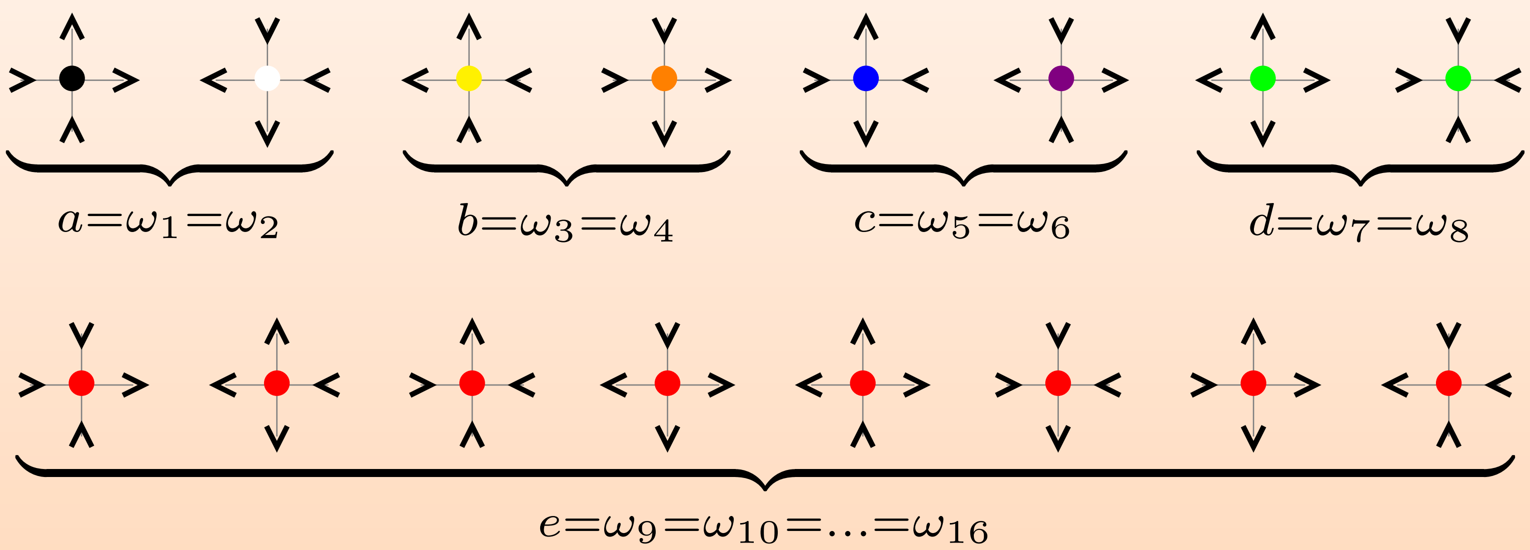


$\Longrightarrow$ Ground state cannot be reached with this procedure

ASI.

‘Thermal’ Artificial spin ice

Special parametrisation of the 16-vertex model:



$$a = b = \exp(-\beta\epsilon_a)$$
$$c = \exp(-\beta\epsilon_c)$$
$$e = \exp(-\beta\epsilon_e)$$
$$d = \exp(-\beta\epsilon_d)$$

Energy of each vertex = approximation of the dipolar interactions

$\Rightarrow$ High energy to the defects

Analytical calculation on the tree of plaquettes

+

Continuous Time Monte Carlo simulations

Our canonical temperature: $\beta = -\dfrac{\ln a}{\epsilon_a}$

Equilibrium.

Sixteen-vertex model

Allow defects $\Rightarrow$ $d = e \neq 0$

Ice rules $\longrightarrow$ collective paramagnet with $\xi_{eq} = \infty$

~~Ice rules~~ $\longrightarrow$ disordered paramagnet ? $\infty > \xi_{eq} > L$

How can we observe a difference?

***Non-equilibrium relaxation** from an completely ordered state towards equilibrium in different phases*

$$M_+(t) \begin{cases} \sim exp(-t/\tau) & \text{disordered} \\ \sim t^{-\lambda} & \text{critical} \quad \xi_{eq} = \infty \\ \sim M_+^{eq}\, exp(-t/T) & \text{ordered} \end{cases}$$

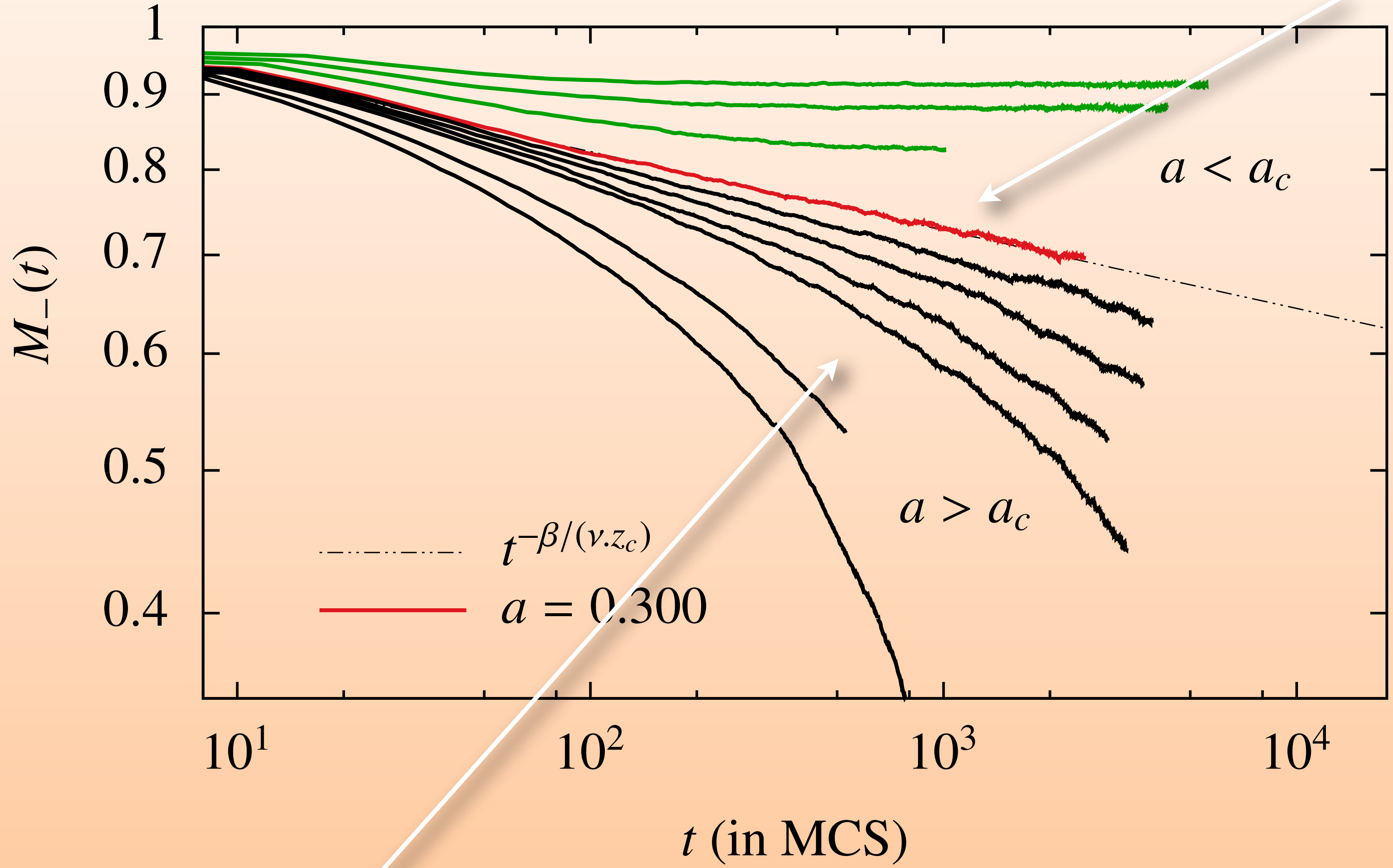
power-law behaviour inside the SL phase

TEST criticality

ASI.

'Thermal' Artificial spin ice

A single critical point



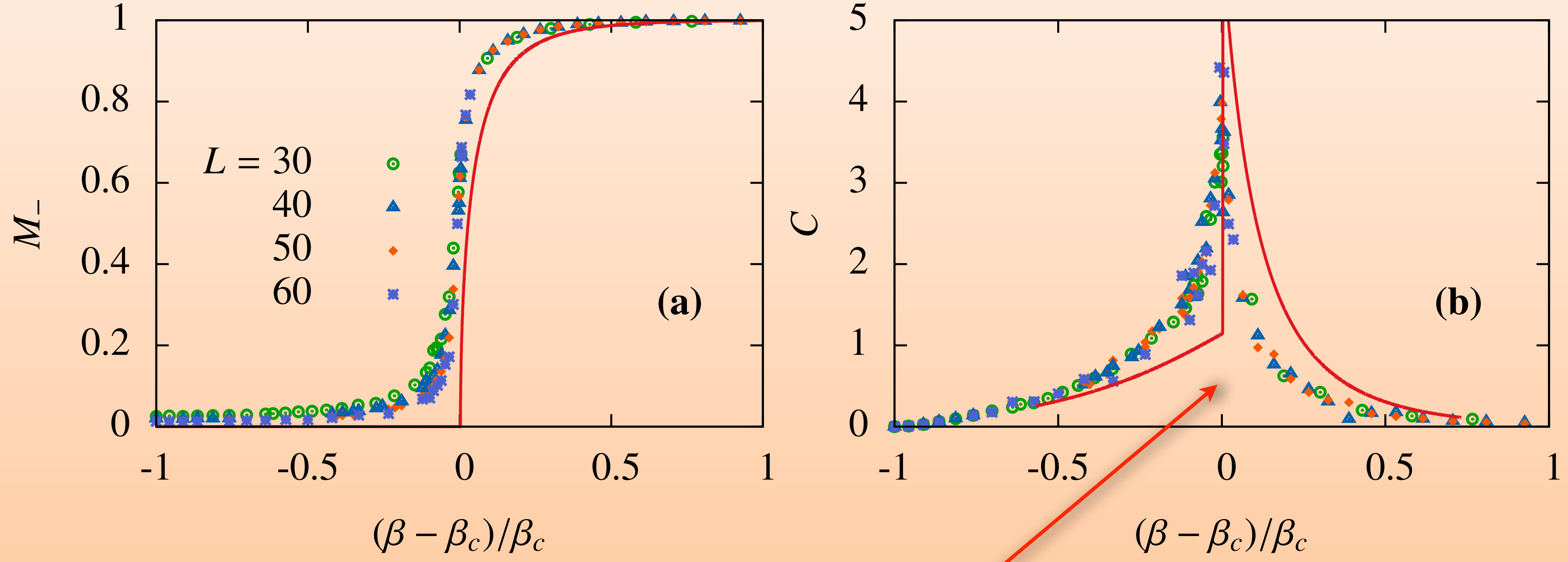
Non-critical correlation in the PM phase

$\Longrightarrow$ The spin-liquid phase is lost as soon as a finite density of defects is present

ASI.

‘Thermal’ Artificial spin ice

Numerics vs. Cavity calculation

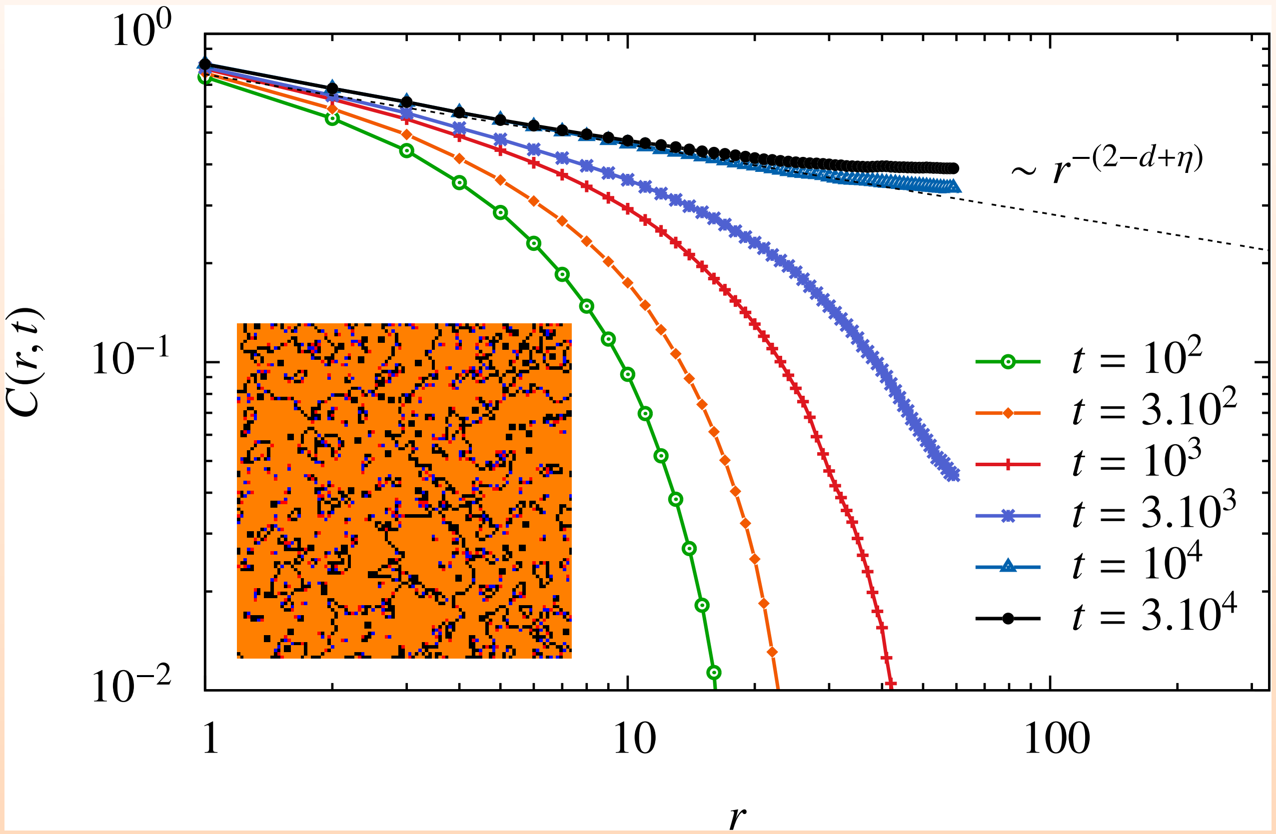


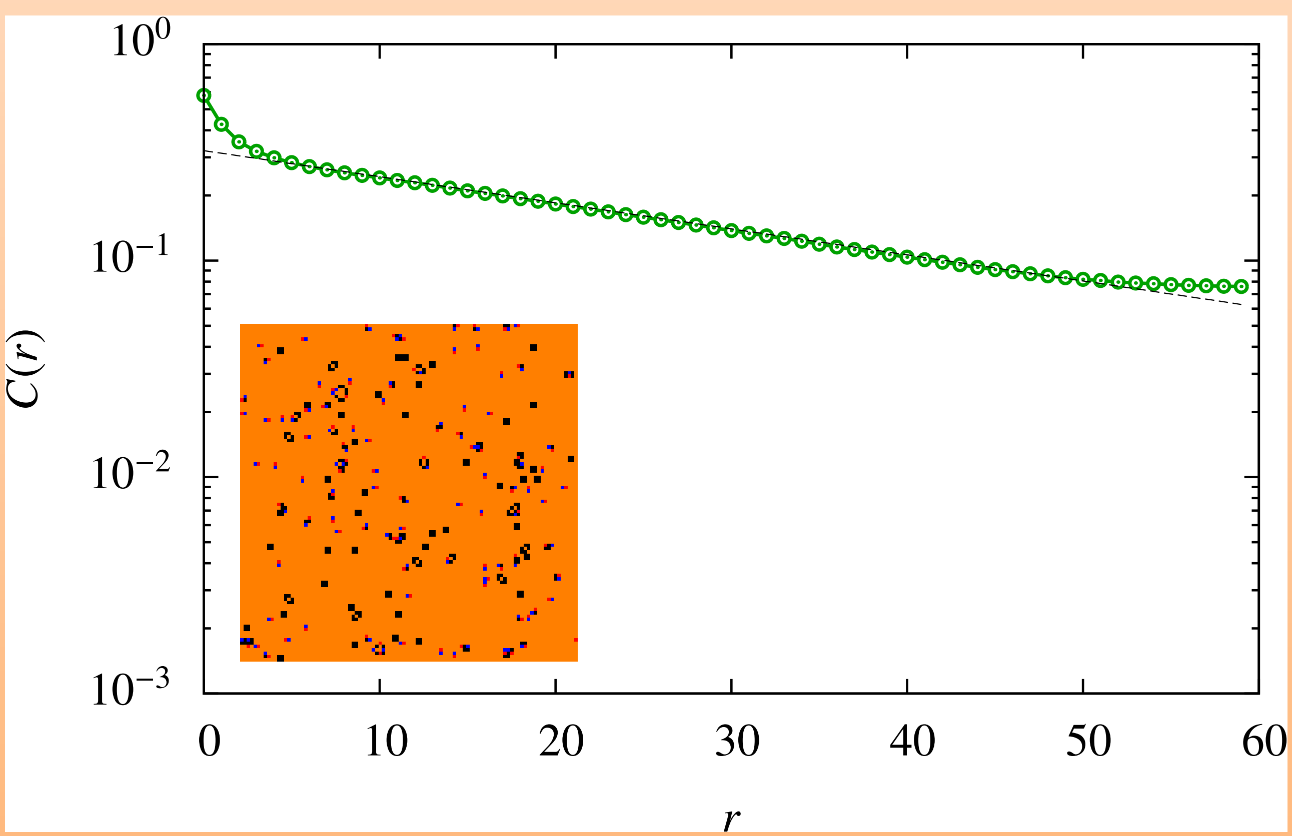
Second order phase transition at $\beta_c \approx 2.65$

→ Compare with experiments

Levis, Cugliandolo, Foini, Tarzia In preparation *(2012)*

‘Thermal’ Artificial spin ice





Two-point space-time correlations:

$$C(r,t) \propto \langle \sum_{i,j} S_{i,j}(t) S_{i+r,j+r}(t) \rangle$$

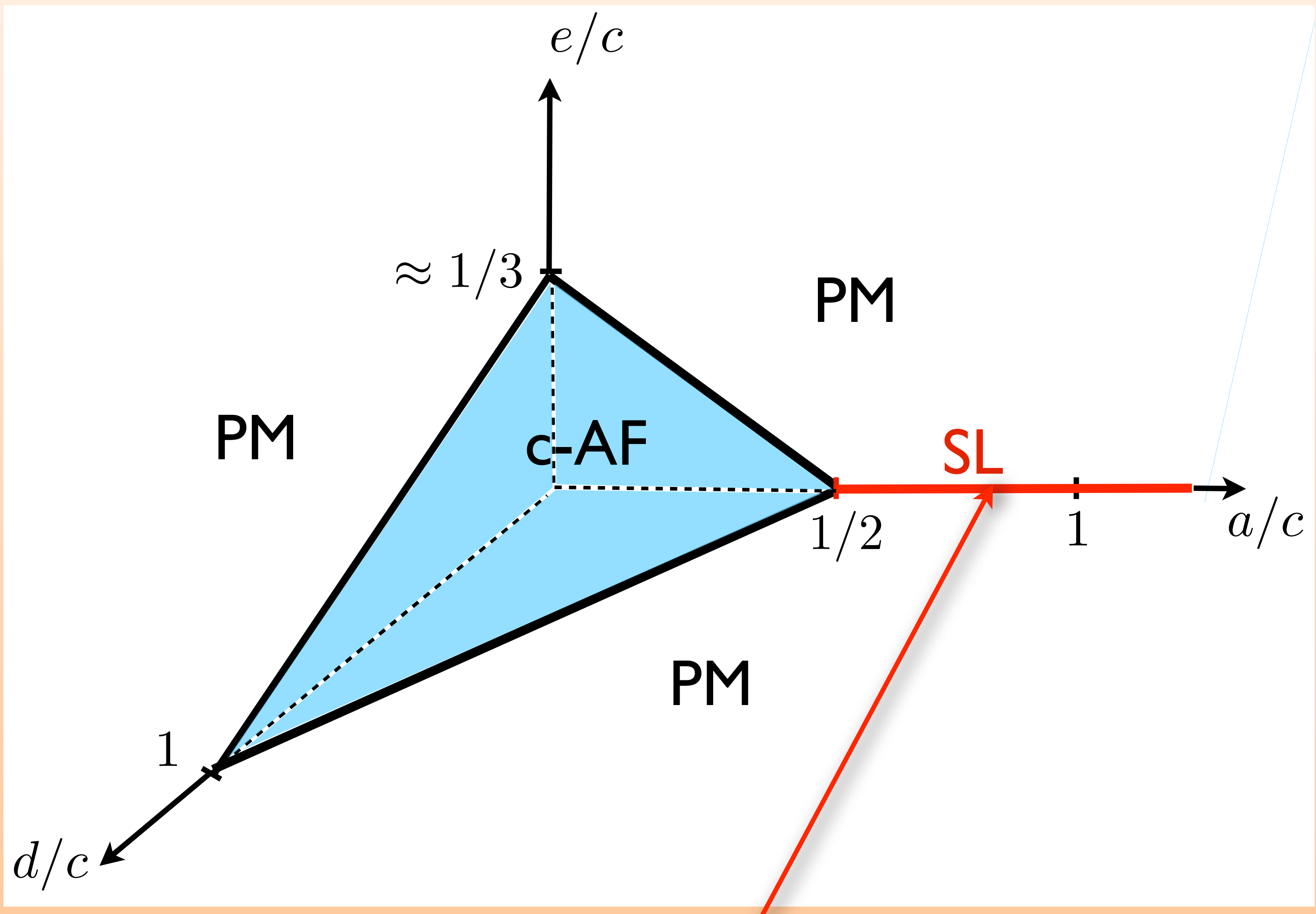
$\beta \approx \beta_c$

Slowing down close to the critical point

$\beta > \beta_c$

'Thermal' Artificial spin ice

Phase diagram of ASI (extensive numerics + cavity)



The criticality of the SL phase is lost as soon as $d, e > 0$

Conclusion & outlook.

1. Investigate the variation of critical exponents in the 16V model by a real-space Renormalisation Group approach.

2. Extend the mappings between integrable vertex models (6V and 8V) into quantum spin chains (XXZ and XYZ resp.) to the unconstrained 16V model. Then, use our understanding on the classical model to predict the behaviour of the quantum system?

3. Link between configurational temperatures in ASI and the canonical temperature? In experiments:
- properties of the domain walls?
- thermal equilibration?
- growth rate (link with a cooling rate)

Conclusion & outlook.

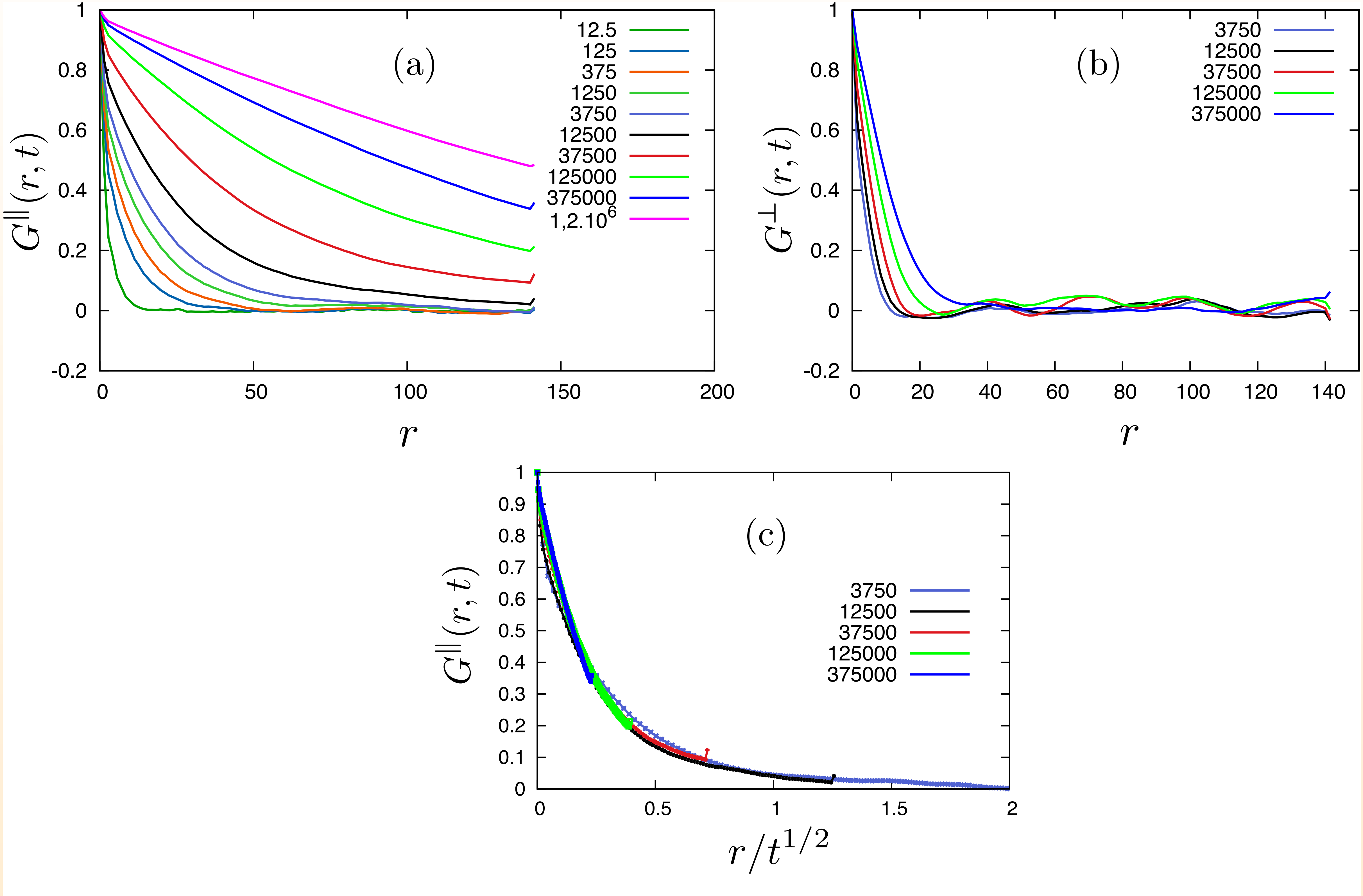
4. Defects' motion in 2D (square or hexagonal).
Prepare the system into an 'ice-rule' configuration, then follow the motion of a pair of defects. Interaction?
Model for collective transport properties?

5. Effect of boundary conditions on the dynamics of hardly constrained models.

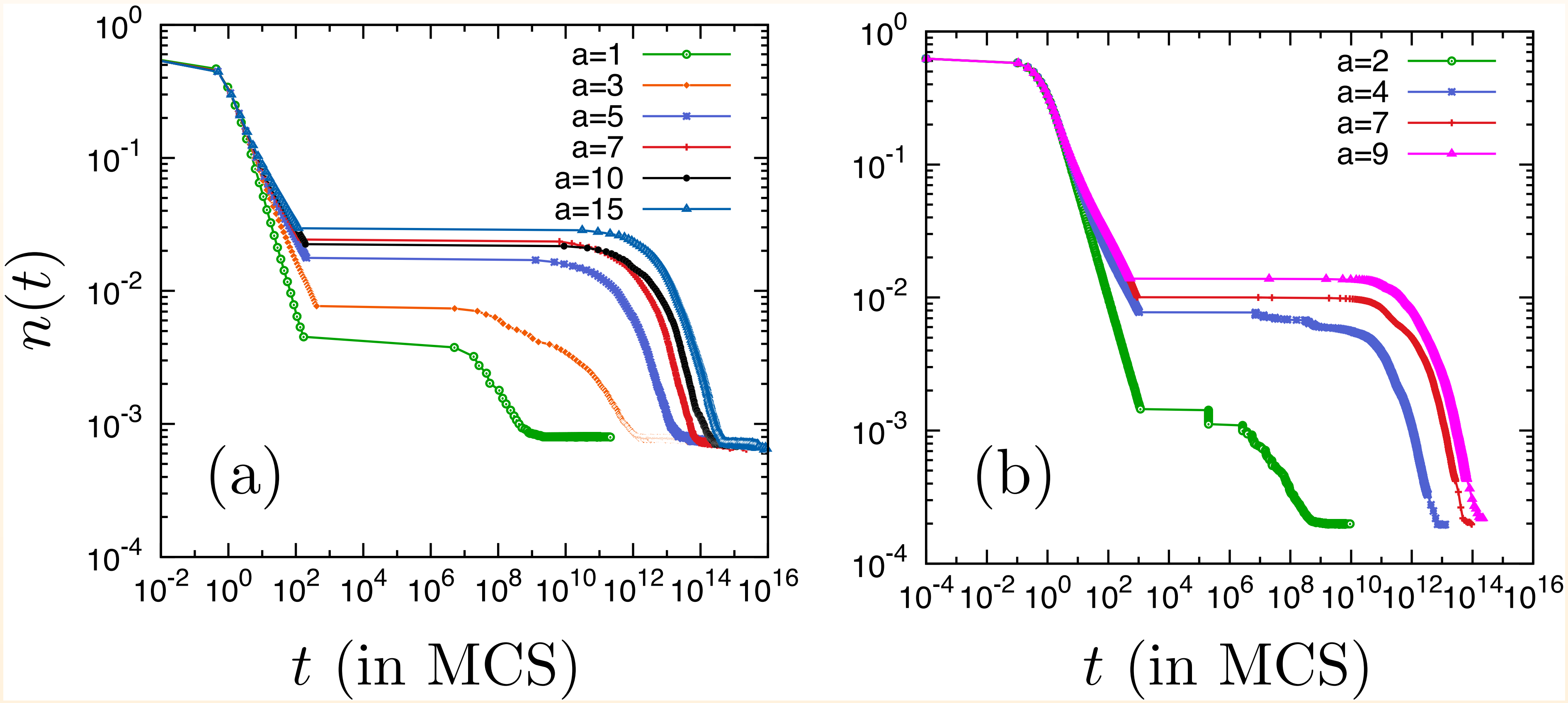
6. In particular, study the relaxation dynamics of the six-vertex model with fixed boundary conditions using local updates. Topological glass?

7. 1. Use the same BP and CTMC approach to study the 16V model in an external field.
Extend the BP approach to deal with other geometrically frustrated stat. models (AF Ising, Kagome ice, colouring models, etc.).

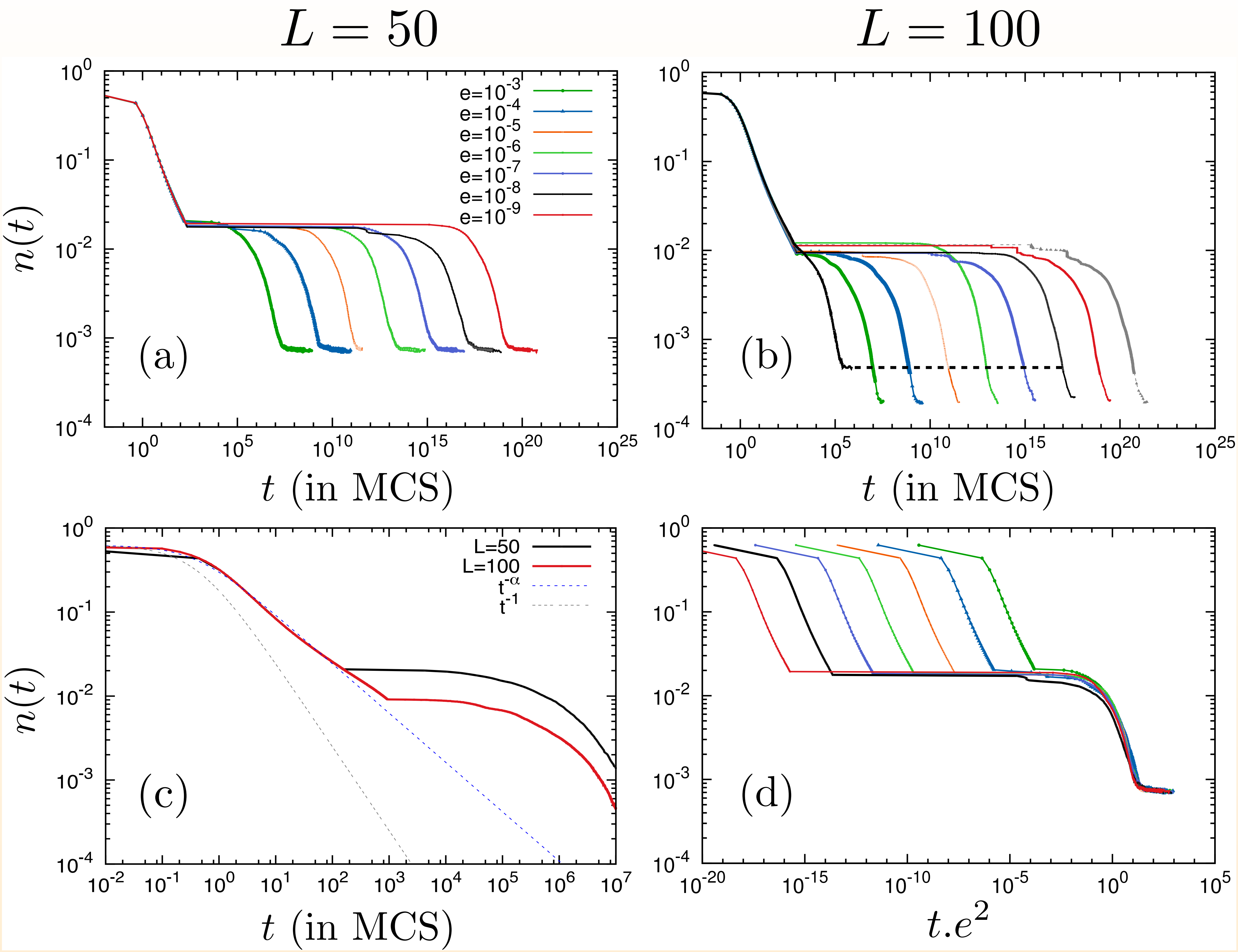
$$L = 200,\ a = 5,\ b = 1,\ d = e = 10^{-5}$$



$L = 50$ $\qquad$ $L = 100$

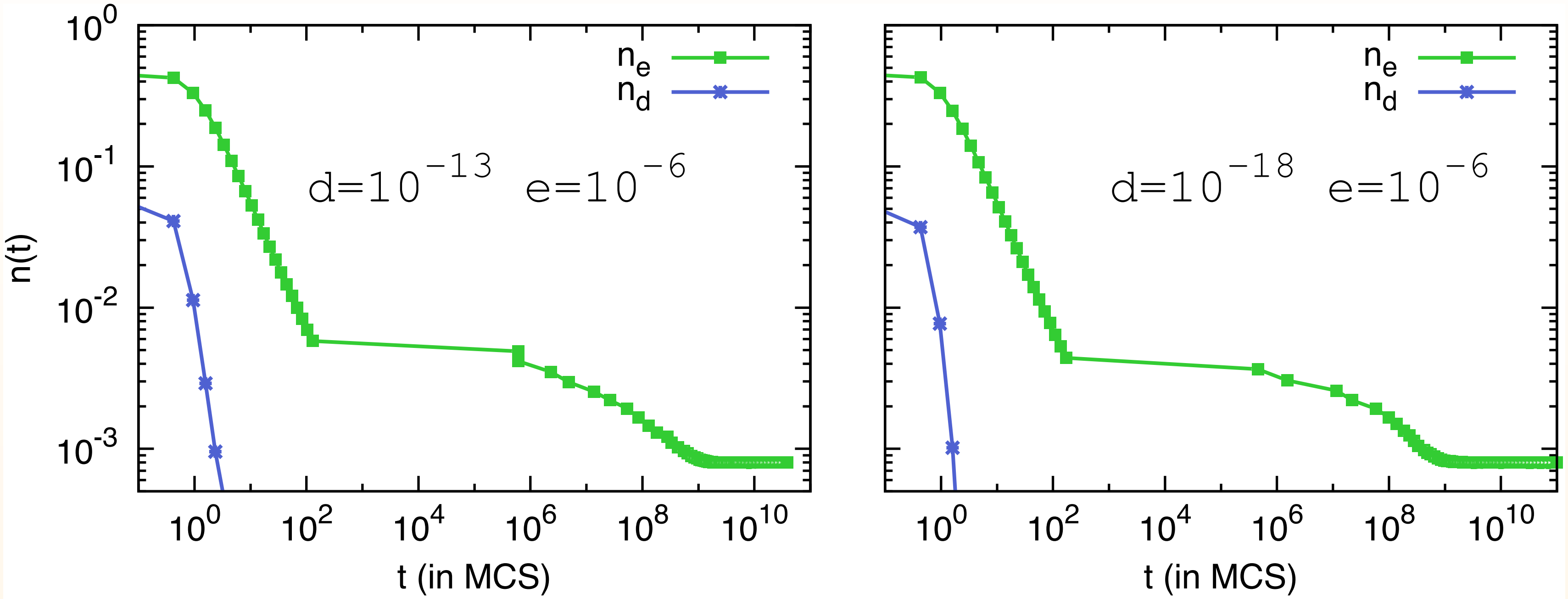


$d = 10^{-18}$, $e = 10^{-6}$



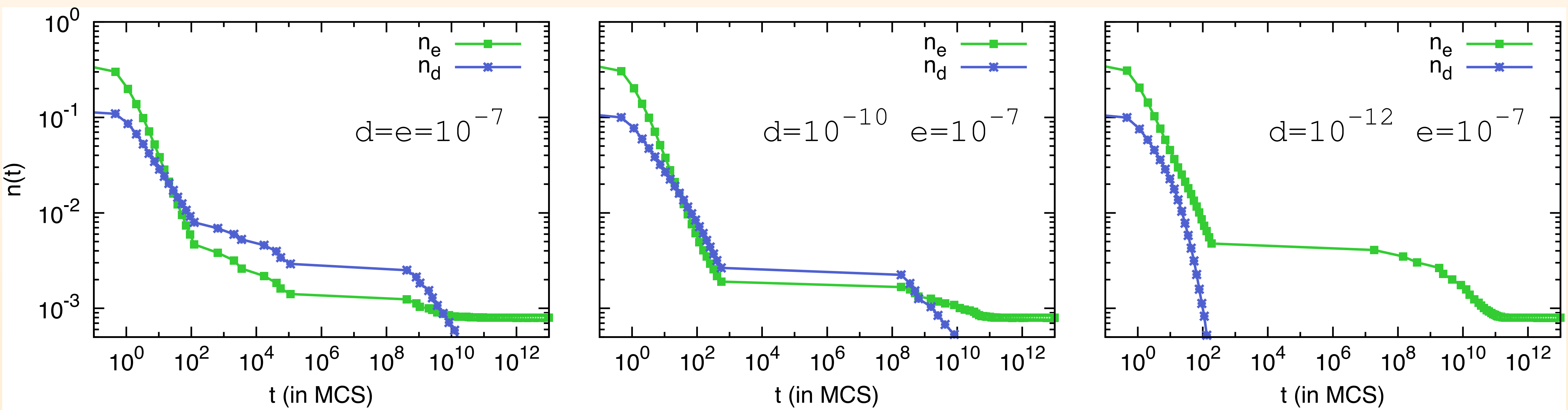
$L = 50$
$L = 100$
e=10^-3
e=10^-4
e=10^-5
e=10^-6
e=10^-7
e=10^-8
e=10^-9
(a)
(b)
(c)
(d)
$n(t)$
t (in MCS)
L=50
L=100
t^-α
t^-1
$t.e^2$

$$a = 5,\ b = 1,\ d = e^2$$



n(t)
t (in MCS)
n_e
n_d
d=10^{-13} e=10^{-6}
d=10^{-18} e=10^{-6}

(a) (b)



n(t)
t (in MCS)
n_e
n_d
d=e=10^{-7}
d=10^{-10} e=10^{-7}
d=10^{-12} e=10^{-7}

(a) (b) (c)