CROSSING A PHASE TRANSITION OUT OF EQUILIBRIUM

Alberto Sicilia Garcia

Phases of matter

Matter can be found in many different phases.

• Characterize the phases of matter is a central problem in many fields of physics.

Some examples:

Energy scale.

- Cosmology: Phases of matter in the first instants of the universe.
- QCD: gluon-plasma phase transition.
- Biophysics: phases of polymers in a solution.
- Condensed matter: insulator/conductor/superconductor.

Out of equilibrium dynamics

- **Great advances have been made in the study of equilibrium properties of phase transitions.**

- **A natural step forward: what happens if we force the system to cross out of equilibrium a phase transition?**

  How?

  - Changing abruptly a control parameter (ex. the temperature).
  - Changing abruptly a microscopical parameter (ex. a coupling constant).

- **The system will evolve on time until reaching its new equilibrium state. Slow dynamics ("aging").**

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- **Note that this is only a particular example of an out of equilibrium process. One can also study, for example, out of equilibrium stationary states:**

  - Coupling the system at two thermal resevoirs at different temperatures.
  - Constantly injecting energy to the system (“shaking”).

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Defect formation

- **When crossing a phase transition out of equilibrium, defects are created.**

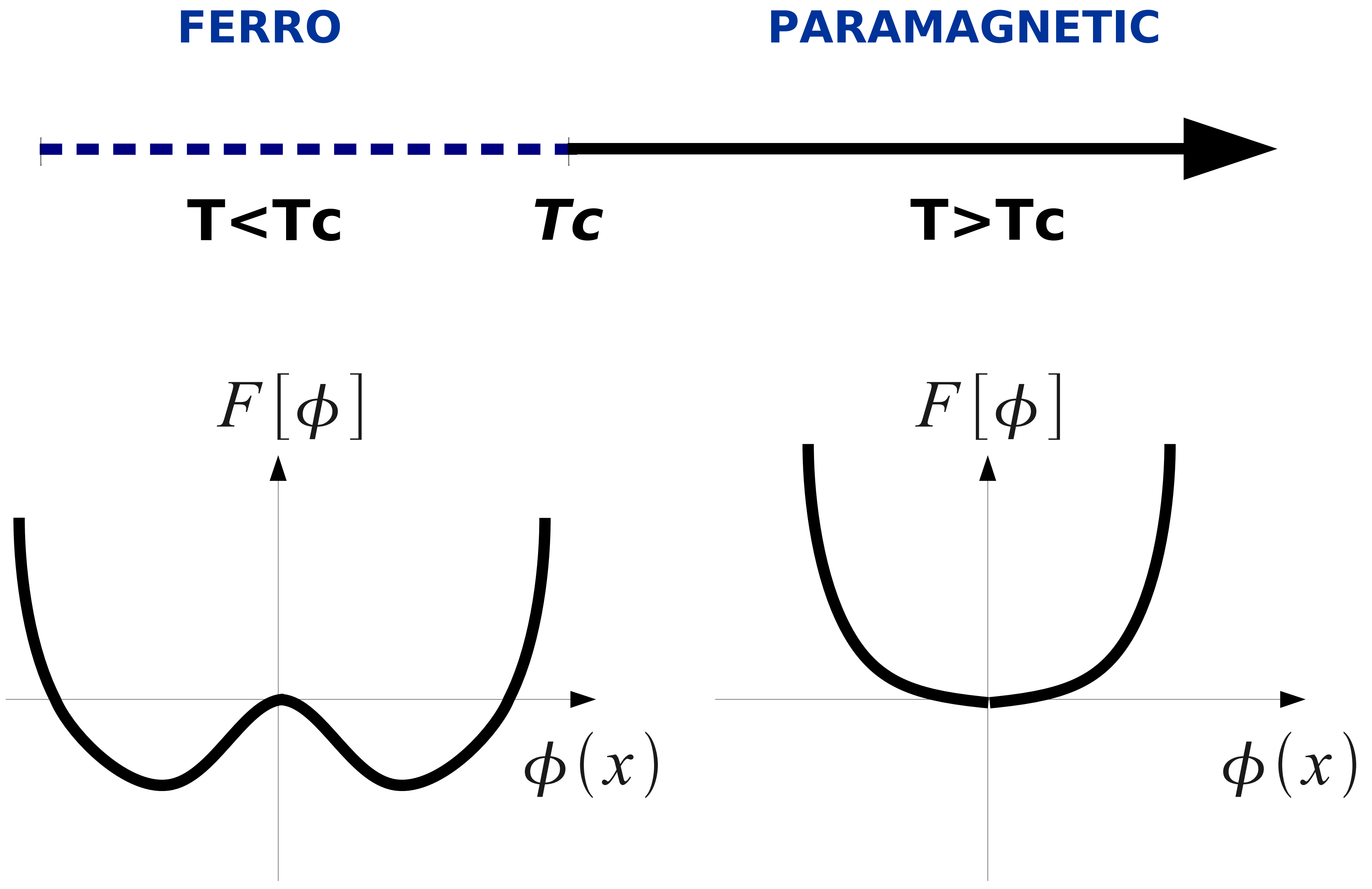
- **On the subsequent time evolution, defects will progressively disappear until the system reaches the new equilibrium state.**

- **A possible way to study the out of equilibrium evolution of the system is then to focus on defect dynamics.**

But...

- What defects are?
- Why are they created when crossing a phase transition?

Defect formation



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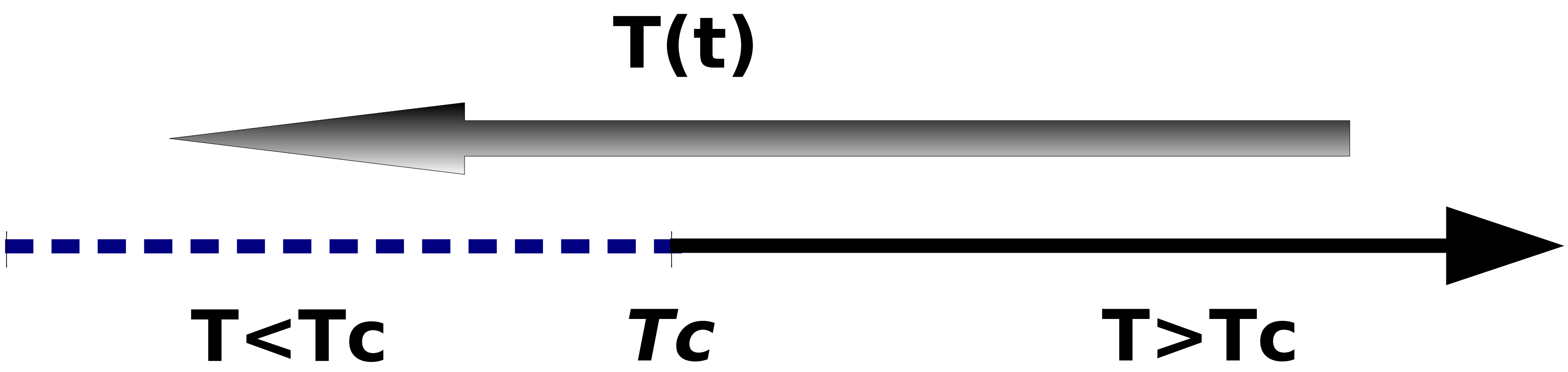

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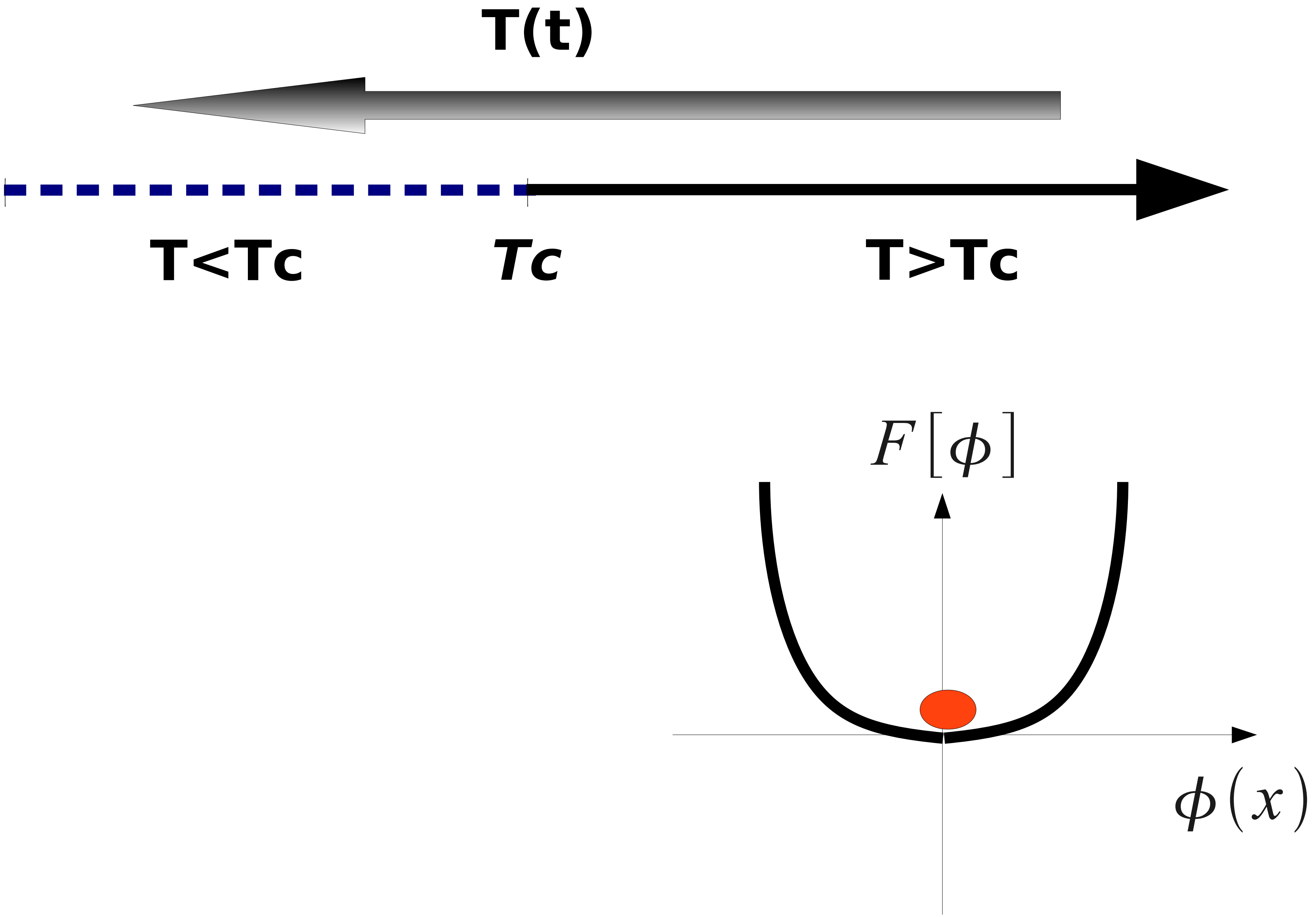

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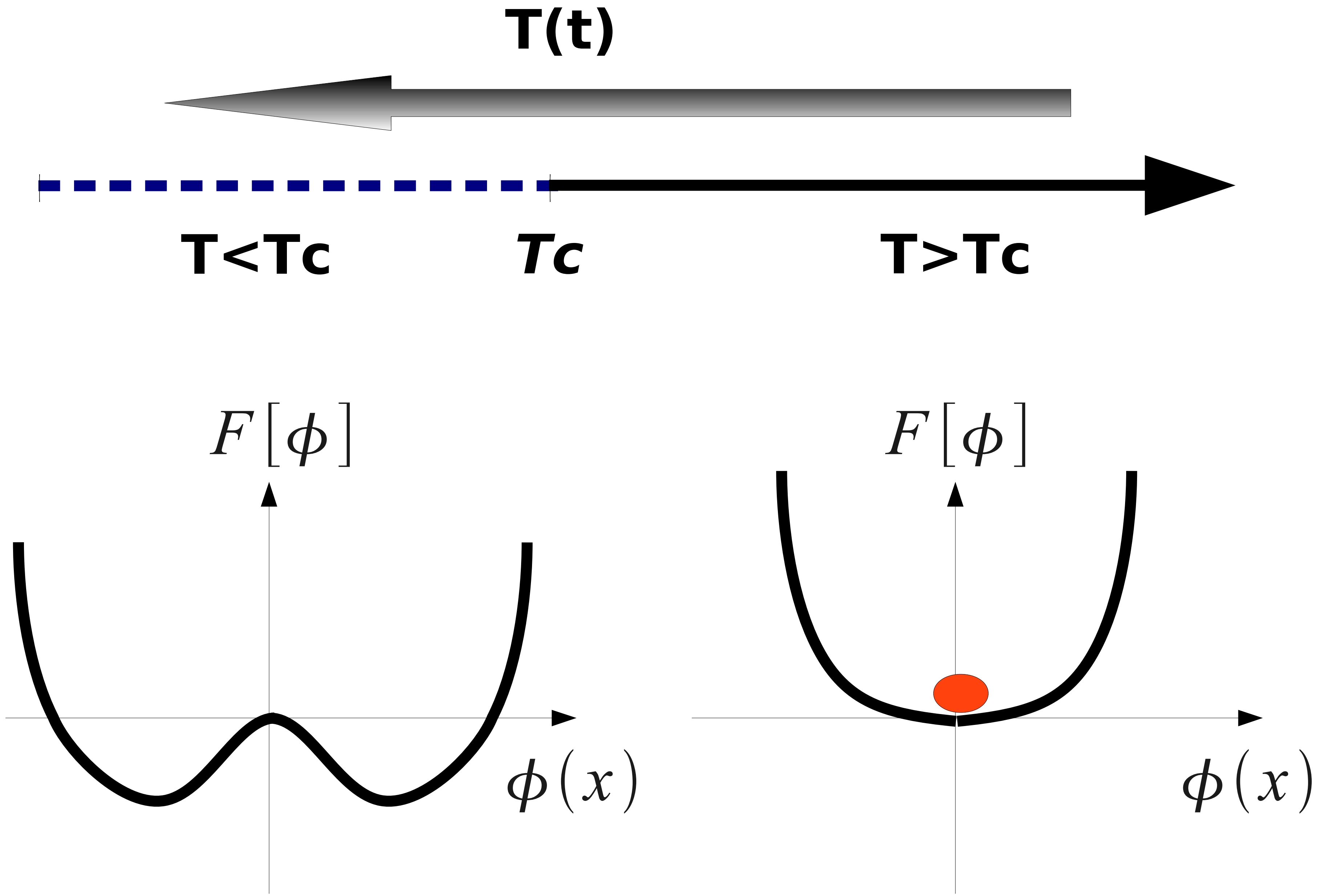

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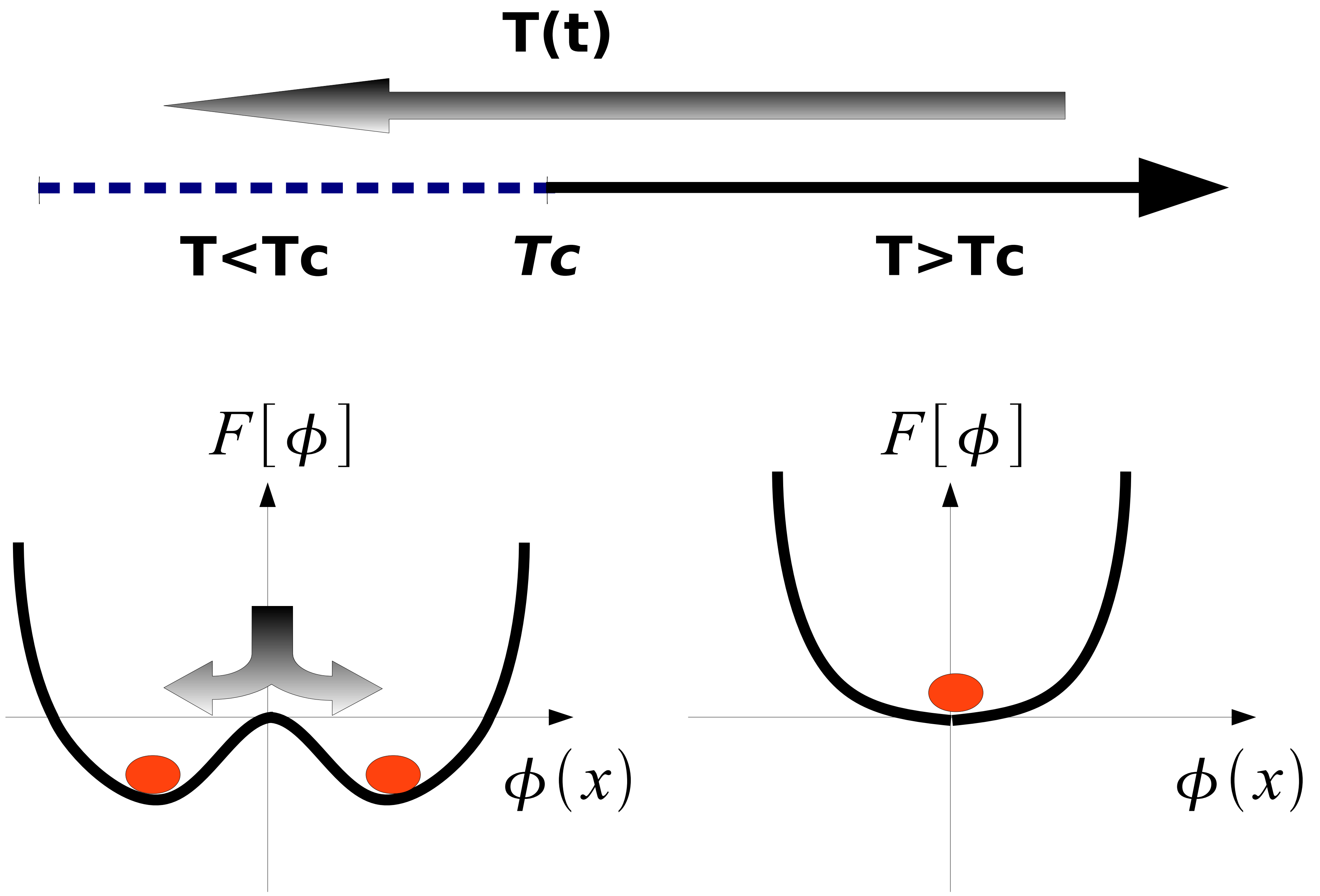

Defect formation

- **Below Tc, the order parameter must choose a vacuum.**

- **But in different regions of the space, the order parameter chooses different vacua...**

$\Rightarrow$ **Formation of TOPOLOGICAL defects.**

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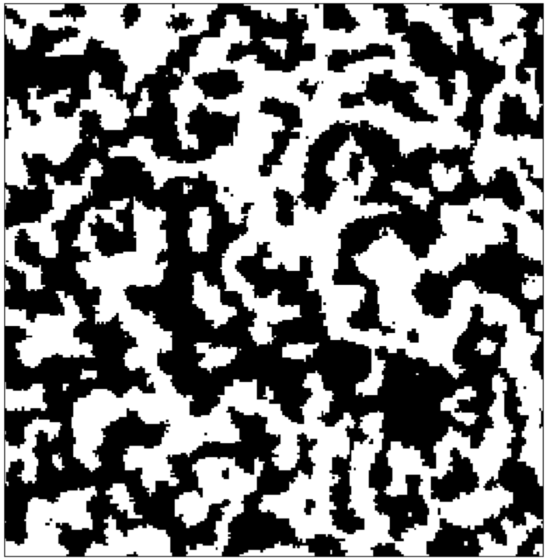
- **An Example of topological defects.**



Domain Walls: Closed hypersurfaces in d-1 dimensions.

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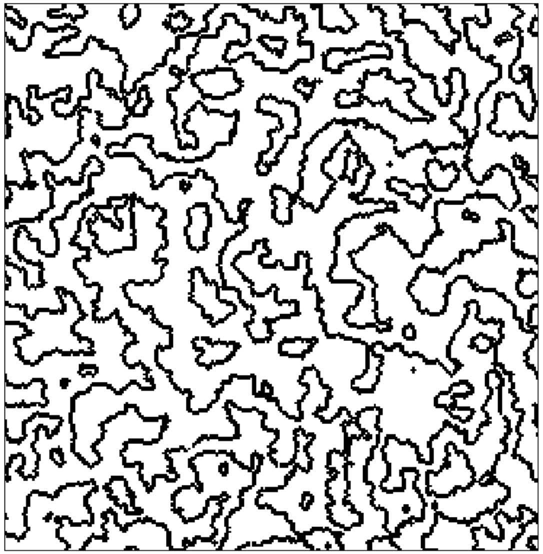

Domain Walls: Closed hypersurfaces in d-1 dimensions.

Defect formation

➢ **Depending on...**

- The nature of the order parameter (space and field dimensionalities, scalar/vector, real/complex...).

- How the symmetry is broken.

$\Rightarrow$ **Different kinds of topological defects!**

- Domain walls.

- Strings.

- Monopoles.

- Textures.

GOAL OF THIS THESIS:

“Gain some understanding in the dynamics of systems forced to cross a phase transition out of equilibrium.”

In particular, study this problem from a geometrical point of view: focusing on defect dynamics.

Particular problems we studied

An equilibrium prelude:

- Geometrical properties of critical parafermionic models.

  with R. Santachiara and M. Picco.

Out-of-equilibrium dynamics:

- Geometry of domain growth in 2d.

  with J. Arenzon, A. Bray, LFC, I. Dierking and Y. Sarrazin.

- Relaxation in spatially extended chaotic systems: coupled map lattices.

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- Revision of the classical Kibble-Zurek mechanism.

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SUMMARY OF THE TALK

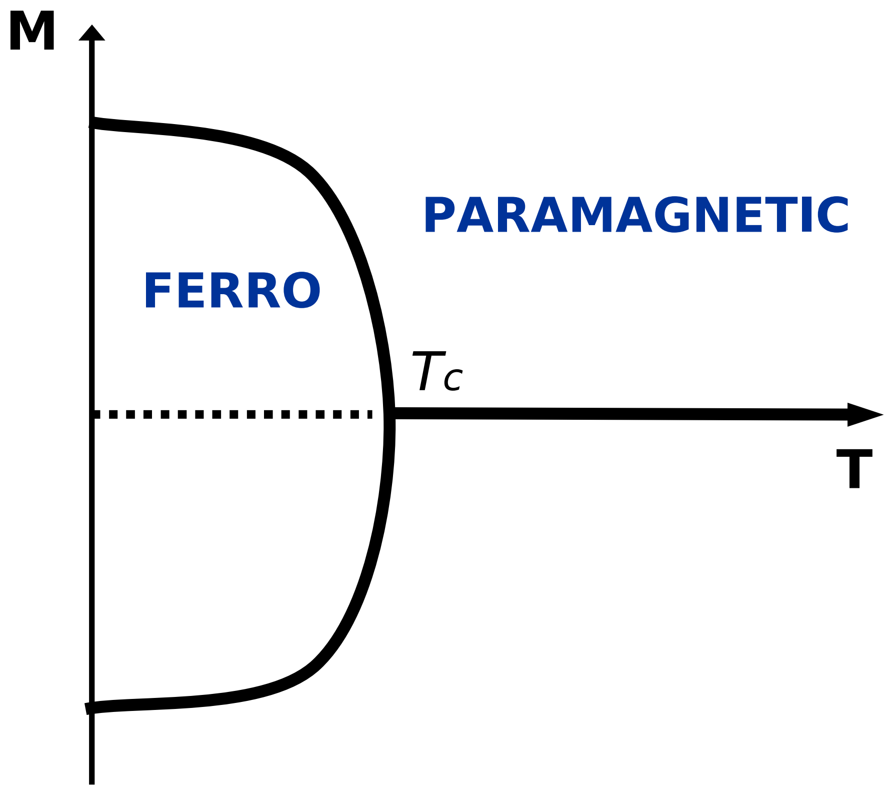
1.- Phenomenology of “domain growth” (“coarsening”, “phase ordering dynamics”).

2.- OUR WORK: Geometrical description of 2d domain growth.

3.- Conclusions.

1.- PHENOMENOLOGY OF DOMAIN GROWTH

What is domain growth?



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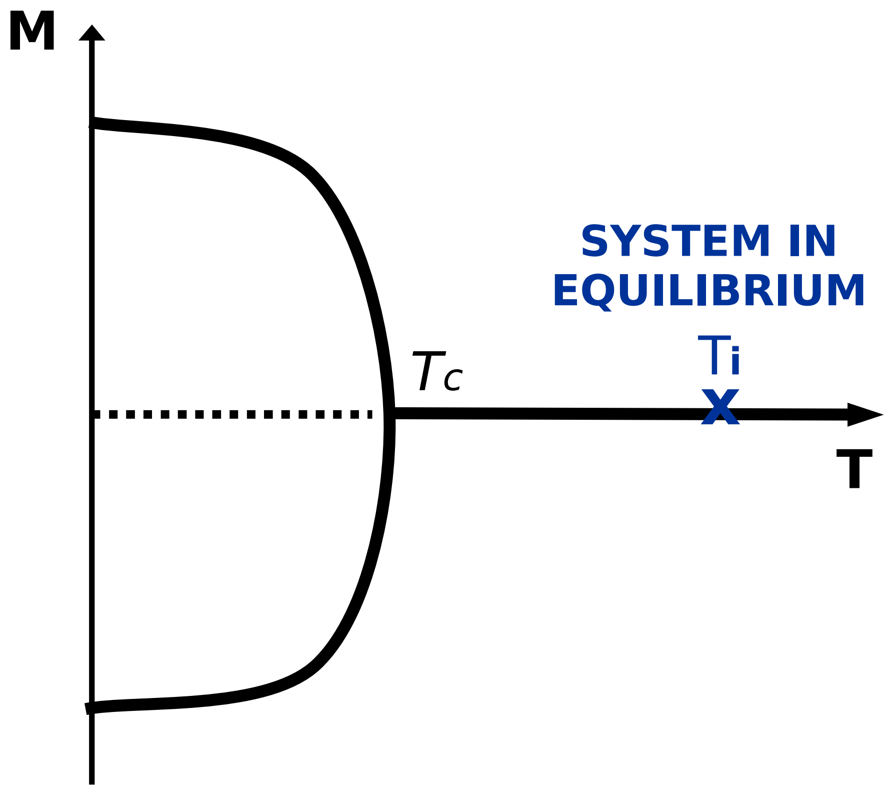

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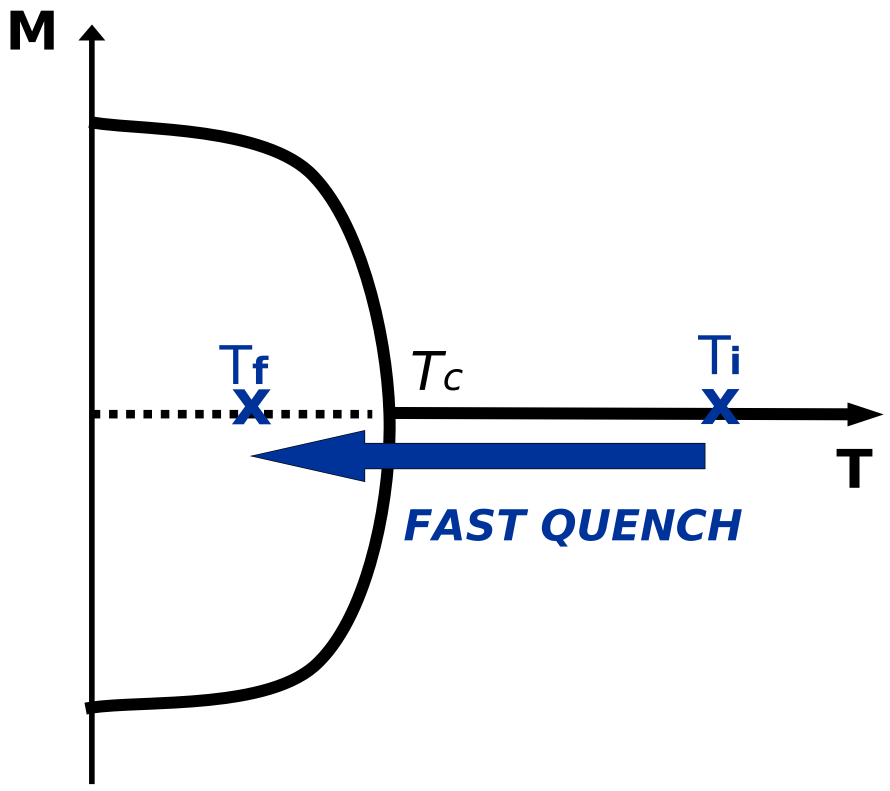

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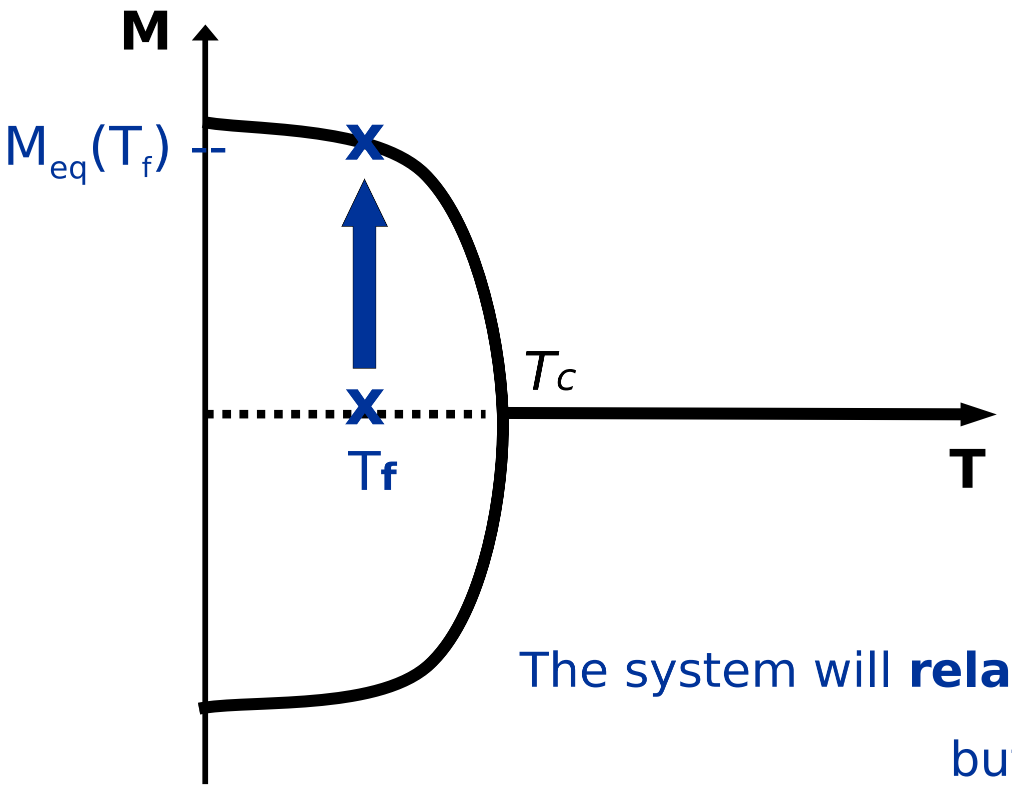

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but **HOW**???

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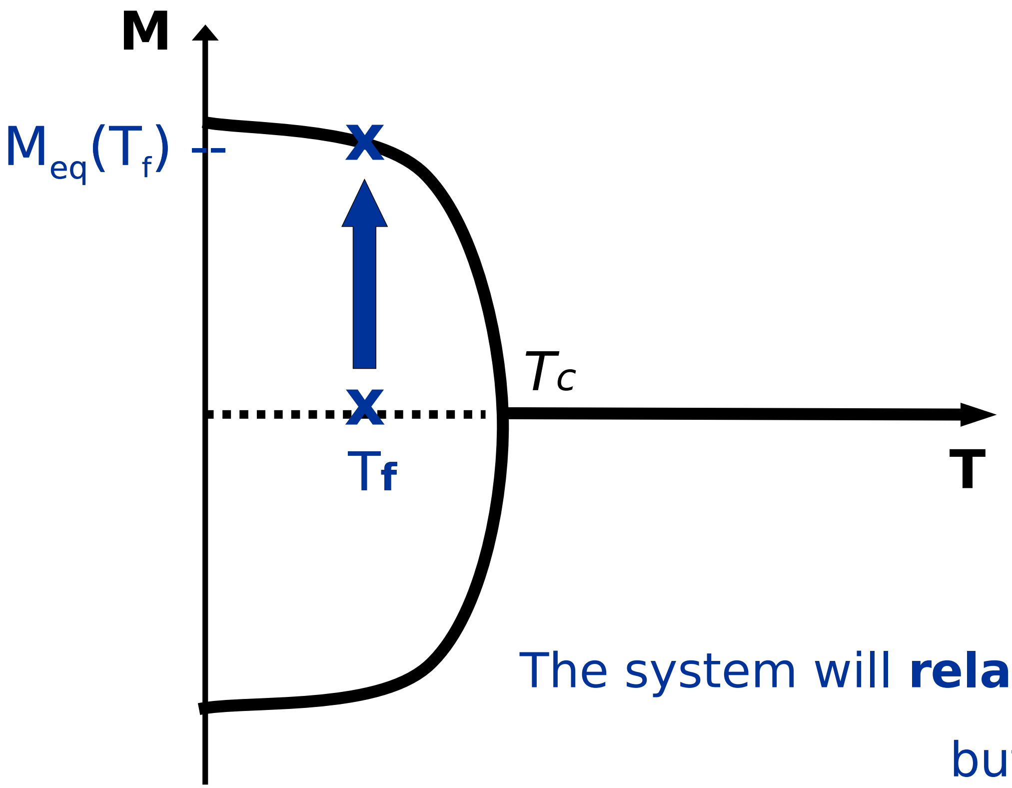

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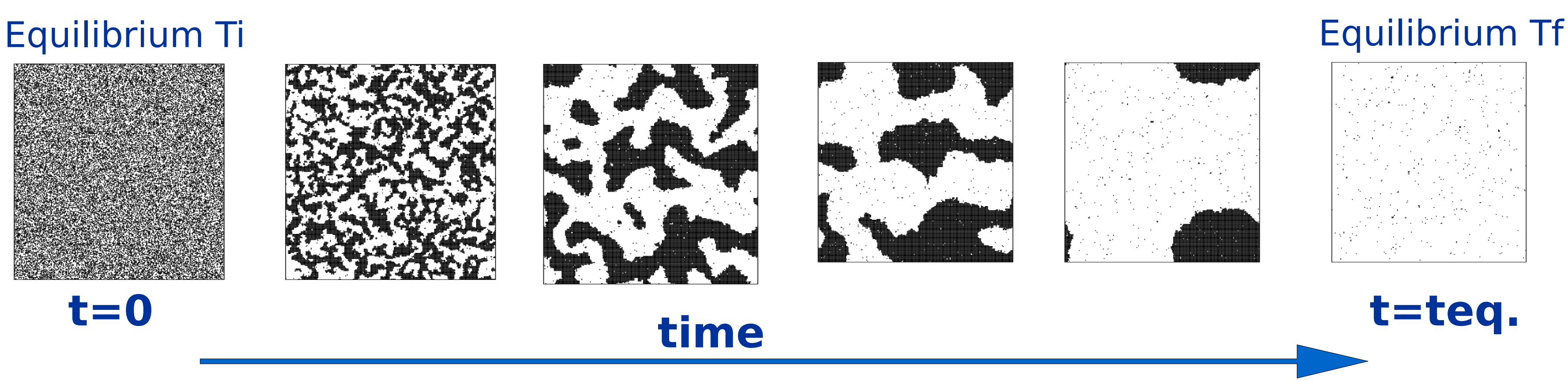

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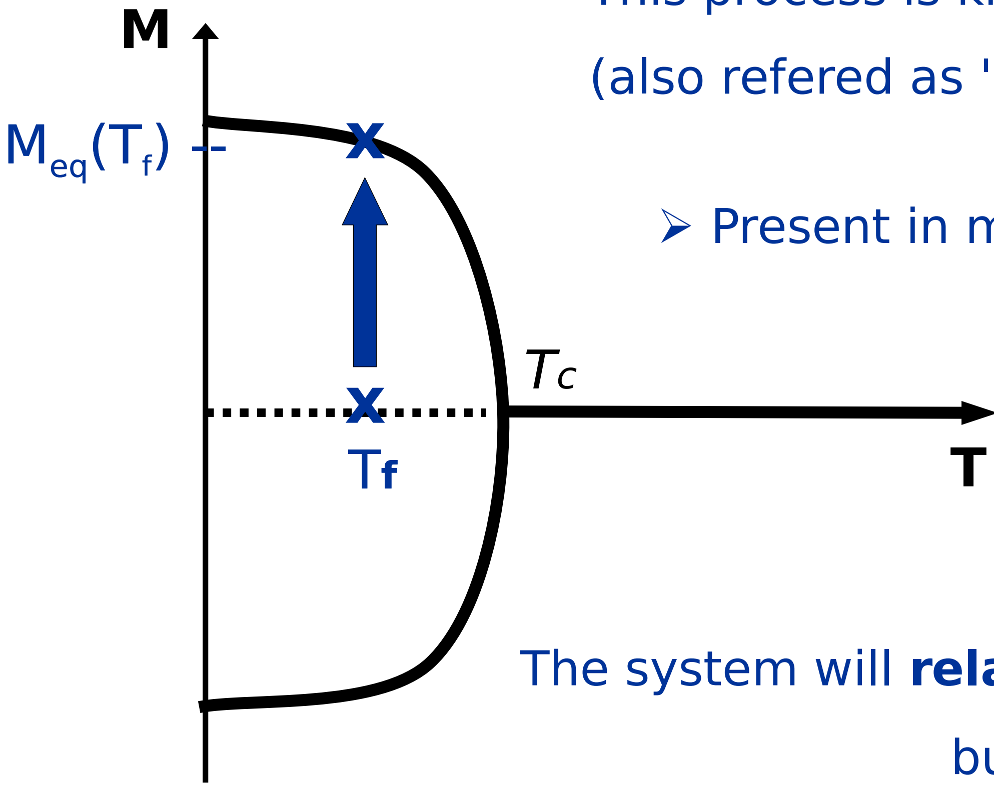

This process is known as **DOMAIN GROWTH**

(also refered as 'phase ordering' or 'coarsening').

- Present in many different systems.

The system will **relax to the new equilibrium state**...

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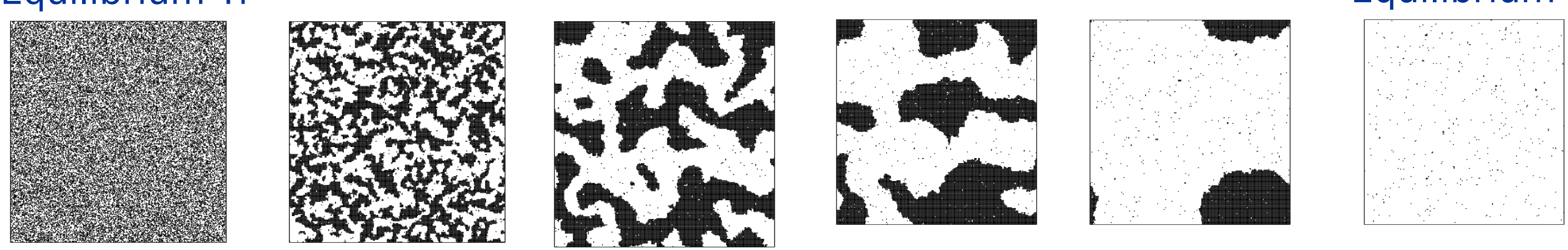
Equilibrium Ti

Equilibrium Tf



t=0

time

t=teq.

What is domain growth?

DYNAMICAL SCALING HYPOTHESIS

- There exists a **characteristic length scale R(t):**

  If we measure all distances in units of R(t), the system is statiscally the same at any time.

$$R(t) \sim t^{1/2}$$

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- **There was no analitycal proof in 2d.**

3.- OUR WORK:

GEOMETRICAL DESCRIPTION OF 2D DOMAIN GROWTH

Mathematical formalism

- Mathematical model we used: Langevin equation.

$$\frac{\partial \phi(\vec{x},t)}{\partial t} = -\frac{\delta H[\phi]}{\delta \phi(\vec{x},t)} + \eta(\vec{x},t)$$

$$\langle \eta(\vec{x},t) \rangle = 0 \qquad \langle \eta(\vec{x},t)\eta(\vec{x}',t') \rangle = 2T\delta(\vec{x}-\vec{x}')\delta(t-t')$$

$$H[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{r}{2}\phi^2 + \frac{g}{4}\phi^4 \right]$$

- There are other mathematical models one could consider.

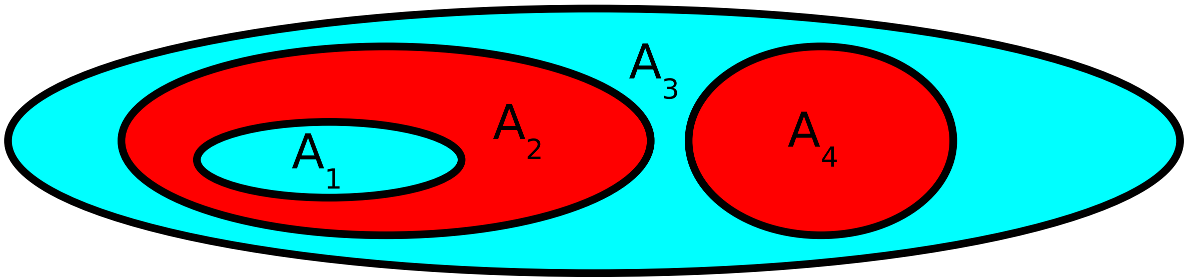
  Examples:

  - Instead of a coarse-grained field, a more microscopical approach.
  - Instead of a continuous order parameter, a lattice model.
  - Instead of a Langevin equation, a master equation.
  - Instead of stochastic dynamics, deterministic dynamics (Hamilton eqs.).
  - Instead of a classical system, a quantum system.

Distribution of areas in quenches to $T=0$

Geometrical Description of 2D Domain Growth

- **Our defects are the domain walls (a.k.a. “hulls”).**
- **Two different areas associated to each domain wall.**



➢ **Domain area.**

There are 4 domains with areas:

A_1 A_2 A_3 A_4

➢ **Hull enclosed area.**

There are 4 hulls with enclosed areas:

A_1 A_1+A_2 A_4 $A_1+A_2+A_3+A_4$

$\mathbf{n_d(A,t)}$ and $\mathbf{n_h(A,t)}$ Area distributions densities (per unit area of the system).

Distribution of areas in a quench to T=0

- From Langevin equation at T=0, Allen and Cahn (1979),

$$v = -\frac{\lambda_h}{2\pi}\,\kappa$$

- The velocity of a wall in each point is proportional to its local curvature.

- Velocity points in the direction of reducing curvature.

Coarsening does not proceed by coalescence of small domains.

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- Integrating the Allen-Cahn eq. around a hull:

$$\oint v\,dl = -\frac{\lambda_h}{2\pi}\oint \kappa\,dl$$

$$\frac{dA_h}{dt} = -\frac{\lambda_h}{2\pi}\oint \kappa\,dl$$

???

Distribution of areas in a quench to T=0

$$\frac{dA_h}{dt} = -\frac{\lambda_h}{2\pi} \oint \kappa \, dl$$???

- We can use the Gauss-Bonnet th.

$$\int_M K_g \, dS + \oint_{\partial M} \kappa \, dl = 2\pi \chi(M)$$

- Hull-enclosed areas are flat 2d manifolds with no holes.

$$K_g = 0 \qquad \chi(M) = 1 \qquad \oint_{\partial M} \kappa \, dl = 2\pi$$

- Evolution equation for hull-enclosed areas:

$$\frac{dA_h}{dt} = -\lambda_h \qquad A_h(t, A_i) = A_i - \lambda_h (t - t_i)$$

- Hull enclosed areas **evolve independently one from another**.
- They **all shrink at a constant rate**, independently of their size.

Distribution of areas in a quench to T=0

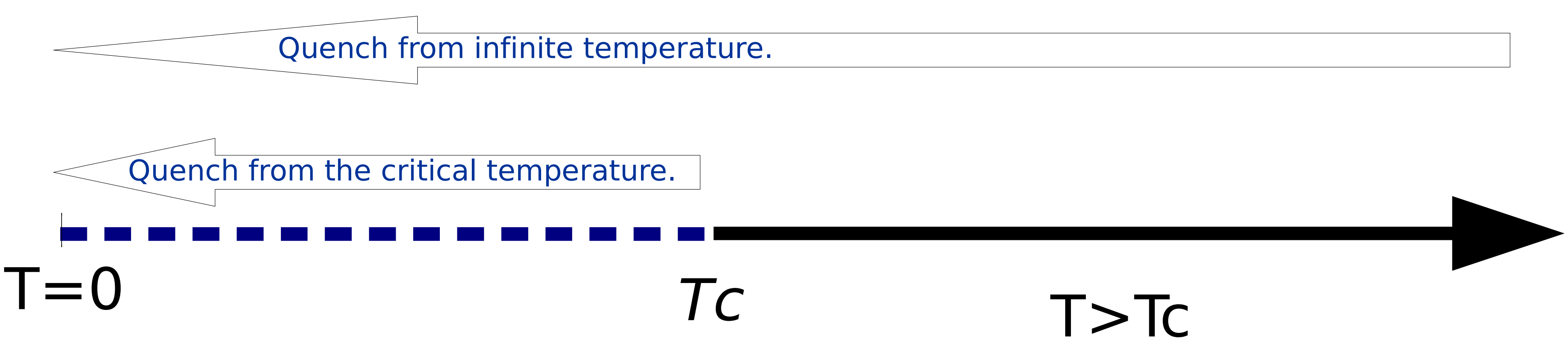
$$n_h(A,t) = \int_0^\infty dA_i\, \delta(A - A_i + \lambda_h(t - t_i))\, n_h(A_i, t_i)$$
$$= n_h(A + \lambda_h(t - t_i), t_i)\ .$$

- If you **know the distribution at initial time**, when the system is quenched, you **know the distribution at any time during the evolution**.

- Remember, we are quenching our system to T=0.

- Before the quench, the system is in equilibrium at some temperature $T>T_c$.

- The two extreme cases are:



Distribution of areas in a quench to T=0

➢ We are lucky, because the equilibrium distributions for both initial cases are known (Cardy and Ziff, 2002):

$$n_h(A,0) \sim \begin{cases} 2c_h/A^2 & \text{Critical percolation} \Leftrightarrow T_0 = \infty \\ c_h/A^2 & T_0 = T_c \end{cases} \qquad c_h = 1/8\pi\sqrt{3}$$

➢ Pluging-in these initial distributions into the evolution equation:

$$n_h(A,t) = \frac{2c_h}{(A+\lambda_h t)^2} \qquad \text{Quench to T=0 from } T_0 = \infty$$

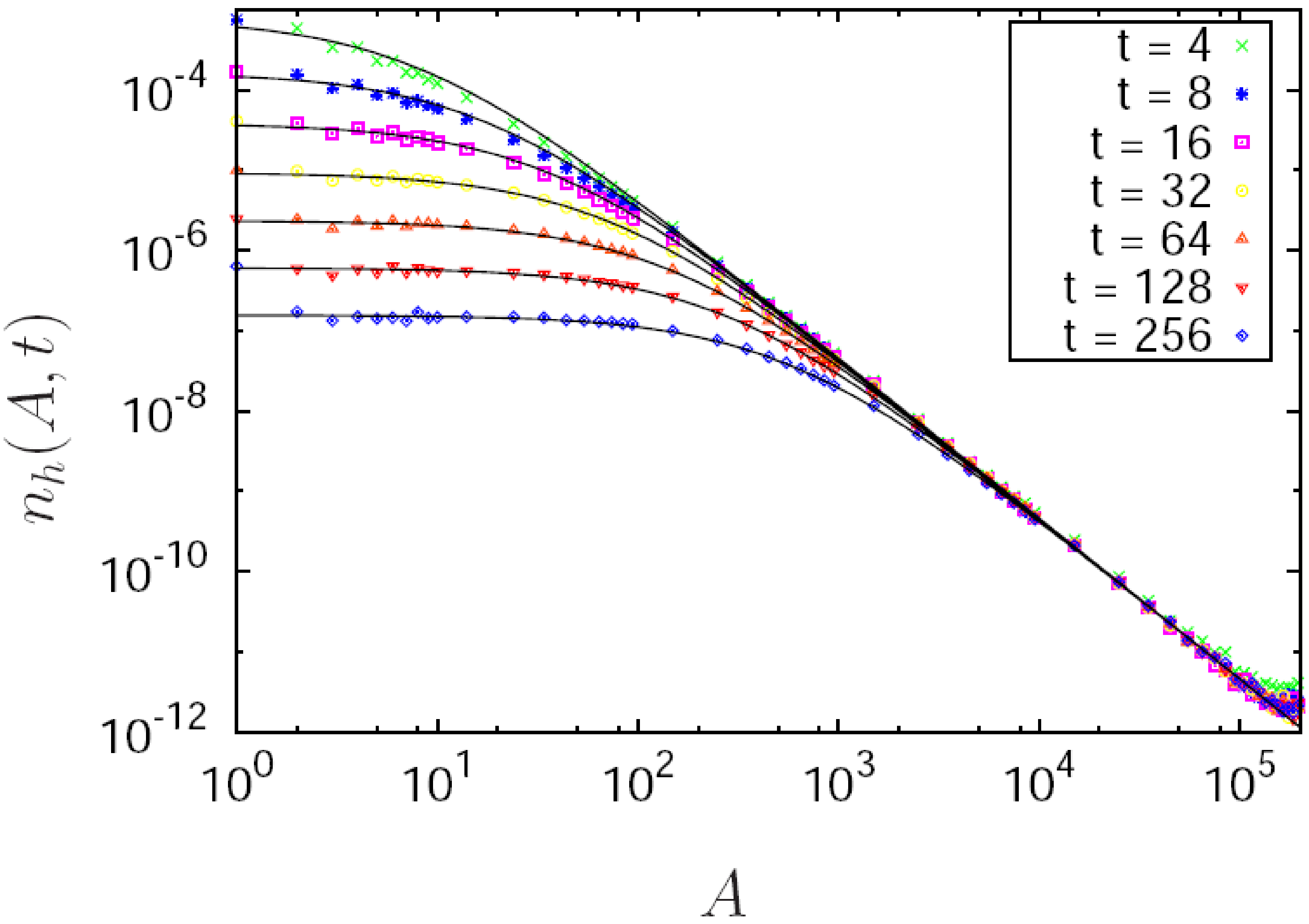
$$n_h(A,t) = \frac{c_h}{(A+\lambda_h t)^2} \qquad \text{Quench to T=0 from } T_0 = T_c$$

➢ The distributions have the scaling form,

$$n_h(A,t) = t^{-2} f(A/t)$$

Scaling dynamical hypothesis is recovered from calculation!

Numerical verification of our exact result



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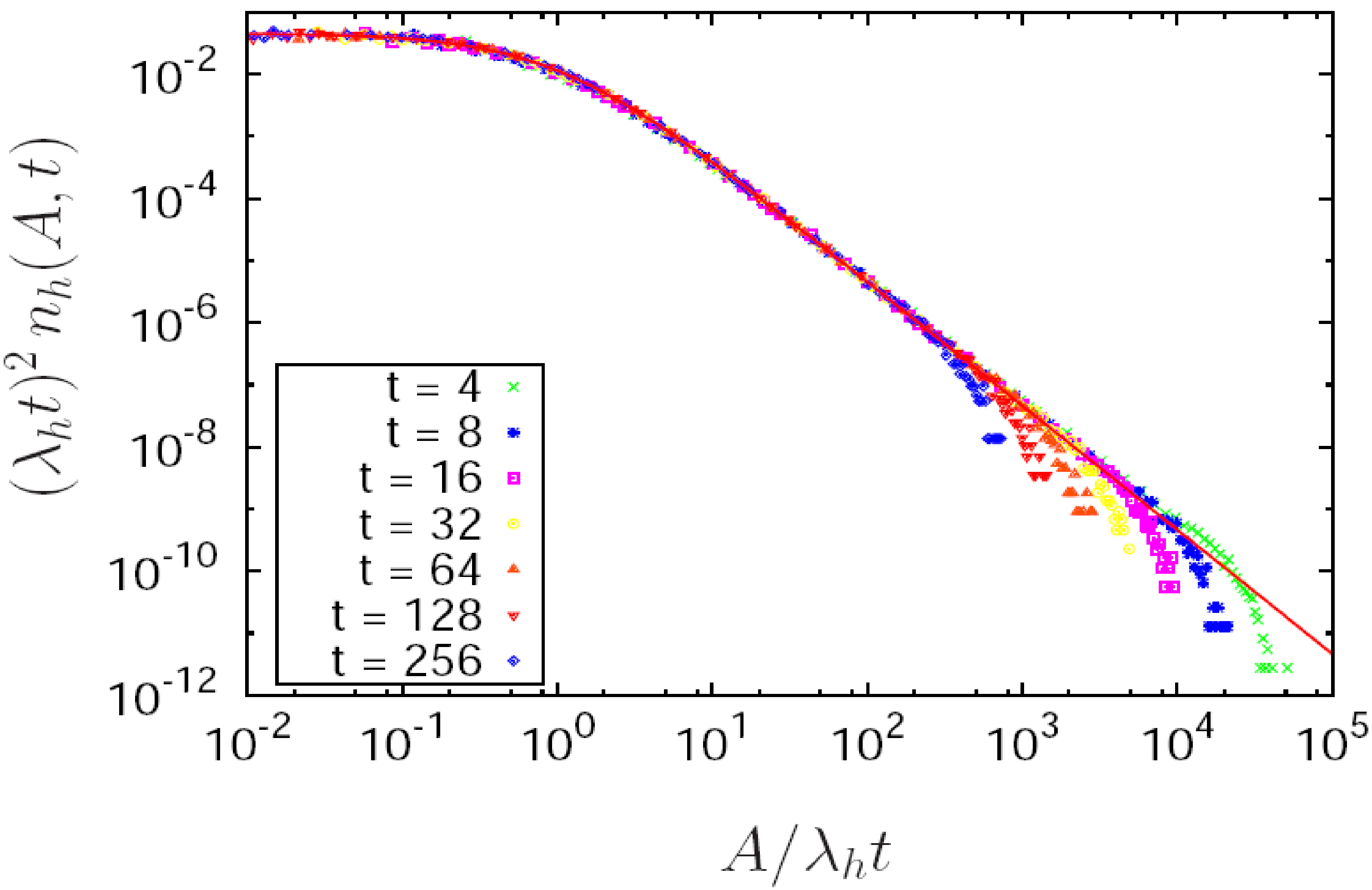

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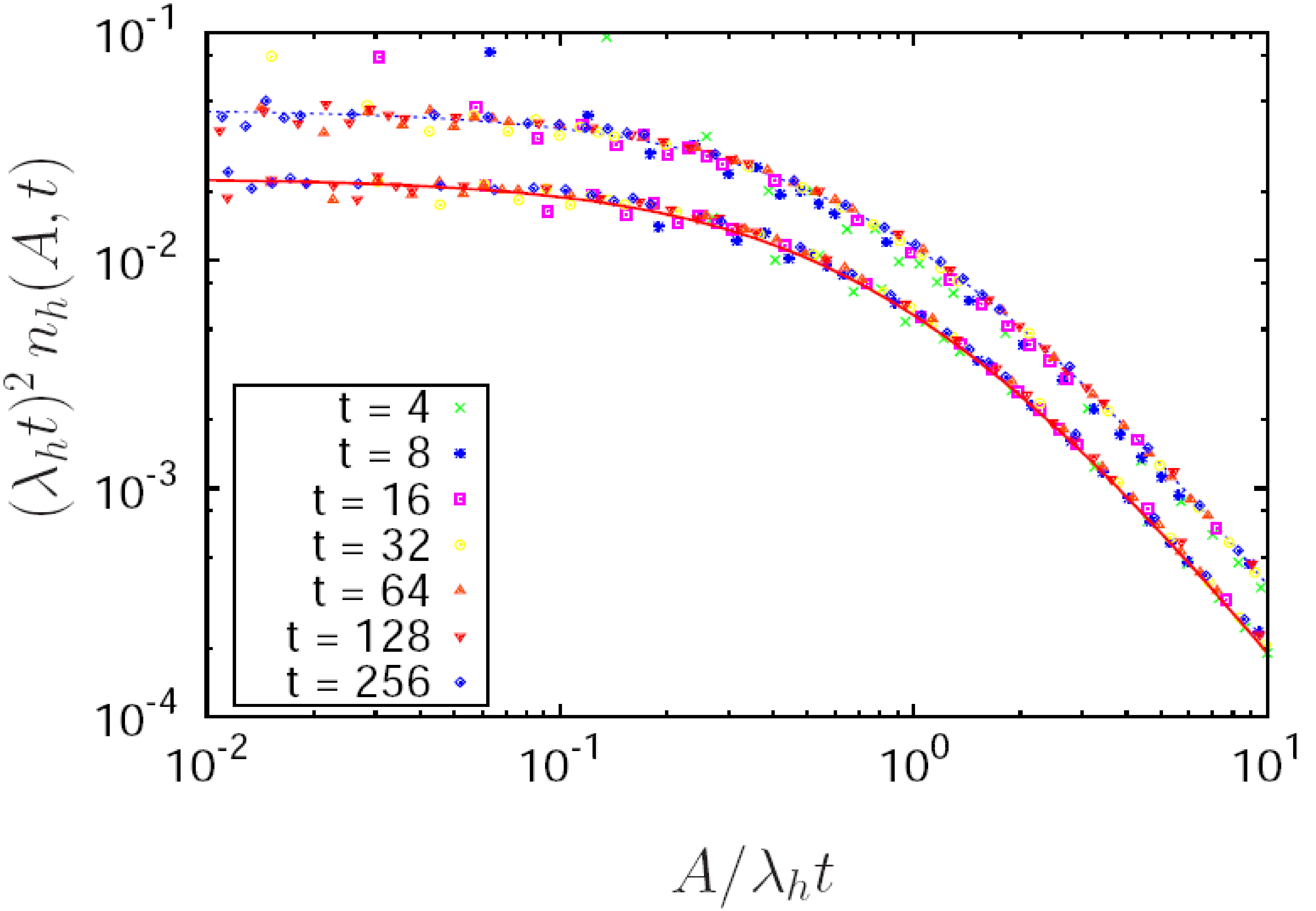
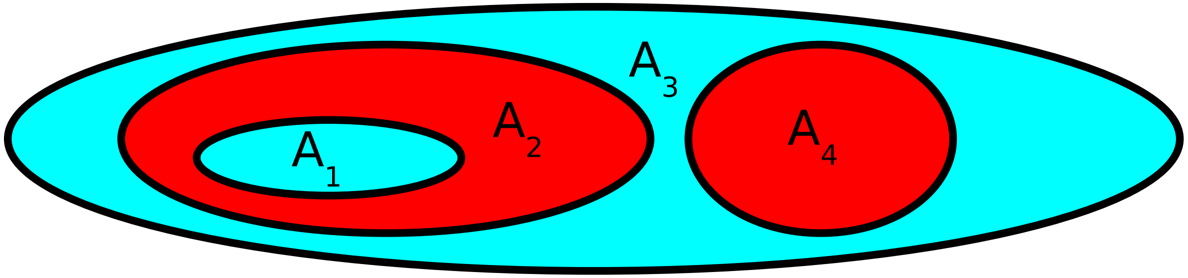


Geometrical Description of 2D Domain Growth

What happens for the domain areas?



- The evolution of a domain area will depend, in general on the movement of several walls.

For example, the evolution of A_3 :

$$\frac{dA_3}{dt} = -\lambda_h + \lambda_h + \lambda_h$$

In general :

$$\frac{dA_d(t)}{dt} = -\lambda_h \left[1 - \nu(t)\right]$$

Domain can grow/shrink/remain constant.

Distribution of areas in a quench to T=0

- Initial distributions are also known for domain areas.

$$n_d(A,0) \sim \frac{2c_d A_0^{\tau'-2}}{A^{\tau'}} \qquad \tau' = \frac{187}{91} \approx 2.055$$

$$n_d(A,0) \sim \frac{c_d A_0^{\tau-2}}{A^{\tau}} \qquad \tau = \frac{379}{187} \approx 2.027$$

- Taking advantage of the smallness of c_h one can (approximately) show:

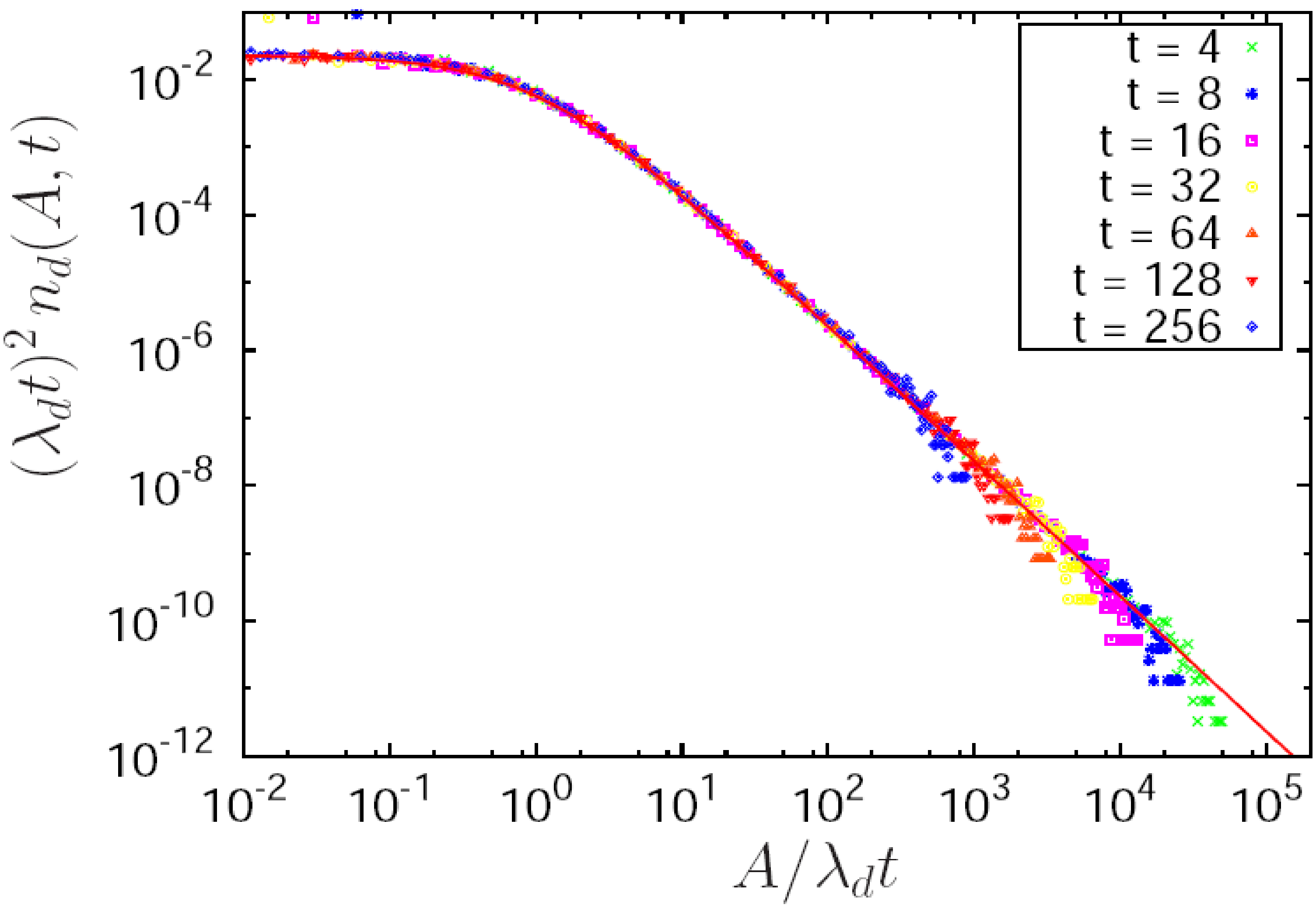
$$n_d(A,t) \sim \frac{2c[\lambda t]^{\tau-2}}{[A+\lambda t]^{\tau}}$$

- Two important sum rules:

$$N_d(t) \equiv \int_0^\infty dA\, n_d(A,t) = \int_0^\infty dA\, n_h(A,t) \equiv N_h(t)$$

$$\int_0^\infty dA\; A\, n_d(A,t) = 1$$

Distribution of areas in a quench to T=0

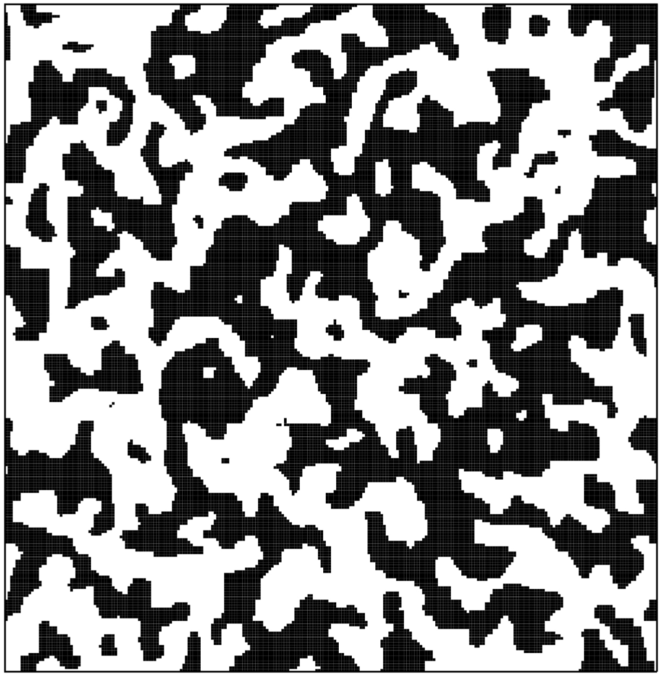


Effects of finite working temperature

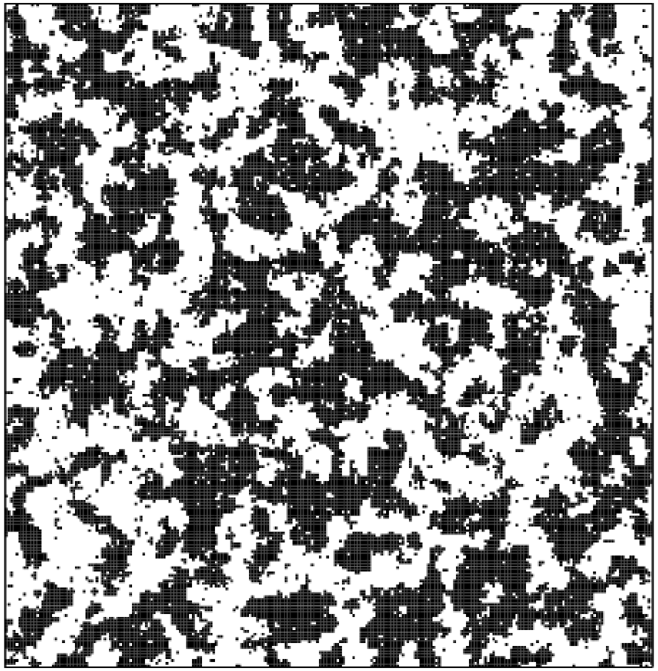
Domain area distributions during coarsening at T≠0

time = 32 MCS

T = 0



T = 1.5



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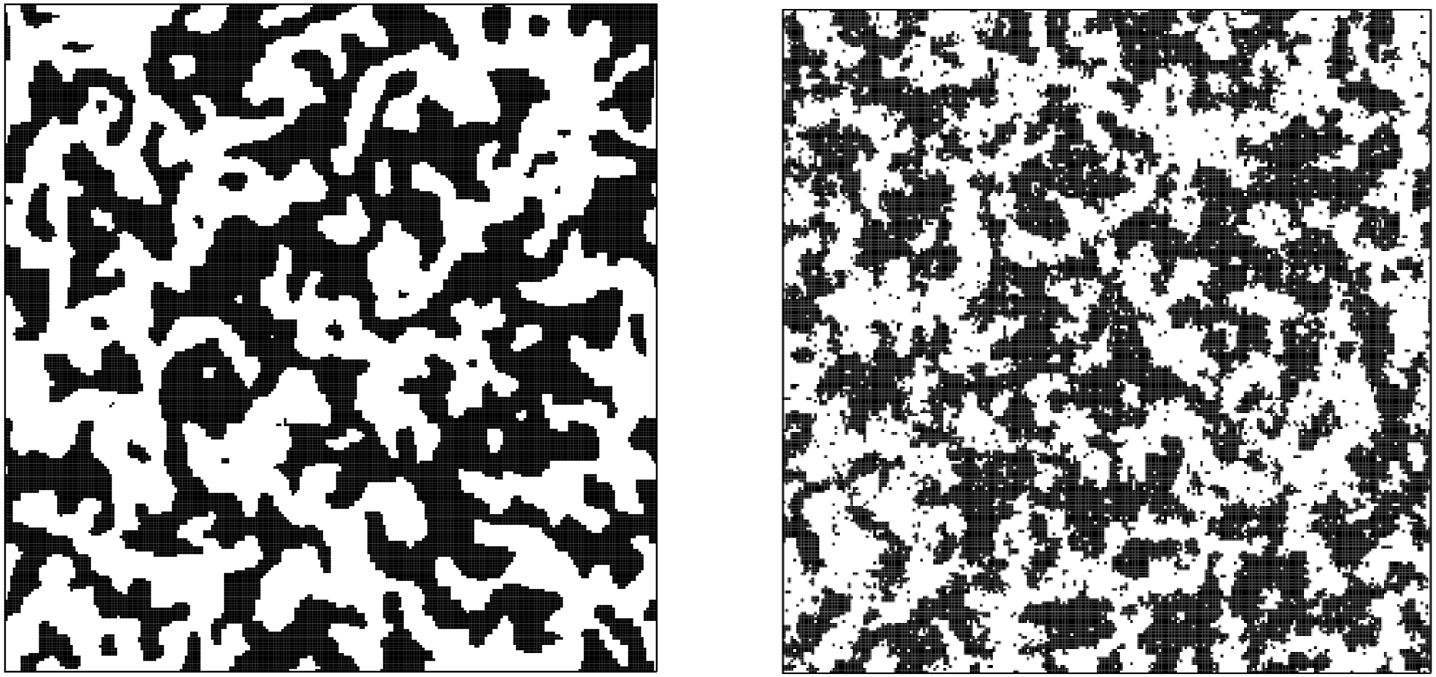

EFFECTS OF TEMPERATURE:

- Roughens the domain walls.
- Generates 'thermal domains' (not formed by coarsening).

Domain area distributions during coarsening at T≠0

- Distribution of domains areas during coarsening at T=0.

$$n_d(A,t) \sim \frac{2c[\lambda t]^{\tau-2}}{[A+\lambda t]^{\tau}} \quad \text{where} \quad \tau = \frac{187}{91} \quad c = \frac{1}{8\pi\sqrt{3}}$$

$$\sqrt{A_{av}} \sim R(t) \sim [\lambda\ t]^{1/2}$$

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- Temperature dependence enters through prefactor λ → λ(T)

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- Dynamical RG: dynamics at T<Tc controlled by T=0 fixed point.

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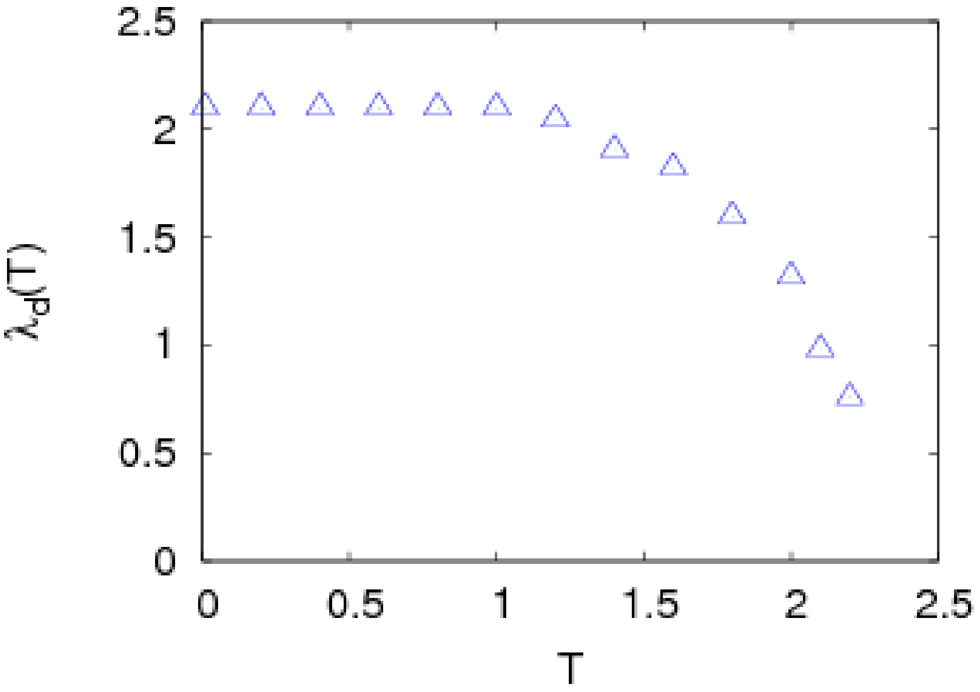
- How to measure λ(T)?

From the spatial-correlation function

$$C(r,t) \equiv \frac{1}{N}\sum_{i=1}^{N}\langle s_i(t)s_{i+r}(t)\rangle$$

which scales with $R(t)$

$$C(r,t) \sim m^2(T) f\left(\frac{r}{R(t)}\right)$$



Domain area distributions during coarsening at T≠0

- Let us test our prediction:

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- Simulations in the 2d Ising Model.

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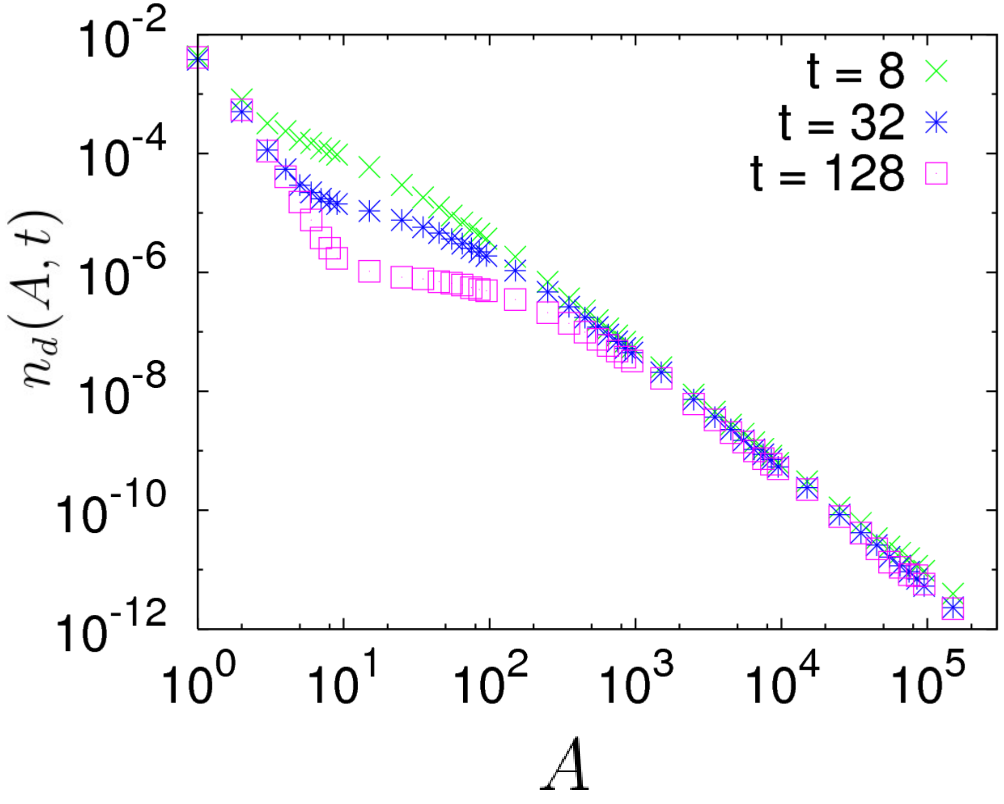

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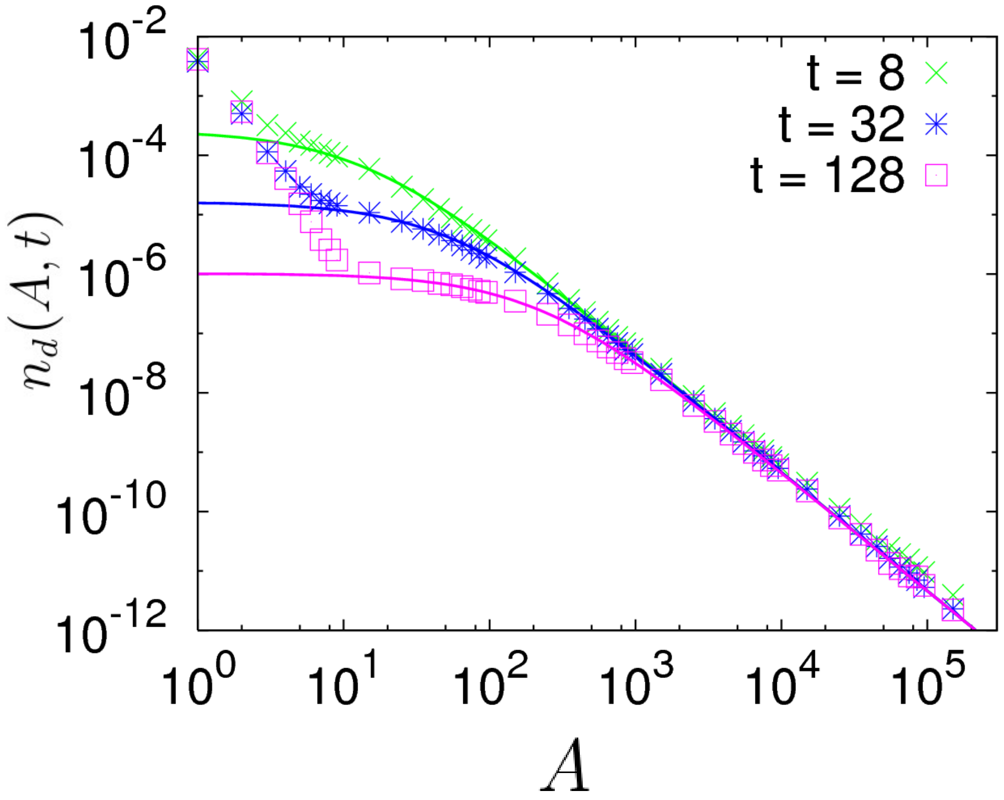

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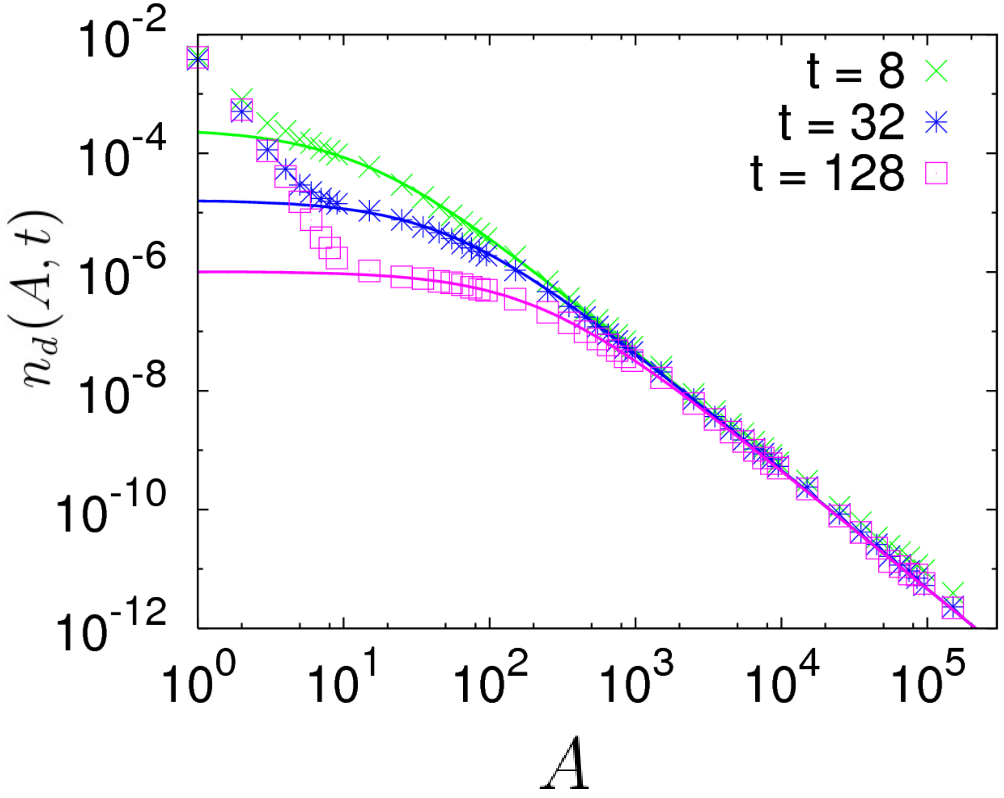
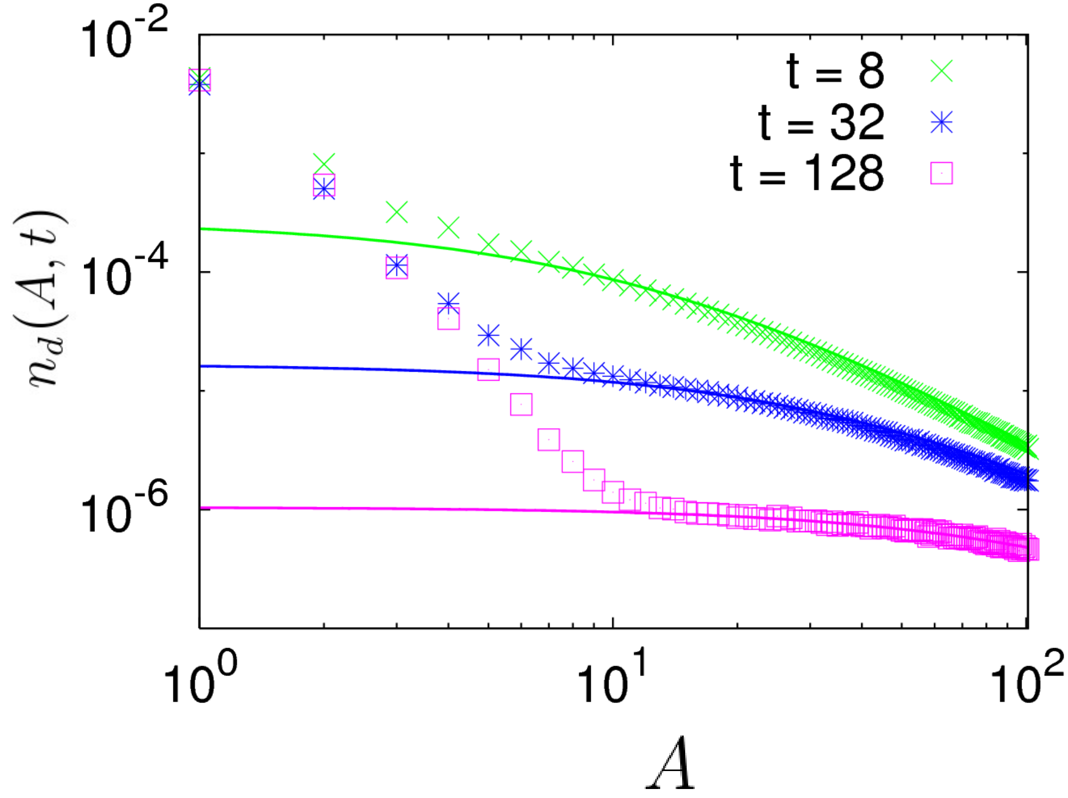


Zoom on the small domain region:



Domain area distributions during coarsening at T≠0

- Let us test our prediction:

$$n_d(A,t) \sim \frac{2c[\lambda(T)t]^{\tau-2}}{[A+\lambda(T)t]^{\tau}} \qquad \text{where} \quad \tau = \frac{187}{91} \quad c = \frac{1}{8\pi\sqrt{3}}$$

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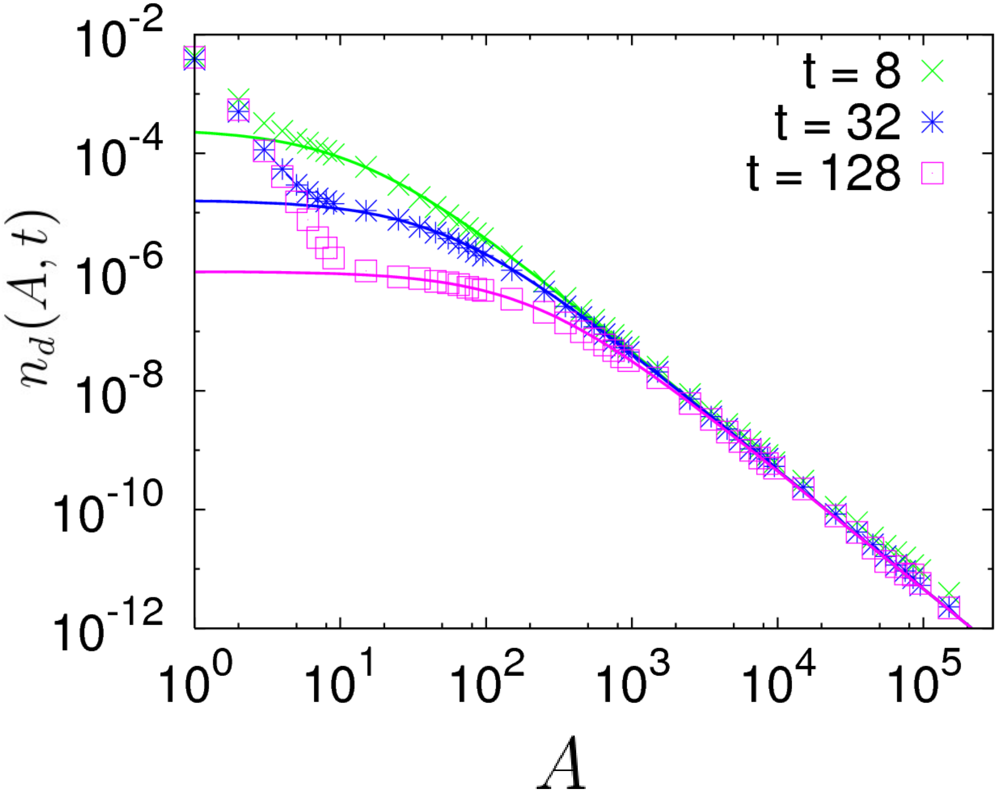

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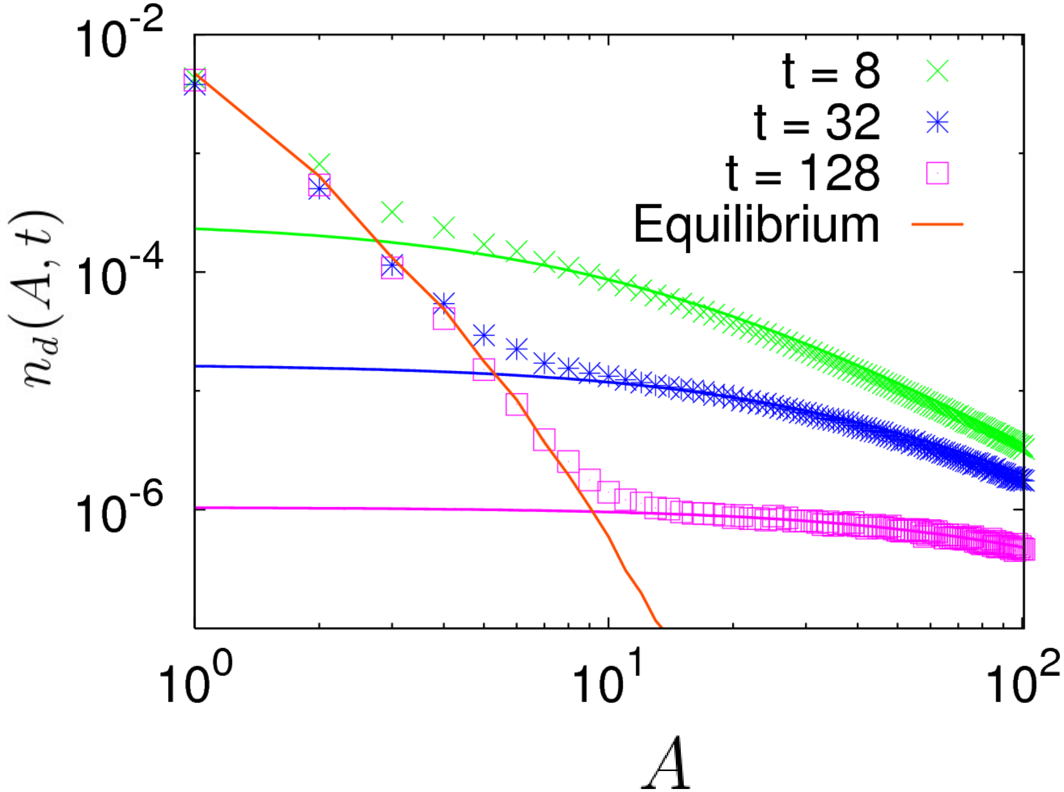
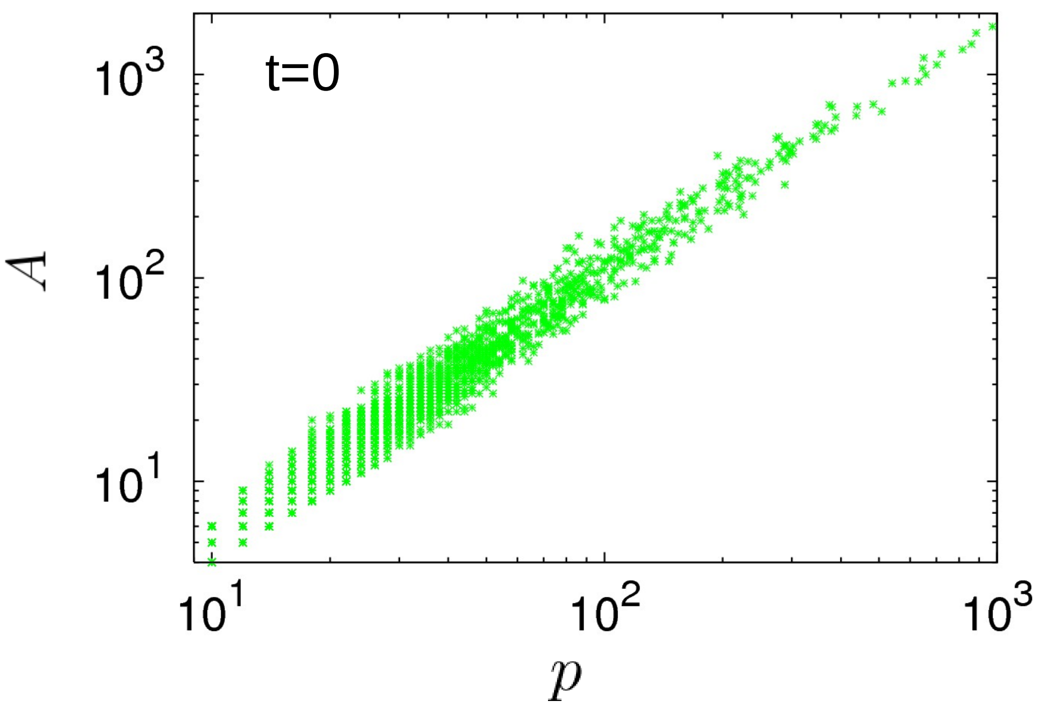


Fractal structure of the domains

Domain growth morphology: area vs. perimeter.



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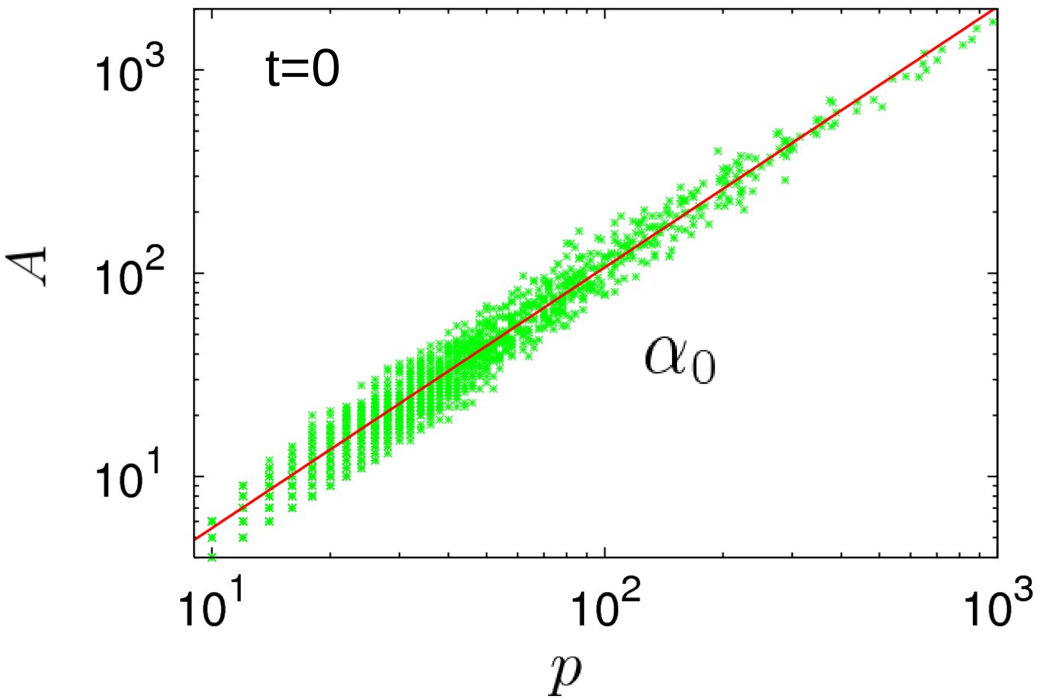
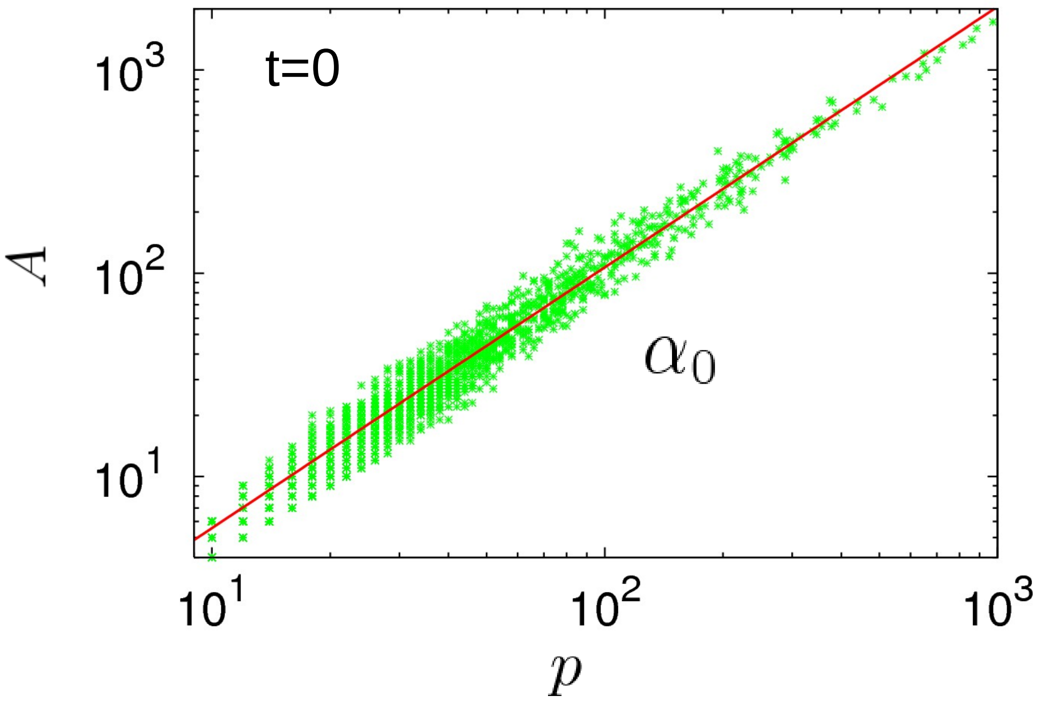
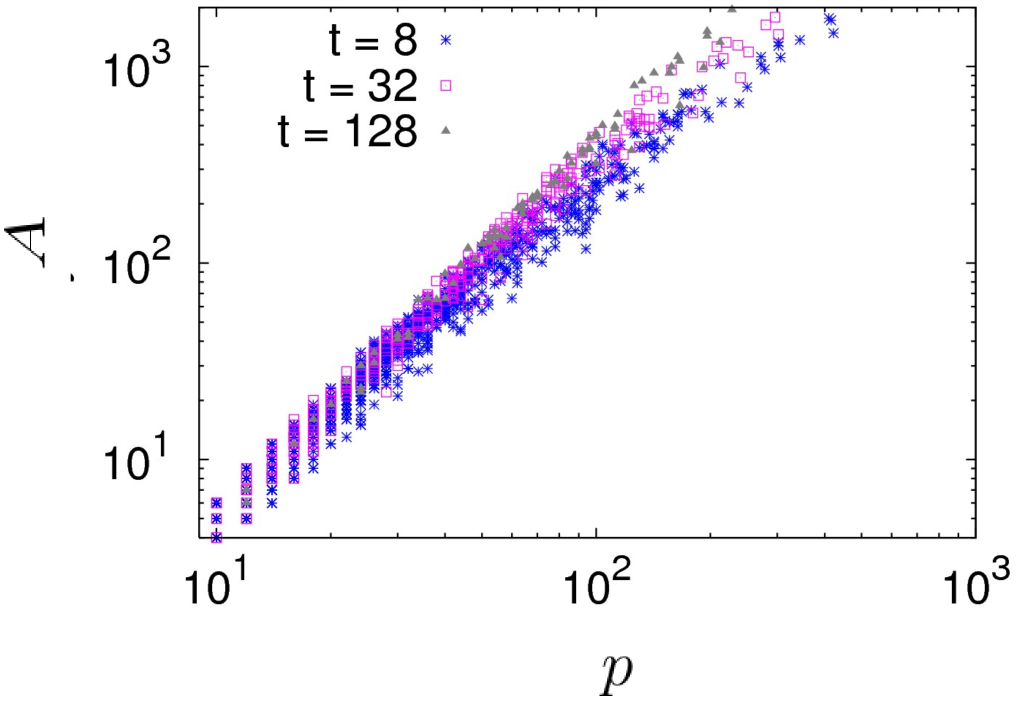


$$A = c_0 \, p^{\alpha_0}$$

$$\alpha_0 = 1.4$$

Domain growth morphology: area vs. perimeter.





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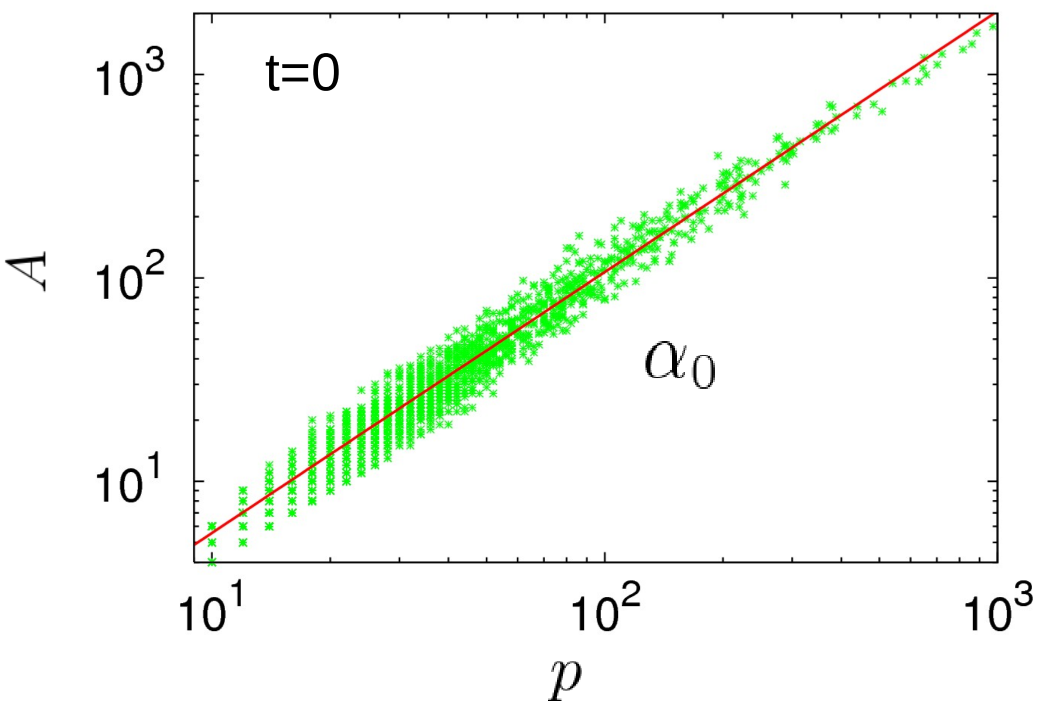

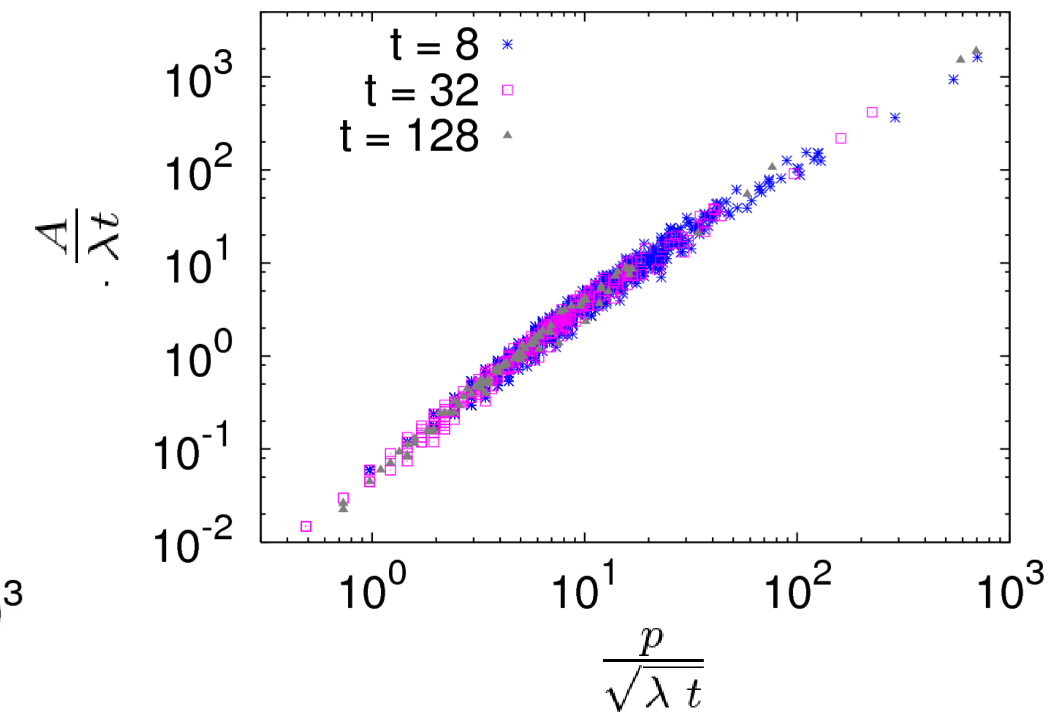

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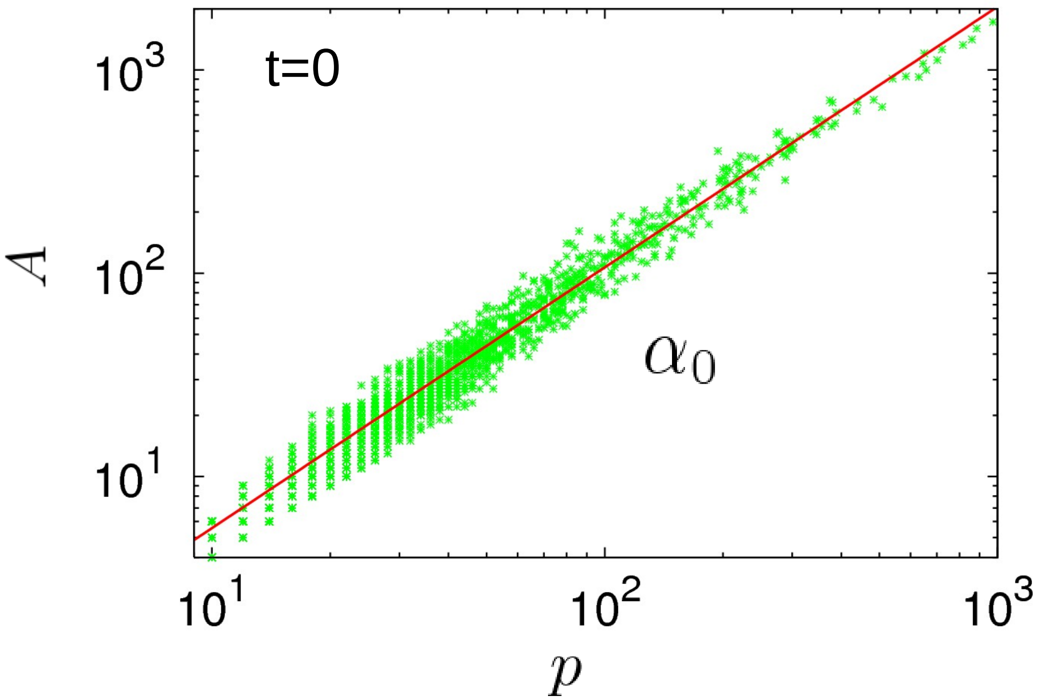

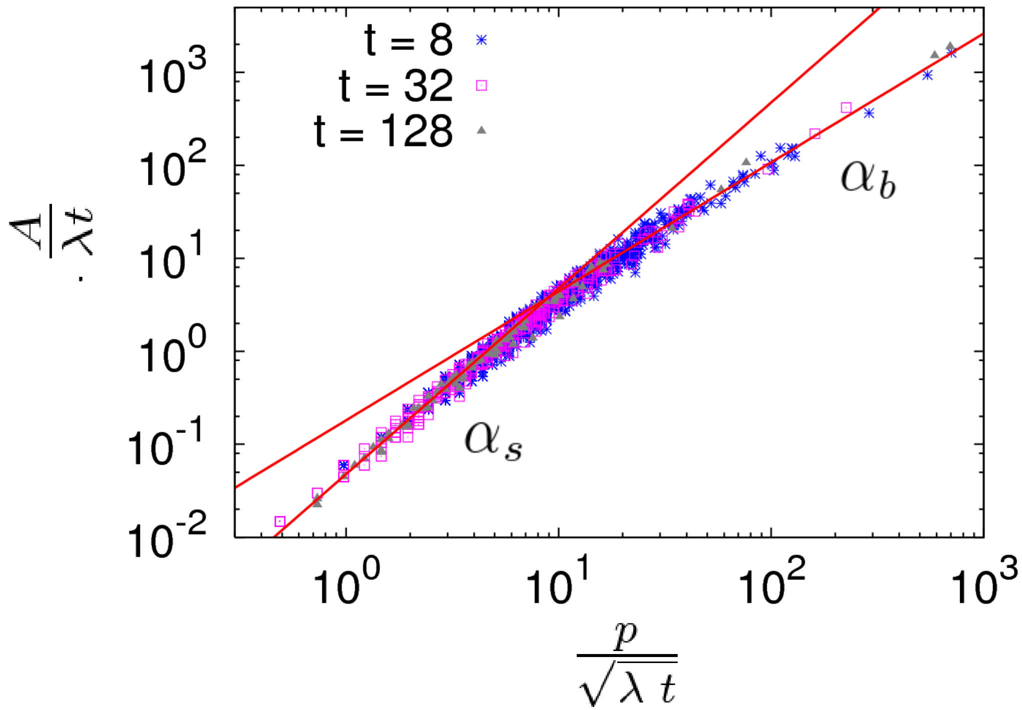

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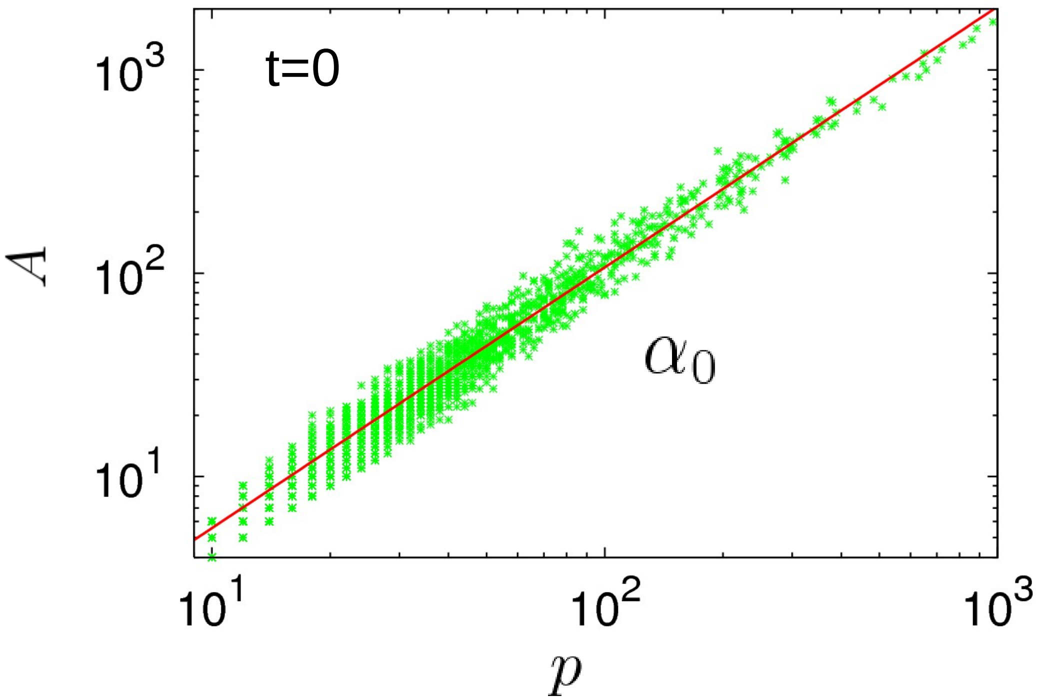

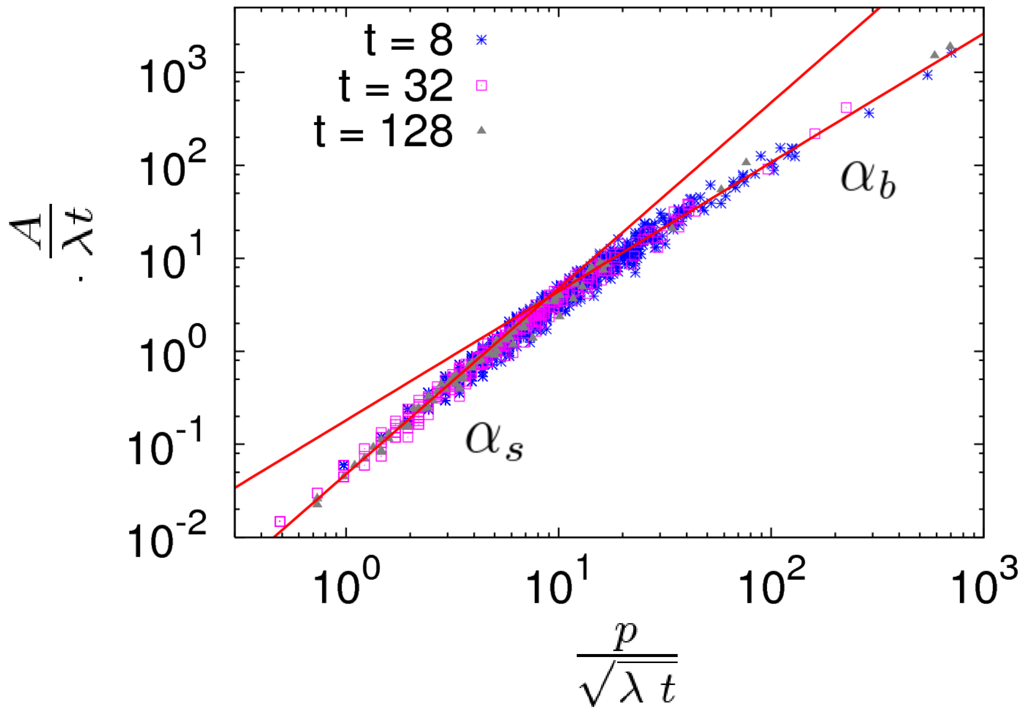

$$A = c_0\ p^{\alpha_0}$$

$$\alpha_0 = 1.4$$

$$\frac{A}{\lambda t} = c_s \left(\frac{p}{\sqrt{\lambda t}}\right)^{\alpha_s} \qquad A < A^* \sim \lambda t$$

$$\frac{A}{\lambda t} = c_b \left(\frac{p}{\sqrt{\lambda t}}\right)^{\alpha_b} \qquad A > A^* \sim \lambda t$$

$$\boxed{\begin{aligned} \alpha_s &= 2.0 \\ \alpha_b = \alpha_0 &= 1.4 \end{aligned}}$$

- **Regular**
- **Fractal** as the initial state

UNROUGHENING TRANSITION!

Distributions of domain perimeters

We know

- Distributions of areas
- Relation area - perimeter

➡ **Distributions of perimeters**

Distributions of domain perimeters

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- Distributions of areas
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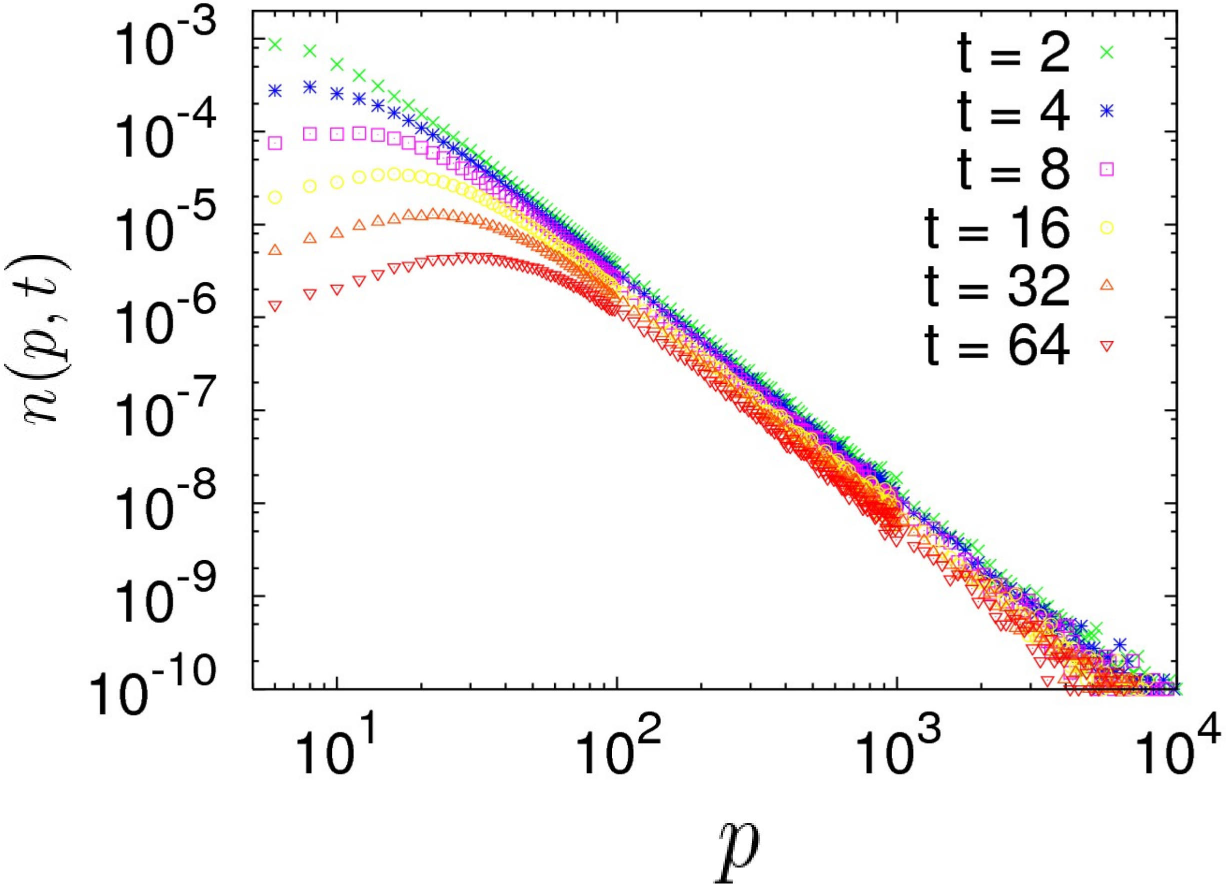
➡ **Distributions of perimeters**



Distributions of domain perimeters

We know

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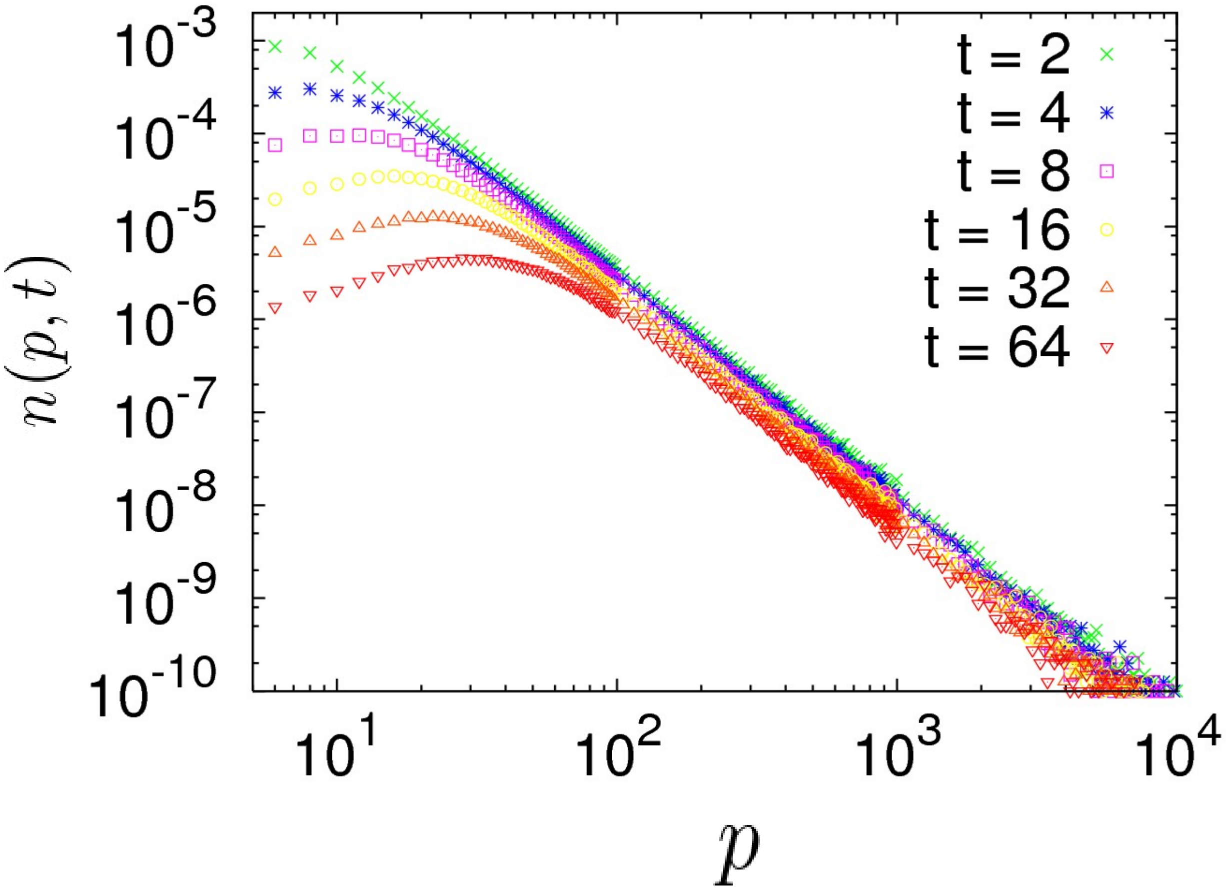
➡ **Distributions of perimeters**

$$(\lambda\, t)^{3/2}\, n(p,t) = f\left(\frac{p}{\sqrt{\lambda\, t}}\right)$$



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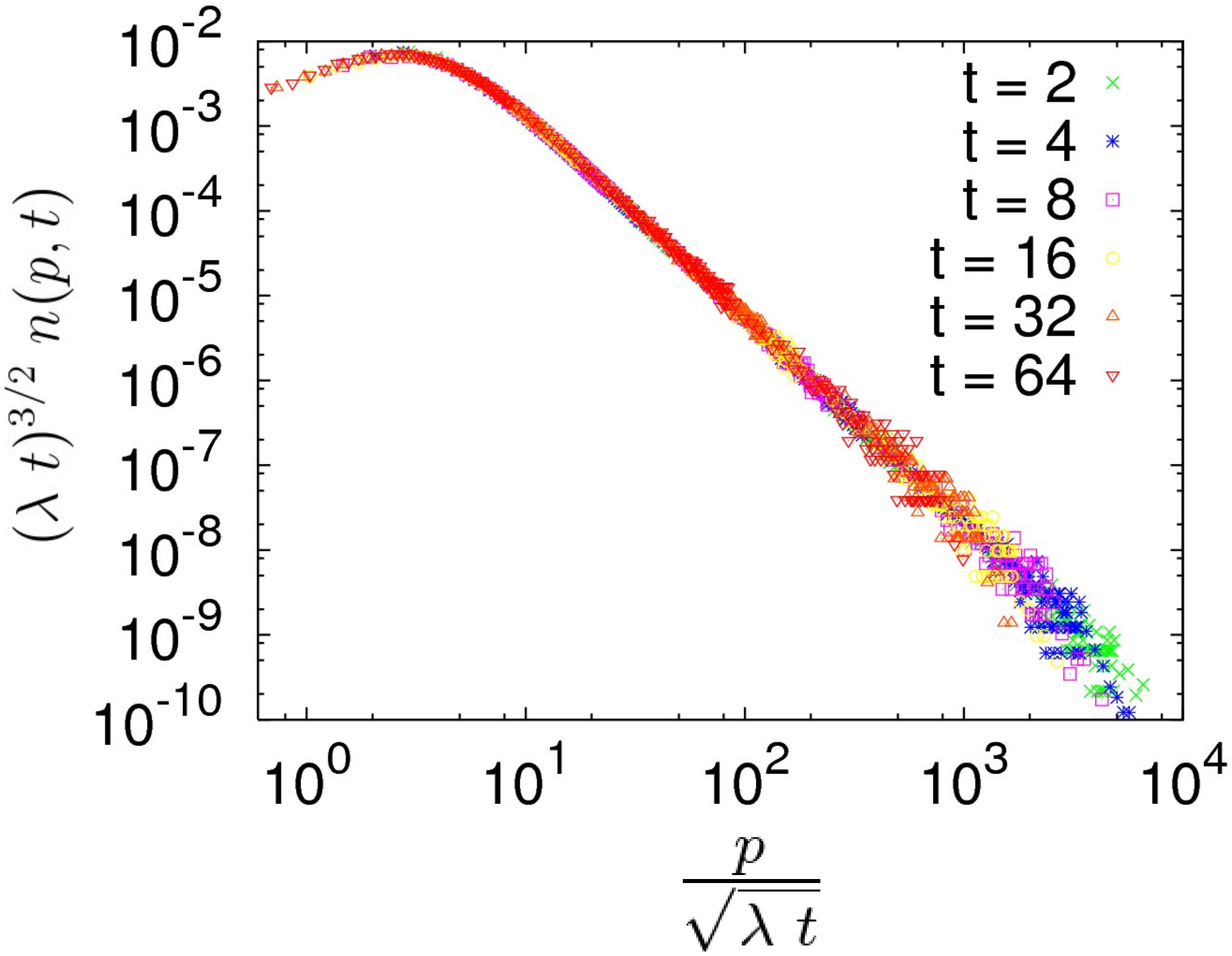

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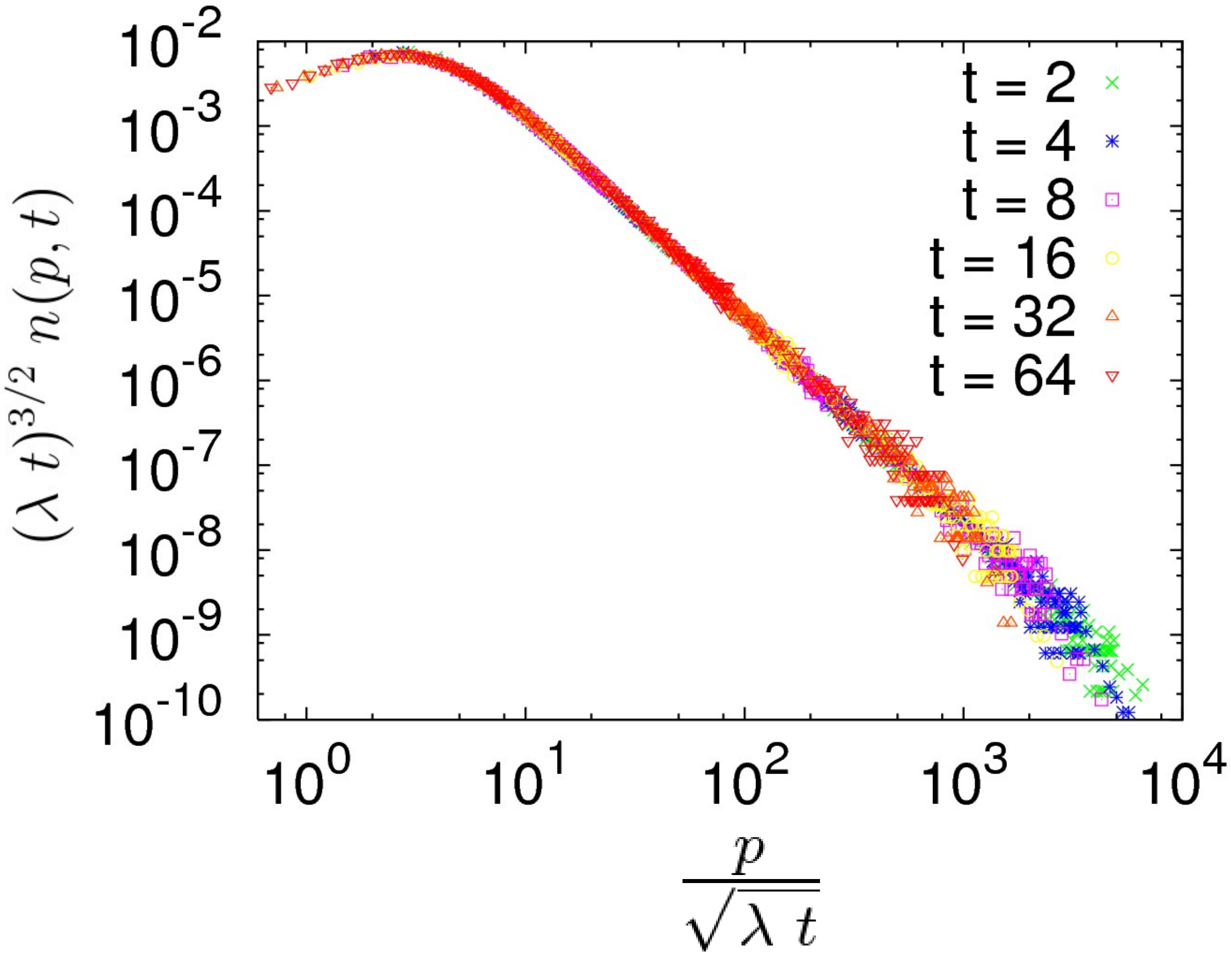

$$(\lambda\, t)^{3/2}\, n(p,t) = f\left(\frac{p}{\sqrt{\lambda\, t}}\right)$$

if $A < A^* \sim \lambda t$

$$f(x) = \frac{\alpha_s\ c_s\ c\ x^{(\alpha_s - 1)}}{[1 + c_s\ x^{\alpha_s}]^{\tau}}$$

if $A > A^* \sim \lambda t$

$$f(x) = \frac{\alpha_b\ c_b\ c\ x^{(\alpha_b - 1)}}{[1 + c_b\ x^{\alpha_b}]^{\tau}}$$

Distributions of domain perimeters

We know

- Distributions of areas
- Relation area - perimeter

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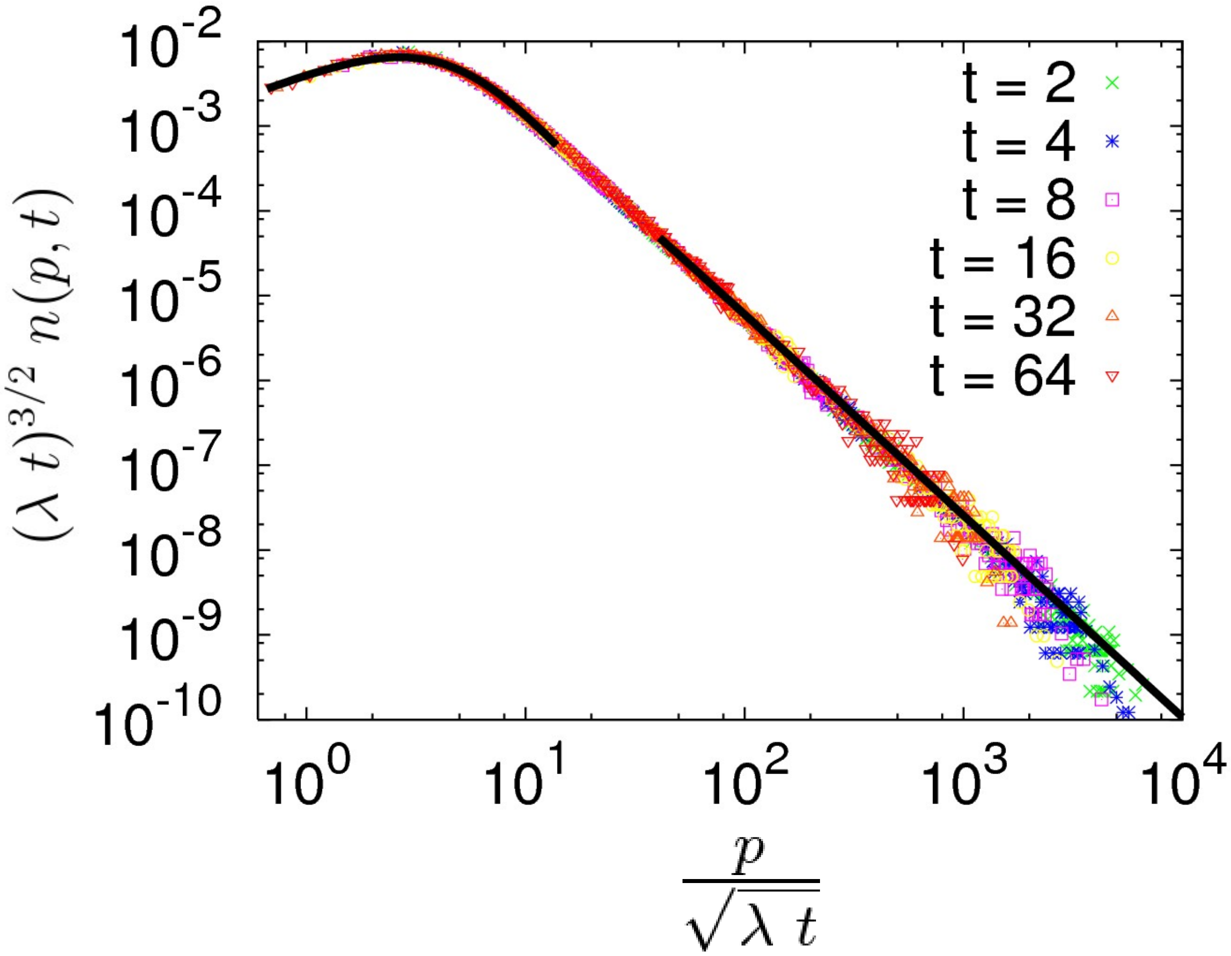

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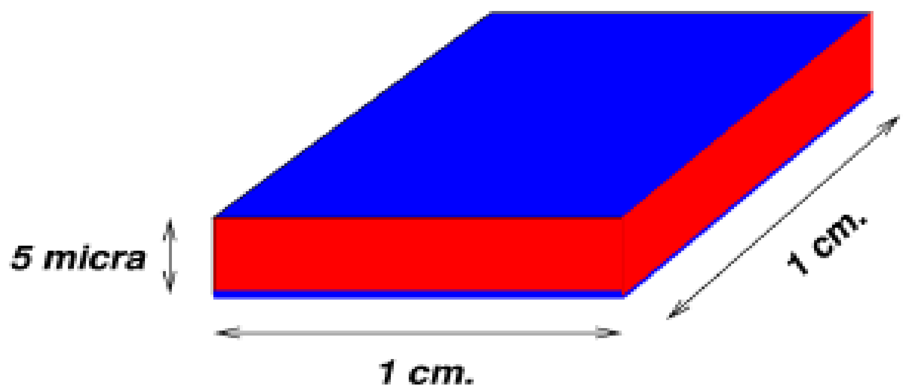
Experiments in liquid crystals

Experimental application of our result

- “bent core” liquid crystals.

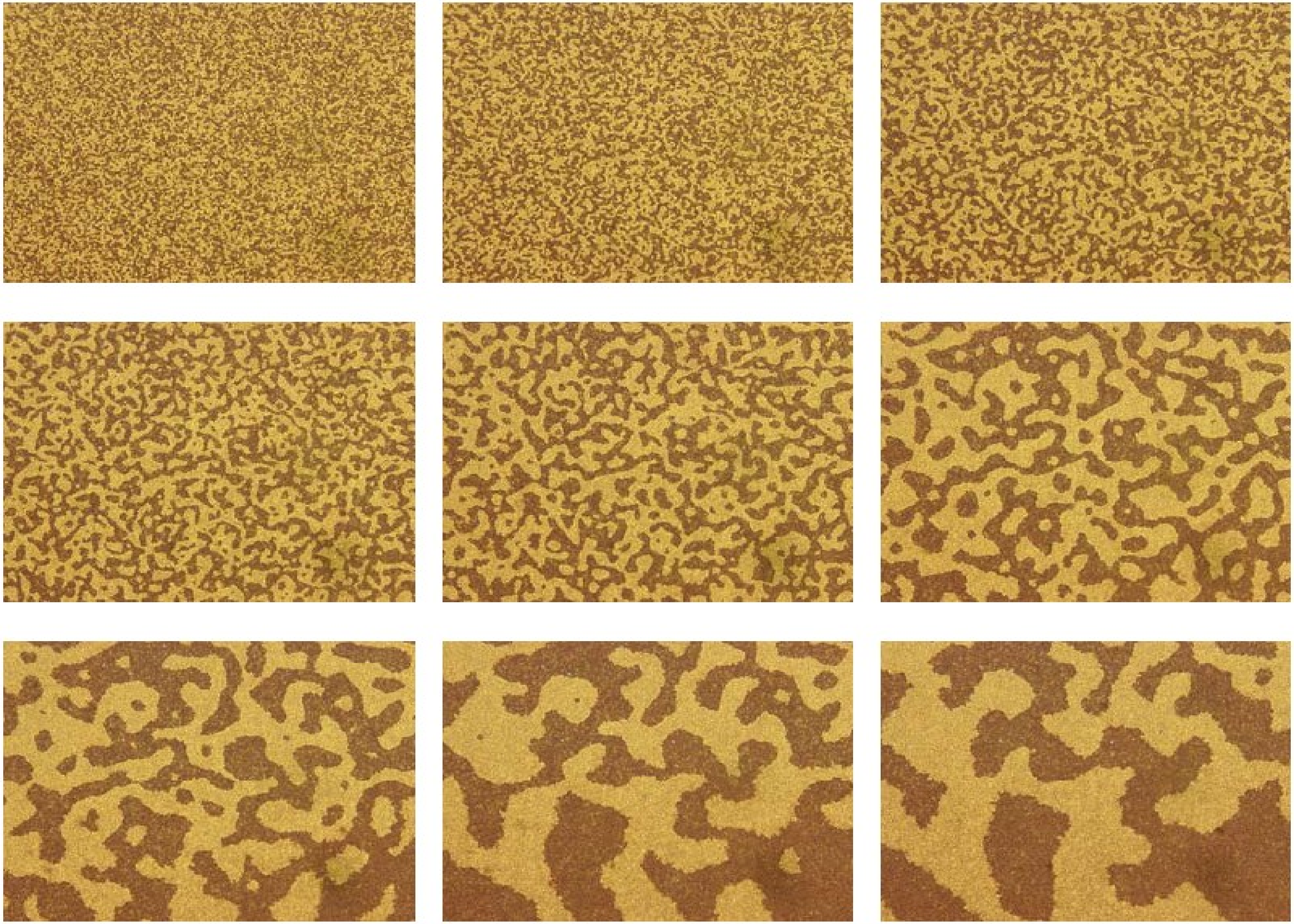
- Quasi 2-d cell filled with the liquid crystal



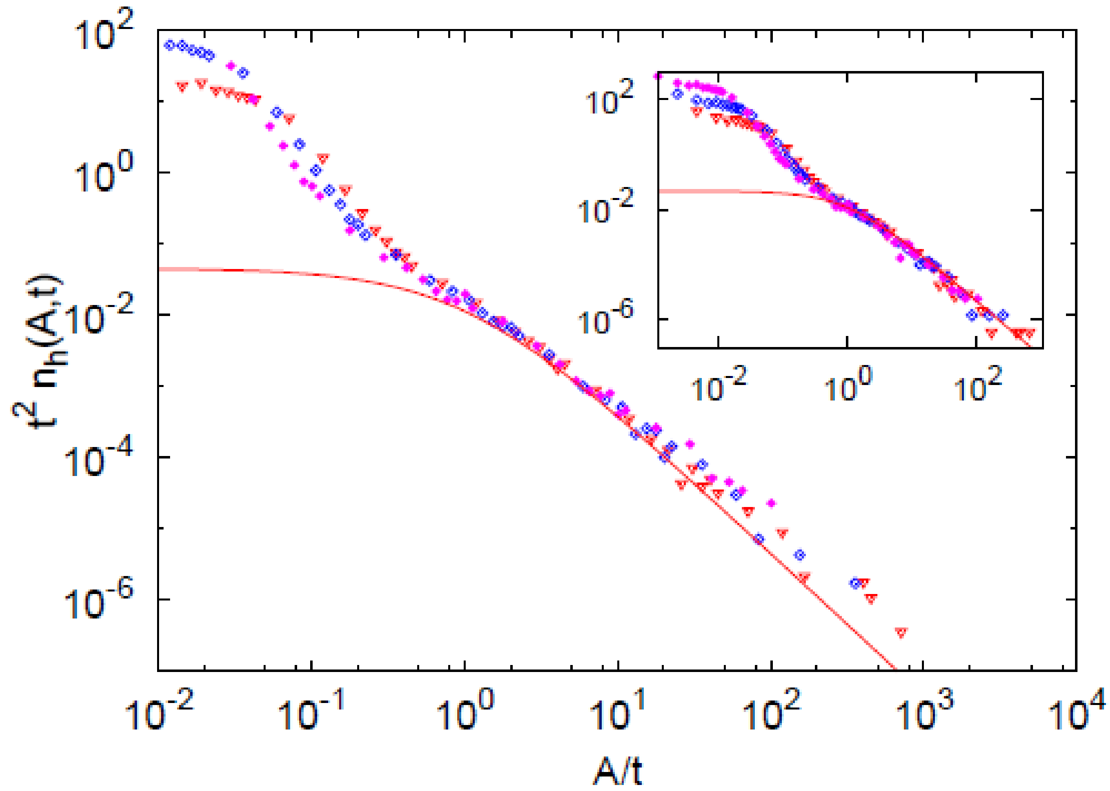
- Application of an electric field ⇒ Growth of chiral domains

Right handed domains Left handed domains

Experimental application of our result



Experimental application of our result



CONCLUSIONS

Conclusions and Future Work

- Proof of Dynamical Scaling Hypothesis in 2D.
- Analytic, Numerical and Experimental results.
- Role played by different initial conditions.
- Role played by the working temperature.
- Geometrical Picture of 2D Coarsening.

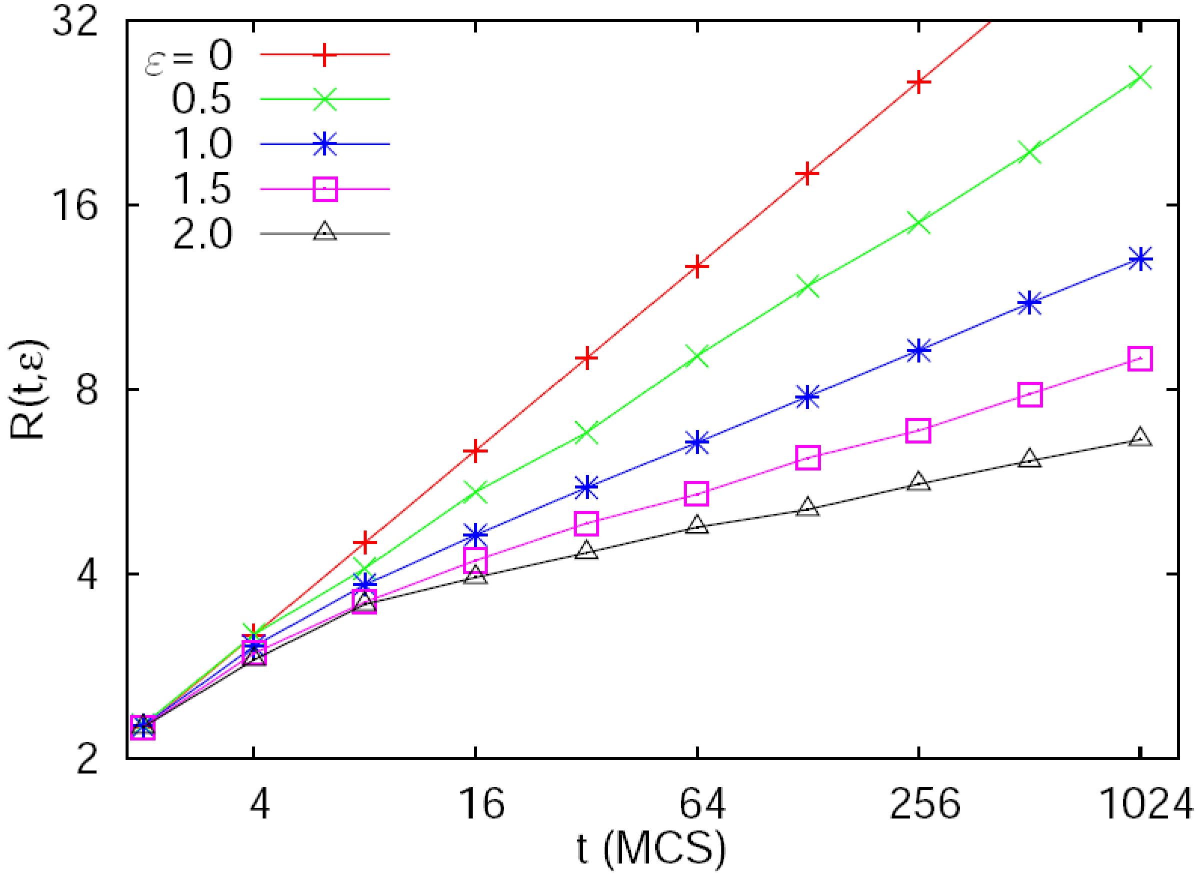
- Refs. for the results presented today: **PRL 98, 145701 (2007), PRE 76, 061116 (2007), EPL 82, 10001 (2008), PRL 101, 197801 (2008).**

Effects of weak quenched disorder

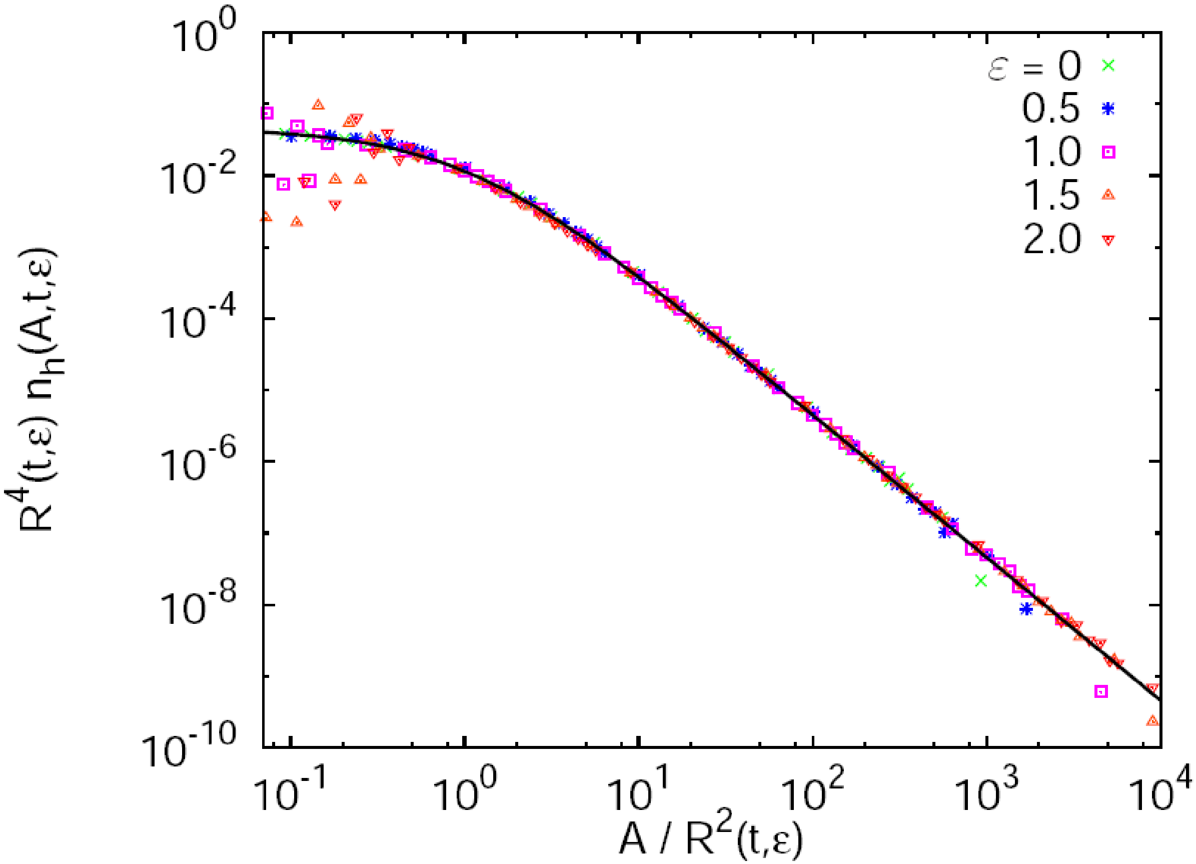
Effects of the disorder

Effects of the disorder

$$J_{ij} = ran\left[1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}\right]$$



DISTRIBUTION OF DOMAIN SIZES



Effects of the disorder

DISTRIBUTION OF DOMAIN PERIMETERS