Random matrices, quantum chaos, and localization

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In this 4-hours lecture, I aim to provide a bird's-eye view on chaos in quantum many-body systems and Anderson localization, using random matrices as a tool to understand their physics.

The intent is pedagogical rather than exhaustive, and I list below a few references where these concepts are explored in greater depth. In particular, this review [1] offers a contained and colloquial introduction to quantum chaos and ETH, whereas this second review [2] provides a more comprehensive discussion of these topics (at the cost of being more lengthy). These slides [7] summarize the same main ideas, following a storyline similar to the one adopted in my lecture. A great introduction to random matrices can be found in this book [4]; see also these lecture notes [6], which are based on a course given by Pierpaolo Vivo, one of the authors of [4]. Unfortunately, there are not many pedagogical introductions around to the topic of Anderson and many-body localization; in this lecture, I initially follow [5], and then the storyline summarized in these slides [8].

Throughout the lecture I propose several exercises, the solutions to which (presented in class) can also be found in the references given below.

Syllabus

- Reminder on the concept of thermalization in classical many-body systems (ergodicity, typicality).
- Reminder on classical integrable and chaotic systems (Liouville-Arnold theorem, Lyapunov exponents).
- Problems with the naive application of these concepts to quantum mechanics.
- How does a quantum system reflect the regular/chaotic behavior of its classical counterpart? Berry-Tabor conjecture (spectrum of quantum systems whose classical counterpart is integrable).
- Exercise [4, 6]. Consider a set of i.i.d. random energy levels E_n . Derive the distribution P(s) of the spacing $s = E_{n+1} E_n$ between adjacent levels.
- More complex Hamiltonians: why introducing random matrices. Generalities on the Gaussian ensembles.
- Exercise [6]. Take a symmetric 2×2 matrix H whose entries are Gaussian random variables. Write the joint pdf $\mathcal{P}(H)$ of its *entries*, and deduce the compatibility between the two requirements of *independent Gaussian entries*, and *rotational invariance* of the ensemble.
- Joint distribution of the eigenvalues in the Gaussian Orthogonal Ensemble (GOE), eigenvalue repulsion.
- Exercise [4]. Use such joint pdf to derive P(s) for the GOE in the 2 × 2 case. Show that it follows the so-called Wigner's surmise.
- Bohigas-Giannoni-Schmit conjecture (spectrum of quantum systems whose classical counterpart is chaotic).
- Exercise [3, 4]. Given the distribution $\mathcal{P}_{\psi}(\psi)$ of a generic eigenvector in the GOE, marginalize it to compute the pdf $p(y = |\psi_i|^2)$ of each single component (known in the thermodynamic limit as the Porter-Thomas distribution). Compute its first two moments.
- Structure of the matrix elements of an observable averaged over the GOE.
- Exercise [2]. Prove (using the Porter-Thomas distribution) that the fluctuations of such observables scale in the GOE as $\sim 1/N$, where N is the size of the system.

- Reminder on the density matrix in quantum mechanics. Time evolution (projection on the diagonal ensemble), apparent clash with ergodicity.
- Statement of the eigenstate thermalization hypothesis (ETH), and its prediction for the expectation value of (a certain class of) observables.
- Exercise [2]. Use the ETH ansatz to show that fluctuations in the observables are suppressed exponentially with the system size.
- Thouless energy and random-matrix limit.
- Cases in which ETH is known to fail: integrable systems, phase transitions, quantum scars, and localized systems. (Brief explanation of what a quantum scar is.)
- Anderson localization, statement of the problem and state of the art.
- Exercise [5]. Consider the infinite-disorder limit. Compute the return probability of a quantum particle initially localized at the origin, and the local density of states.
- Exercise [5]. Consider the zero-disorder limit. Compute the return probability and the local density of states (first for a finite system size, then take the thermodynamic limit).
- Diagnostics of localization: multifractal exponents, spectral statistics (gap ratio).
- Exercise [3]. Use the Porter-Thomas distribution to compute the IPR in the GOE.
- Many-body localization in a nutshell. Analogies with the problem of Anderson localization on a random network.
- Another way to break ETH: multifractality.
- Simplest toy model for multifractal wave functions: the Rosenzweig-Porter ensemble.

References

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