Effective temperatures

Leticia F. Cugliandolo

Sorbonne Universités, Université Pierre et Marie Curie
Laboratoire de Physique Théorique et Hautes Energies
Institut Universitaire de France

leticia@lpthe.jussieu.fr
www.lpthe.jussieu.fr/~leticia/seminars


Beg Rohu, France, 2017
Plan

1. Equilibrium temperature.

2. An ‘effective temperature’ for certain out of equilibrium systems.
   
   LFC, J. Kurchan & L. Peliti 97
   
   — Measurement and properties.
   — Relation to entropy: Edwards’ measure.
   — Fluctuation theorems.
   — Ratchets

3. Quenches of isolated systems

1. Equilibrium temperature.

2. An ‘effective temperature’ for certain out of equilibrium systems.
   - Measurement and properties.
   - Relation to entropy: Edwards’ measure.
   - Fluctuation theorems.
   - Ratchets

3. Quenches of isolated systems

Temperature

Statistical mechanics definition

- Isolated system \( \Rightarrow \) conserved energy \( \mathcal{E} \)
- Ergodic hypothesis

\[
S = k_B \ln \mathcal{N}
\]

\[
\beta \equiv \frac{1}{k_B T} = \frac{\partial S}{\partial \mathcal{E}} \bigg|_{\mathcal{E}}
\]

Microcanonical definition

\[
\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}
\]

Neglect \( \mathcal{E}_{int} \) (short-range int.)

\[
\mathcal{E}_{syst} \ll \mathcal{E}_{env}
\]

\[
p_{eq}({\mathcal{E}_{syst}}) = g({\mathcal{E}_{syst}}) e^{-\beta \mathcal{E}_{syst}} / Z
\]

Canonical ensemble
Properties & measurement

Connection with thermodynamics

— Relation to entropy.
— Control of heat-flows: $\Delta Q$ follows $\Delta T$.
— Partial equilibration – transitivity:

$$T_A = T_B, T_B = T_C \Rightarrow T_A = T_C.$$ 

thermometers for systems in good thermal contact ($\Delta Q$)

Whatever we identify with a temperature should have these properties
In and out of equilibrium

Take a **mechanical point of view** and call \( \{ \vec{r}_i \}(t) \) the variables

e.g. the particles’ coordinates \( \{ \vec{x}_i(t) \} \) and momenta \( \{ \vec{p}_i(t) \} \)

Choose an initial condition \( \{ \vec{r}_i \}(0) \) and let the system evolve.

- For \( t_w > t_{eq} \) : \( \{ \vec{r}_i \}(t) \) reach the equilibrium pdf and thermodynamics and statistical mechanics apply. **Temperature** is a well-defined concept.

- For \( t_w < t_{eq} \) : the system remains out of equilibrium and thermodynamics and (Boltzmann) statistical mechanics **do not** apply.

**Is there a quantity to be associated with a temperature?**
Dynamics in equilibrium

Conditions

Take an open system coupled to an environment

Necessary:

— The bath should be in equilibrium same origin of noise and friction.

— Deterministic force:

  conservative forces only, \( \vec{F} = -\vec{\nabla}V \).

— Either the initial condition is taken from the equilibrium pdf, or the latter should be reached after an equilibration time \( t_{eq} \):

\[
P_{eq}(v, x) \propto e^{-\beta (\frac{mv^2}{2} + V)}
\]
Plan

1. Equilibrium temperature.

2. An ‘effective temperature’ for certain out of equilibrium systems.

   — Measurement and properties.
   — Relation to entropy: Edwards’ measure.
   — Fluctuation theorems.
   — Ratchets

3. Quenches of isolated systems

Two-time observables

Correlations

The two-time correlation between $A[\vec{x}(t)]$ and $B[\vec{x}(t_w)]$ is

$$C_{AB}(t, t_w) \equiv \langle A[\vec{x}(t)]B[\vec{x}(t_w)] \rangle$$

average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise in Langevin dynamics, etc.)
The perturbation couples linearly to the observable $B[\vec{x}(t_w)]$

$$E \rightarrow E - hB[\vec{x}(t_w)]$$

The linear instantaneous response of another observable $A[\vec{x}(t)]$ is

$$R_{AB}(t, t_w) \equiv \frac{\delta \langle A[\vec{x}(t)] \rangle}{\delta h(t_w)} \bigg|_{h=0}$$

The linear integrated response is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t')$$
Rue de Fossés St. Jacques et rue St. Jacques
Paris 5ème Arrondissement.
Fluctuation-dissipation

In equilibrium

\[ P(\vec{r}, t_w) = P_{eq}(\vec{r}) \]

- The dynamics are stationary

\[ C_{AB} \rightarrow C_{AB}(t-t_w) \text{ and } R_{AB} \rightarrow R_{AB}(t-t_w) \]

- The fluctuation-dissipation theorem between spontaneous \((C_{AB})\) and induced \((R_{AB})\) fluctuations

\[ R_{AB}(t-t_w) = -\frac{1}{k_B T} \frac{\partial C_{AB}(t-t_w)}{\partial t} \theta(t-t_w) \]

holds and implies \((k_B = 1 \text{ henceforth})\)

\[ \chi_{AB}(t-t_w) \equiv \int_{t_w}^{t} dt' \ R_{AB}(t, t') = \frac{1}{T}[C_{AB}(0) - C_{AB}(t-t_w)] \]
Fluctuation-dissipation

Linear relation between $\chi$ and $C$

$$P(\vec{r}, t_w) = P_{eq}(\vec{r})$$

- The dynamics are stationary

$$C_{AB} \rightarrow C_{AB}(t - t_w) \quad \text{and} \quad R_{AB} \rightarrow R_{AB}(t - t_w)$$

- The fluctuation-dissipation theorem between spontaneous ($C_{AB}$) and induced ($R_{AB}$) fluctuations

$$R_{AB}(t - t_w) = -\frac{1}{k_B T} \frac{\partial C_{AB}(t - t_w)}{\partial t} \theta(t - t_w)$$

holds and implies ($k_B = 1$ henceforth)

$$\chi_{AB}(t - t_w) \equiv \int_{t_w}^{t} dt' \ R_{AB}(t, t') = \frac{1}{T} [C_{AB}(0) - C_{AB}(t - t_w)]$$
Fluctuation-dissipation

Solvable glasses: \( p \) spin-models & mode-coupling theory

- Stochastic dynamics of a particle in an *infinite dimensional* space under the effect of a quenched random potential.

- A fully-connected (Curie approximation) spin model with as many ferromagnetic as antiferromagnetic couplings.

Sketch of the **separation of time-scales** in the out of equilibrium relaxation
A quench from $T_0 \to \infty$ (gas) to $T < T_g$ (glass)

Parametric construction

$t_w$ fixed

$t_{w1} < t_{w2} < t_{w3}$

$t : t_w \to \infty$ or

$\tau \equiv t - t_w : 0 \to \infty$

used as a parameter

Note that $T^* > T$

LFC & Kurchan 93
Fluctuation-dissipation

Proposal

For non-equilibrium systems, relaxing slowly towards an asymptotic limit (cfr. threshold in $p$ spin models) such that one-time quantities [e.g. the energy-density $E(t)$] approach a finite value

$$\lim_{t_w \to \infty} \chi(t, t_w) = f_{\chi}(C')$$

For weakly forced non-equilibrium systems in the limit of small work

$$\lim_{\epsilon \to 0} \chi(t, t_w) = f_{\chi}(C')$$

And the effective temperature is

$$-\frac{1}{T_{\text{eff}}} \equiv \frac{\partial \chi}{\partial C'}$$

LFC & Kurchan 94
A short-time regime with FDT?

A general property proven by a bound for Langevin dynamics

\[ |\chi(t, t_w) - C(t, t) + C(t, t_w)| \leq K \left( -\frac{1}{N} \frac{d\mathcal{H}(t_w)}{dt_w} \right)^{1/2} \]

with the “H-function”

\[ \mathcal{H}(t_w) = \int d\vec{x} d\vec{v} P(\vec{x}, \vec{v}, t_w) [k_B T \ln P(\vec{x}, \vec{v}, t_w) + H(\vec{x}, \vec{v})] \]

and its time variation

\[ \frac{d\mathcal{H}(t_w)}{dt_w} = -\langle \vec{f}(t_w) \cdot \vec{v}(t_w) \rangle - \sum_i g_i(t_w) \]

where the first term is the work done by eventual non-potential forces \( \vec{f} \) and the second term is a sum of positive terms

\[ g_i(t_w) = \gamma_0 \int d\vec{x} d\vec{v} \frac{(mv_i P + T \partial_{v_i} P)^2}{m^2 P} \geq 0 \]

LFC, Dean & Kurchan 97
FDT & effective temperatures

Can one interpret the slope as a temperature?

(1) Measurement with a thermometer with

- Short internal time scale $\tau_0$, fast dynamics is tested and $T$ is recorded.
- Long internal time scale $\tau_0$, slow dynamics is tested and $T^*$ is recorded.

(2) Partial equilibration

(3) Direction of heat-flow

LFC, Kurchan & Peliti 97
Glassy dynamics

Non stationary relaxation & separation of time-scales

Density-density correlation

\[ C(t, t_w) \]

Correlation

Density response

\[ \chi(t, t_w) = \int_{t_w}^{t} dt' \ R(t, t') \]

Time-integrated linear response

Analytic solution to a mean-field model LFC & J. Kurchan 93
Glassy dynamics

Fluctuation-dissipation relation: parametric plot

Analytic solution to a mean-field model LFC & J. Kurchan 93
FDT & effective temperatures

Sheared binary Lennard-Jones mixture

Left: the kinetic energy of a tracer particle (the thermometer) as a function of its mass ($\tau_0 \propto \sqrt{m_{tr}}$)

$$\frac{1}{2} m_{tr} \left\langle v_z^2 \right\rangle = \frac{1}{2} k_B T_{\text{eff}}$$

Right: $\chi_k (C_k)$ plot for different wave-vectors $k$, partial equilibrations.

J-L Barrat & Berthier 00
FDT & effective temperatures

Role of initial conditions

$T^* > T$ found for quenches from the disordered into the glassy phase

(Inverse) quench from an ordered initial state, $T^* < T$

2d XY model or O(2) field theory

Binary Lennard-Jones mixture

Berthier, Holdsworth & Sellitto 01

Gnan, Maggi, Parisi & Sciortino 13
Fluctuations

All subregions in space tend to have the temperature in the same time-scale, e.g. $C_r < q_{ea}$

Simulations

3d Edwards-Anderson spin-glass

Castillo, Chamon, LFC, Iguain & Kennett 02
A harmonic and an unharmonic oscillator driven out of equilibrium by two baths with different time-scales and temperatures.
Ratchets

Asymmetric particle immersed in an ageing glass

\[ \langle \Delta x_0(t) \rangle \equiv \langle x_0(t) - x_0(0) \rangle \]

Gradenigo, Sarracino, Villamaina, Grigera, Puglisi 10
Ratchets

Asymmetric particle immersed in an ageing glass

Quenches to $T = 0.67, 0.53, 0.42, 0.31 \ T_{MCT}$

Gradenigo, Sarracino, Villamaina, Grigera, Puglisi 10
FDT & FTs

Fluctuations $\Delta s = s(t) - s(t_w)$ in ROM model

Crisanti, Picco & Ritort 13
Experiments

Ageing glycerol

Grigera & Israeloff 99
Experiments

Beads and hairpins

\[ C(t) \equiv (\langle x(t) \rangle - x(t)) x(0) \]

\[ \chi(t) \equiv (x(t) - x(0)) / \delta f \]

\[ \tau_0 < \tau_s \]

\[ \tau_e > \tau_s \]

\[ \chi(C) \]

\[ T \quad \text{Large } t \]

\[ T \quad \text{No } I_{\text{eff}} \]

\[ \text{Small } t \]

\[ \text{Large } t \]

\[ \text{Small } t \]

Dietrich et al 15
Experiments

Beads and hairpins

Dietrich et al 15
Closed classical system

\[ p = 2 \text{ spherical model *** preliminary ***} \]

Foini, LFC, Gambassi, Konik 16-17

LFC, Lozano, Nessi, Picco & Tartaglia
Plan

1. Equilibrium temperature.

2. An ‘effective temperature’ for certain out of equilibrium systems.
   - Measurement and properties.
   - Relation to entropy: Edwards’ measure.
   - Fluctuation theorems.
   - Ratchets

3. Quenches of isolated systems

Dissipative quantum glasses

Quantum $p$-spin coupled to a bath of harmonic oscillators

Out of equilibrium decoherence

LFC & Lozano 98
Isolated quantum systems

Quantum quenches

- Take an isolated quantum system with Hamiltonian $H_0$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $H_0$.
- Unitary time-evolution with $U = e^{-\frac{i}{\hbar}Ht}$ with a Hamiltonian $H$.

Does the system reach some steady state?

Note that it is the ergodic theory question posed in the quantum context.

Motivated by cold-atom experiments & exact solutions of 1d quantum models.

Are at least some observables described by thermal ones?

When, how, which?
Take a closed system, $H_0$, in a given state, $|\psi_0\rangle$, and suddenly change a parameter, $H$. The unitary evolution is ruled by $H$.

e.g. $H = \int d^d x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla}\phi)^2 + r\phi^2 + \lambda\phi^4 \right\}$
Quantum quench

Setting

Take a closed system, $H_0$, in a given state, $|\psi_0\rangle$, and suddenly change a parameter, $H$. The unitary evolution is ruled by $H$.

e.g. $H = \int d^dx \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + r \phi^2 + \lambda \phi^4 \right\}$

$r > 0 \quad \rightarrow \quad r < 0$
Quantum quenches

Questions

Does the system reach a thermal equilibrium density matrix?

Under which conditions?

- non-integrable vs integrable systems; role of initial states; non critical vs. critical quenches

- Definition of $T_e$ from $\langle \psi_0 | H | \psi_0 \rangle = \langle H \rangle_{T_e} = \text{Tr} \ H e^{-\beta_e H}$

  Just one number, it can always be done

- Comparison of dynamic and thermal correlation functions, e.g.

  $C(r, t) \equiv \langle \psi_0 | \phi(\vec{x}, t) \phi(\vec{y}, t) | \psi_0 \rangle$ vs. $C(r) \equiv \langle \phi(\vec{x}) \phi(\vec{y}) \rangle_{T_e}$

  Calabrese & Cardy; Rigol et al; Cazalilla & Iucci; Silva et al, etc.

Proposal: put qFDT to the test to check whether $T_{\text{eff}} = T_e$ exists
Fluctuation-dissipation theorem

Classical dynamics in equilibrium

The classical FDT for a stationary system with \( \tau = t - t_w \) reads

\[
\chi(\tau) = \int_0^\tau dt' R(t') = -\beta [C(\tau) - C(0)] = \beta [1 - C(\tau)]
\]

choosing \( C(0) = 1 \).

Linear relation between \( \chi \) and \( C \)

Quantum dynamics in equilibrium

The quantum FDT reads

\[
\chi(\tau) = \int_0^\tau d\tau' R(\tau') = \int_0^\tau d\tau' \int_{-\infty}^{\infty} \frac{id\omega}{\pi \hbar} e^{-i\omega \tau'} \tanh \left( \frac{\beta \hbar \omega}{2} \right) C(\omega)
\]

Complicated relation between \( \chi \) and \( C \)
Fluctuation-dissipation theorem

Quantum SU(2) Ising chain

The initial Hamiltonian

\[ H_{\Gamma_0} = - \sum_i \sigma_i^x \sigma_{i+1}^x + \Gamma_0 \sum_i \sigma_i^z \]

The initial state \( |\psi_0\rangle \) ground state of \( H_{\Gamma_0} \)

Instantaneous quench in the transverse field \( \Gamma_0 \rightarrow \Gamma \)

Evolution with \( H_{\Gamma} \).

Iglói & Rieger 00

Reviews: Karevski 06; Polkovnikov et al. 10; Dziarmaga 10

Observables: correlation and linear response of local longitudinal and transverse spin, etc.

Specially interesting case \( \Gamma_c = 1 \) the critical point. Rossini et al. 09
Quantum quench $T_{\text{eff}}$ from FDT? Longitudinal spin

Insets

$e^{-\tau/\tau_C}$

$\tau^{-2} \sin(4\tau + \phi)$

$C^x(\tau) \simeq A_C e^{-\tau/\tau_C} [1 - a_C \tau^{-2} \sin(4\tau + \phi_C)]$

$R^x(\tau) \simeq A_R e^{-\tau/\tau_R} [1 - a_R \tau^{-2} \sin(4\tau + \phi_R)]$

Foini, LFC & Gambassi 11
Quantum quench

$T_{\text{eff}}$ from FDT?

For sufficiently long-times such that one drops the power-law correction

$$-\beta_{\text{eff}}^x \sim \frac{R^x(\tau)}{d_{\tau} C^x_+(\tau)} \sim -\frac{\tau_C A_R}{A_C}$$

A constant consistent with a classical limit but

$$T_{\text{eff}}^x(\Gamma_0) \neq T_e(\Gamma_0)$$

Moreover, a complete study in the full time and frequency domains confirms that

$$T_{\text{eff}}^x(\Gamma_0, \omega) \neq T_{\text{eff}}^z(\Gamma_0, \omega) \neq T_e(\Gamma_0)$$

(though the values are close).

Fluctuation-dissipation relations as a probe to test thermal equilibration

No equilibration for generic $\Gamma_0$ in the quantum Ising chain
Summary

\[ T_{\text{eff}} \text{ from FDT} \]

- Analysis of \textit{fluctuation-dissipation relations} in closed or open classical and quantum systems.

- \( T_{\text{eff}} \) calculated for dissipative classical and quantum \textit{mean-field} models – large \( N \), large \( d \) or with self-consistent closure approximations.

  A \textit{finite dimensional} solvable model with the phenomenology discussed is missing. (This is probably the same as finding a solvable glass)

- Discussion of the \textit{thermodynamic meaning} of \( T_{\text{eff}} \).

- A better understanding of the microscopic origin of \( T_{\text{eff}} \) is lacking.

- Use of \textit{fluctuation-dissipation relations} to check for Boltzmann equilibrium (application to quantum quenches).
The generic **Langevin equation** for a particle in $1d$ is

$$m\dddot{x}(t) + M'[x(t)] \int_{-\mathcal{T}}^{t} dt' \, \Gamma(t - t') M'[x(t')] \dot{x}(t') = F(t) + \xi(t) M'[x(t)]$$

with the coloured noise

$$\langle \xi(t) \xi(t') \rangle = T \, \Gamma(t - t')$$

The dynamic generating functional is a path-integral

$$Z_{\text{dyn}}[\eta] = \int dx_{-\mathcal{T}} d\dot{x}_{-\mathcal{T}} \int DxDi\dot{x} \, e^{-S[x,i\dot{x};\eta]}$$

with $i\dot{x}(t)$ the ‘response’ variable.

$x_{-\mathcal{T}}$ and $\dot{x}_{-\mathcal{T}}$ are the initial conditions at time $-\mathcal{T}$.

**Martin-Siggia-Rose-Jenssen-deDominicis formalism**
Fluctuation-dissipation

A proof

The action has a deterministic part (Newton) that includes the initial condition and a dissipative part that depends upon $\Gamma$:

$$S = S_{\text{det}} + S_{\text{diss}}$$

The transformation

$$x(t) \rightarrow x(-t) \quad i\dot{x}(t) \rightarrow i\dot{x}(-t) + \beta \dot{x}(-t)$$

leaves $S_{\text{diss}}$ and the path-integral measure invariant.

$S_{\text{det}}$ is also invariant if $P(x_{-\tau}, \dot{x}_{-\tau}) = P_{eq}(x_{-\tau}, \dot{x}_{-\tau})$, and $F = V'[x]$

The FDT valid for Newton or Langevin dynamics

$$R_{AB}(t, t_w) + R_{AB}(t_w, t) = \beta \partial_{t_w} C_{AB}(t, t_w)$$

and higher-order extensions are Ward identities of this symmetry.

The fluctuation theorems can also be proven in this way.
Fluctuation theorems

Take a system \textit{in equilibrium} and drive it into a \textbf{non-equilibrium steady state} with a perturbing force. The \textbf{fluctuations of ‘entropy production rate’}

\[ p \equiv (\tau \sigma_+)^{-1} \int_{-\tau/2}^{\tau/2} dt \frac{W(S_t)}{T} \]

where \( S_t \) is the trajectory of the system in phase space,

- \( T \) is the temperature of the equilibrated unperturbed system,
- \( W(S_t) \) is the work done by the external forces, and
- \( T \sigma_+ \equiv \int dx P_{st}(x)W(x) \sim \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/s}^{\tau/s} dt W(t) \) is an average over the steady state distribution, satisfy

\[ \xi(p) - \xi(-p) = p \sigma_+ \quad \text{with} \quad \xi(p) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \ln \pi_{\tau}(p) \]

and \( \pi_{\tau} \) the probability density of \( p \).

Cohen, Morriss & Evans 90; Gallavoti & Cohen 93
Take a glass *out of equilibrium* and take it into a **driven steady glassy state** with a perturbing force.

For which entropy production rate does a fluctuation theorem hold?

Since there is no meaning to $T$ but there is to $T_{\text{eff}}$, the proposal is to replace

$$\int_{-\tau/2}^{\tau/2} dt \frac{W(t)}{T} \quad \rightarrow \quad \int_{-\tau/2}^{\tau/2} dt \frac{W(t)}{T_{\text{eff}}(t)}$$

with $T_{\text{eff}}(t)$ the **effective temperature** as measured from

the fluctuation-dissipation relation of the *unperturbed* relaxing system

with its two values $T$ and $T^*$

Zamponi, Bonetto, LFC & Kurchan 05
Is $T_{\text{eff}}$ related to an entropy?

**Configurational entropy**

An exponentially large number of metastable states is reached dynamically.

Curie-Weiss (ferro) Sketch of free-energy landscape

*Threshold level is reached asymptotically*

\[ \lim_{t \to \infty} \mathcal{E}(t) = \mathcal{E}_\infty > \mathcal{E}_{eq}. \]

Well-understood in mean-field models with the **Thouless-Anderson-Palmer** technique.
Is $T_{\text{eff}}$ related to an entropy?

Configurational entropy

$$\Sigma(f) = k_B \ln \mathcal{N}(f) \implies \frac{1}{k_B T^*} = \frac{\partial \Sigma(f)}{\partial f} \bigg|_{f_\infty}$$

$\Sigma$

$\frac{1}{k_B T^*} = \frac{\partial \Sigma}{\partial f} \bigg|_{f_\infty}$

$\Sigma$

$\frac{1}{k_B T^*} = \frac{\partial \Sigma}{\partial f} \bigg|_{f_\infty}$

$\Sigma$

NB $f_{\text{max}} \neq f_\infty \implies$ failure of ‘maximum entropy principles’.

Edwards & Oakshott 89, Monasson 95, Nieuwenhuizen 98

Very sketchy view: many amorphous solid configurations ($\Sigma \leftrightarrow T^*$) and vibrations around them ($f \leftrightarrow T$).
Quantum quench

$T_{\text{eff}}$ from FDT? Longitudinal spin

A quantum quench $\Gamma_0 \to \Gamma$ of the isolated Ising chain

Here: to its critical point $\Gamma = 1$

Linear response and symmetrized correlation of $\sigma^x$

Foini, LFC & Gambassi 11
Summary

- $T_{\text{eff}}$ definition from the analysis of fluctuation-dissipation relations.

- Discussion of thermodynamic meaning.
  Shown for mean-field models – large $N$, large $d$ or, in other words, within the mode-coupling approach to glassy systems.

- Numerical evidence Lennard-Jones silica, soft particles; vortex glasses granular matter; thin magnetic films, active matter, etc.

- Other evidence: extended Arrhenius law for activation (Ilg & J-L Barrat), fluctuation theorems (Zamponi et al), ratchets (Gradenigo et al), etc.

- Experimental results are less clear glycerol, laponite, spin-glasses, etc. (Jabbari-Bonn, Abou-Gallet, Ciliberto et al., Bartlett et al, Hérisson & Ocio, etc.).
Summary

classical context

- The energy density approaches the equilibrium one, typically as $\Delta E \sim t^{-b}$.

- The correlation and linear response functions have highly non-trivial time-dependencies (aging effects, non-exponential relaxations).

- There is an extended time-regime in which correlation and linear response vary "macroscopically" but the effective temperature $T_{\text{eff}} = T^*$ is constant.

- $T^*$ can be related to the variation of a configurational entropy with respect to the energy-density (à la micro-canonic.)

- $T^*$ has intuitive properties: hotter for more disordered, colder for more ordered.

Cases in which this does not hold were exhibited by, e.g., Sollich et al in models with unbounded energy or artificial (emerging?) dynamic rules.
Is $T_{\text{eff}}$ related to an entropy?

**Granular matter**

- **Static** granular matter: blocked states $mgd \gg k_B T$

- Hypotheses to describe **weakly driven** granular matter:
  
  – walk from blocked state to blocked state
  
  – blocked states are visited with equal probability working at fixed $V$ (and $\mathcal{E}$): $P(\{\vec{r}_i\}_{\text{blocked}}) = \text{constant}$.

  – From the entropy of blocked states

    $$S(V, \mathcal{E}) = k_B \ln \# \text{ blocked states}(V, \mathcal{E})$$

    define the temperature

    $$T_{Edw}^{-1} = \frac{\partial S(V, \mathcal{E})}{\partial \mathcal{E}}$$

    and the compactivity

    $$X_{Edw}^{-1} = \frac{\partial S(V, \mathcal{E})}{\partial V}$$

Edwards & Oakeshott 89