SK WITH REPLICAS

1 DEFINITION

$$H_J = -\frac{1}{2}\sum_{i\neq j} J_{ij}\, S_i S_j$$

EACH LINK ADDED ONCE
$N(N-1)$ TERMS IN $\sum_{i\neq j}$
THEN DIVIDED BY 2.

$S_i = \pm 1$ ISING

J_{ij} QUENCHED RANDOM iid GAUSSIAN pdf

$$P(\{J_{ij}\}) = \prod_{ij} p(J_{ij})$$

$$p(J_{ij}) = e^{-J_{ij}^2/2\sigma_J^2} \Big/ \sqrt{2\pi\sigma_J^2}$$

$$[J_{ij}] = 0 \qquad [J_{ij}^2] = \sigma_J^2 = \frac{J^2}{N}$$

GOOD NORMALIZ TO HAVE AN INTERESTING $N\to\infty$ LIMIT AND FINITE T_c

ALSO MEANS $J_{ij} \simeq \frac{J}{\sqrt{N}}$

$h_i^{loc} \sim \sum_j J_{ij} S_j \sim O(1)$

OFTEN $J=1$ MEANING β AS βJ

THIS CALCULATION IS DONE IN GREAT DETAIL IN MANY PLACES. FOR EX IN

H. NISHIMORI'S BOOK

W/ $[J_{ij}] = J_0$ AND ALSO APPLYING AN EXTERNAL FIELD

② REPLICA METHOD - SETTING

$$Z_J = \sum_{\{s_i\}} e^{+\beta \frac{1}{2} \sum_{i\neq j} J_{ij} s_i s_j}$$

$$Z_J^n = \sum_{\{s_i^a\}} e^{\beta \frac{1}{2} \sum_{i\neq j} J_{ij} \sum_a s_i^a s_j^a}$$

$$[Z_J^n] = \sum_{\{s_i^a\}} \int \prod_{i\neq j} \frac{dJ_{ij}}{\sqrt{2\pi\sigma_J^2}} \; e^{-\sum_{i\neq j} \frac{J_{ij}^2}{2\sigma_J^2}}$$

$$e^{\sum_{i\neq j} J_{ij} \left(\frac{\beta}{2} \sum_a s_i^a s_j^a\right)}$$

NOW THE STEPS ARE:

FOR EACH $i \neq j$

- MOVE DOWN THE SUM

$$e^{\sum_{i\neq j} A_{ij}} \longrightarrow \prod_{i\neq j} e^{A_{ij}}$$

- INTEGRATE OVER J_{ij} (GAUSSIAN INT.)

- MOVE UP THE RESULT

$$\prod_{i\neq j} e^{B_{ij}} \longrightarrow e^{\sum_{i\neq j} B_{ij}}$$

③ AVERAGE OVER DISORDER

THE GAUSSIAN INTEGRAL

$$\underbrace{\int \frac{dy}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}+ya-\frac{a^2\sigma^2}{2}}}_{1}\; e^{+\frac{a^2\sigma^2}{2}}$$

$$\frac{1}{2}\left(\frac{y}{\sigma}-a\sigma\right)^2 = \frac{1}{2}\frac{y^2}{\sigma^2}+\frac{a^2\sigma^2}{2}-\frac{2ya\sigma}{2\sigma} \quad ✓$$

SO THE INTEGRAL YIELDS $e^{\frac{a^2\sigma^2}{2}}$

IN OUR CASE $a = \frac{\beta}{\sqrt{2}} \sum_a s_i^a s_j^a$

AND WILL APPEAR SQUARED IN EXPONENTIAL

THE RESULT OF THE AVERAGE IS

$$[Z_J^n] = \sum_{\{s_i^a\}} e^{\sum_{i \neq j} \frac{\beta^2}{4} \underbrace{\left(\sum_a s_i^a s_j^a\right)^2}_{\text{REPLICA INTERACTION}} \sigma_J^2}$$

UNCOUPLED SPACE INDICES ij BUT
COUPLED REPLICA INDICES ab
$\Rightarrow$ INTERACTIONS.

④ REORGANIZE $\sum_{i \neq j} \sum_{ab} s_i^a s_j^a s_i^b s_j^b$

$$\sum_{i \neq j} \left(\sum_a s_i^a s_j^a\right)^2 = \sum_{ab} \sum_{i \neq j} s_i^a s_i^b s_j^a s_j^b$$

$$= \sum_{ab} \sum_{ij} s_i^a s_i^b s_j^a s_j^b - \sum_{ab} \sum_i \underset{\overset{\shortparallel}{1}}{(s_i^a)^2} \underset{\overset{\shortparallel}{1}}{(s_i^b)^2}$$

$$= \sum_{ab} \sum_{ij} s_i^a s_i^b s_j^a s_j^b - n^2 N$$

$$= \sum_{a \neq b} \sum_{ij} s_i^a s_i^b s_j^a s_j^b$$

$$+ \underbrace{\sum_a \sum_{ij} \underset{\overset{\shortparallel}{1}}{(s_i^a)^2} \underset{\overset{\shortparallel}{1}}{(s_j^a)^2}}_{n N^2} - n^2 N$$

$$= \sum_{a \neq b} \left(\sum_i s_i^a s_i^b \right) \left(\sum_j s_j^a s_j^b \right)$$

$$+ n N^2 - n^2 N$$

SECOND TERM $\Theta(n^2)$ VS. FIRST $\Theta(n)$

MOREOVER FIRST $\Theta(N^2)$ SECOND $\Theta(N)$

$\Rightarrow$ DROP 2nd TERM

WE USE $[J_{ij}^2] = \sigma_J^2 = \frac{J^2}{N}$

$$[Z_J^n] = \sum_{\{s_i^a\}} e^{\frac{\beta^2 J^2}{4N}\left\{ \sum_{a\neq b} \left(\sum_i s_i^a s_i^b\right)\left(\sum_j s_j^a s_j^b\right) + nN^2 \right\}}$$

$$= e^{+\frac{\beta^2 J^2}{4} nN}$$

$$\sum_{\{s_i^a\}} e^{\frac{(\beta J)^2}{4} N \sum_{a\neq b} \overbrace{\left(\frac{1}{N}\sum_i s_i^a s_i^b\right)}^{x_{ab}} \underbrace{\left(\frac{1}{N}\sum_j s_j^a s_j^b\right)}_{x_{ab}}}$$

PRODUCT OF TWO IDENTICAL FACTORS x_{ab} WITH (ab) INDICES, EACH INVOLVING A SUM OVER i OR j INDICES.

5 GAUSSIAN DECOUPLING FOR EACH (a,b)

RECALL THAT OVERALL FACTOR REMAINS THE SAME IN THE EXPONENTIAL

$$\int \frac{dq_{ab}}{\sqrt{2\pi\sigma_q^2}} \, e^{-N(\beta J)^2 \frac{q_{ab}^2}{2}} \, e^{+q_{ab}(\beta J)^2 N \cdot \left(\frac{1}{N}\sum_i s_i^a s_i^b\right)}$$

$$= \underbrace{e^{\frac{(\beta J)^2 N}{4}\left(\frac{1}{N}\sum_i s_i^a s_i^b\right)\left(\frac{1}{N}\sum_j s_j^a s_j^b\right)}}_{\text{THIS IS WHAT WE'LL REPLACE ABOVE BY THE INTEGRAL}}$$

$$\sigma_q^2 = \left(N(\beta J)^2\right)^{-1}$$

CHECK $b^2 = (\beta J)^2$ $\quad z = q_{ab}$ $\quad u = \frac{1}{N}\sum_i s_i^a s_i^b$

$$\int \frac{dz}{\sqrt{2\pi\sigma_z^2}} \, e^{-\frac{Nb^2}{4} z^2 + \frac{Nb^2}{2} z u}$$

$$= \int \frac{dz}{\sqrt{2\pi\sigma_z^2}} \, e^{-\frac{Nb^2}{4}(z-u)^2 + \frac{Nb^2 u^2}{4}} \quad \circledast$$

THE PREFACTOR $(2\pi\sigma_q^2)^{-1/2}$:

- DEPENDS ON N
- THERE WILL BE ONE OF THESE FACTORS PER INTEGRAL OVER q_{ab}

$\Rightarrow$ IN TOTAL $\left[2\pi\left(N(\beta J)^2\right)^{-1}\right]^{\#}$

$\#$ NUMBER OF q_{ab} : $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$

$$\left(\frac{2\pi}{N(\beta J)^2}\right)^{\frac{n(n-1)}{2}}$$

BUT THIS IS A PRE-FACTOR. IT DOES NOT GO IN EXP $\rightarrow$ IGNORE

WE HAVE NOW,

$$[Z_J^n] \propto e^{+\frac{\beta^2 J^2}{4} nN}$$

$$\sum_{\{s_i^a\}} \int \prod_{a \neq b} dq_{ab} \; e^{-N(\beta J)^2 \sum_{a \neq b} \frac{q_{ab}^2}{4}}$$

$$e^{\frac{1}{2}(\beta J)^2 \sum_{a \neq b} q_{ab} \; N \cdot \left(\frac{1}{N} \sum_i s_i^a s_i^b\right)}$$

EXCHANGE IT WITH INTEGRAL $\int dq_{ab} \ldots$

HAVE TO CALCULATE THE PARTITION SUM OVER $\{s_i^a\}$ ISING SPINS

⑥ THE INDEX i IS NOW SUPERFLOUS

AS IN THE RFIM CASE,

$$\int \prod_{a\neq b} dq_{ab}\, e^{-\frac{N}{4}(\beta J)^2 \sum_{a\neq b} q_{ab}^2} \sum_{\{s_i^a\}} \prod_{i=1}^{N} e^{\frac{(\beta J)^2}{2} \sum_{a\neq b} q_{ab}\, s_i^a s_i^b}$$

$$= \int \prod_{a\neq b} dq_{ab}\, e^{-\frac{N}{4}(\beta J)^2 \sum_{a\neq b} q_{ab}^2} \left(\sum_{\{s^a\}} e^{\frac{(\beta J)^2}{2} \sum_{a\neq b} q_{ab}\, s^a s^b} \right)^N$$

WE CAN DO THIS SINCE THERE'S NO MORE COUPLING $i \neq j$

WE THEN HAVE

$$[Z_J^n] = e^{+\frac{\beta^2 J^2}{4} nN} \int \prod_{a\neq b} dq_{ab} \; e^{-\frac{N}{4}(\beta J)^2 \sum_{a\neq b} q_{ab}^2} \; e^{N \ln \zeta(q_{ab})}$$

WITH THE NEW PARTITION FUNCTION

$$\zeta(\{q_{ab}\}) \equiv \underbrace{\sum_{\{s^a\}} e^{\frac{(\beta J)^2}{2} \sum_{a\neq b} q_{ab}\, s^a s^b}}$$

FOR THE SPINS $\{s^a\}$ WITH INTERACTIONS

q_{ab} GAUSSIANLY DISTRIBUTED.

BECAUSE OF THE 1st FACTOR

STILL COUPLING $a\neq b$ IN ζ THOUGH.

WHAT IS q_{ab} ? SADDLE POINT EVAL.

$$[Z_J^n] \propto \int \prod_{ab} dq_{ab}\, e^{-N\bar{A}(\{q_{ab}\})}$$

$$0 = \frac{\partial \bar{A}}{\partial q_{ab}} \quad \forall ab$$

$$-\cancel{\frac{(\beta J)^2}{2}}\, q_{ab}^{sp.} + \left.\frac{\partial \ln \zeta(\{q_{ab}\})}{\partial q_{ab}}\right|_{q_{ab}^{sp}} = 0$$

$$\frac{1}{\zeta}\sum_{\{s^a\}} \cancel{\frac{(\beta J)^2}{2}}\, s^a s^b\, e^{\cdots}$$

$$\Rightarrow \quad \boxed{q_{ab}^{sp} = \langle s^a s^b \rangle_\zeta} \quad \forall a \neq b$$

RECALL RFIM FOR
ONE REPLICA INDEX
ORDER PARAM.

WRITE A LANDAU FREE-ENERGY

$$[Z_J^n] = e^{\frac{(\beta J)^2}{4} nN} \int \prod_{a \neq b} dq_{ab} \; e^{-\beta N \underbrace{\overline{f}(\{q_{ab}\})}}$$

LIKE A LANDAU FREE-ENERGY,
FUNCTION OF A STRANGE
SET OF ORDER PARAMETERS $\{q_{ab}\}$

$$-\beta \overline{f}(\{q_{ab}\}) = -\frac{1}{4}(\beta J)^2 \sum_{a \neq b} q_{ab}^2 + \ln \zeta(\{q_{ab}\})$$

$$\zeta(\{q_{ab}\}) = \sum_{\{s^a\}} e^{\frac{(\beta J)^2}{2} \sum_{a \neq b} q_{ab} s^a s^b}$$

THE FACTOR βJ APPEARS IN WEIRD PLACES
COMPARED TO THE USUAL CURIE-WEISS MODEL

SADDLE - POINT

NOW, COMES THE CHANGE IN THE ORDER OF LIMITS

$$-\beta[f_J] = \lim_{N\to\infty} \lim_{n\to 0} \frac{[z^n]-1}{Nn}$$

WE'LL CALCULATE THE INTEGRAL OVER $\{q_{ab}\}$ BY S.P. TAKING $N\to\infty$ FIRST

$$[z^n] \simeq e^{\frac{(\beta J)^2}{4} nN} e^{-\beta N \overline{f}(\{q_{ab}\}^{SP})} \left(\det \frac{\delta \overline{f}(\{q_{ab}\})}{\delta q_{cd}}\right)^{-1/2}$$

RECALL S.P. $\int dz \; e^{-Nf(z)}$

$$\simeq \sqrt{2\pi\sigma^2} \int \frac{dz}{\sqrt{2\pi\sigma^2}} e^{-N\left(f(z^*)+\frac{1}{2}f''(z^*)(z-z^*)^2\right)}$$

$$= e^{-Nf(z^*)} \left(\frac{2\pi}{Nf''(z^*)}\right)^{1/2} \qquad \sigma^2 = \left(Nf''(z^*)\right)^{-1}$$

BUT IT'LL BE "IMPOSSIBLE" TO FIND THE S.P. SOLUTION WITHOUT AN <u>ANSATZ</u>

REPLICA SYMMETRIC → REWRITING $\bar{f}(q)$

PROPOSE

$$Q = \begin{pmatrix} \ddots & & q \\ & \ddots & \\ q & & \ddots \end{pmatrix}$$

$q_{ab} = q$ FOR $a \neq b$

EVALUATE $\bar{f}(\{q_{ab}\})$ AND THE FIND THE EXTREME WRT q.

- FIRST TERM IN $-\beta \bar{f}(q_{ab})$:

$$-\frac{1}{4}\beta J^2 \sum_{a \neq b} q_{ab}^2 = -\frac{1}{4}\beta J^2 q^2 \quad n(\cancel{n}-1)$$

THE $\mathcal{O}(n^2) \to 0$

SECOND TERM IN $\bar{f}(q_{ab})$

$$\ln \zeta(q)$$

WITH

$$\zeta(q) = \sum_{\{s^a\}} e^{\frac{(\beta J)^2}{2} q \sum_{a\neq b} s^a s^b}$$

ONCE AGAIN, WE DECOUPLE THE PRODUCT IN THE EXPONENTIAL

$$\zeta(q) =$$

$$e^{-\frac{(\beta J)^2 q n}{2}} \sum_{\{s^a\}} \int \frac{dz}{\sqrt{2\pi\sigma_z^2}}\, e^{-(\beta J)^2 q \frac{z^2}{2} + z(\beta J)^2 q \sum_a s^a}$$

TO CANCEL THE TERM WITH $a=b$ GENERATED BY THE GAUSSIAN INTEGRAL

$$\left(\sum_a s^a\right)^2 = \sum_{ab} s^a s^b = \sum_{a\neq b} s^a s^b + \sum_a \underbrace{(s^a)^2}_{1}$$

$$= \sum_{a\neq b} s^a s^b + n$$

$$\sigma_z^2 = \left[q(\beta J)^2\right]^{-1}$$

NOW WE SUM OVER $\{s^a = \pm 1\}$

$$e^{-\frac{(\beta J)^2 q n}{2}} \int \frac{dz}{\sqrt{2\pi\sigma_z^2}} e^{-(\beta J)^2 q \frac{z^2}{2}} \prod_a 2\mathrm{ch}\left(z(\beta J)^2 q\right)$$

$$= e^{-\frac{(\beta J)^2 q n}{2}} \int \frac{dz}{\sqrt{2\pi\sigma_z^2}} e^{-(\beta J)^2 q \frac{z^2}{2}} \left[2\mathrm{ch}\left(z(\beta J)^2 q\right)\right]^n$$

$$\zeta = e^{-\frac{(\beta J)^2 q n}{2}} \int \frac{dz}{\sqrt{2\pi\sigma_z^2}} e^{-(\beta J)^2 q \frac{z^2}{2} + n \ln\left[2\mathrm{ch}\left(z(\beta J)^2 q\right)\right]}$$

NOW, EXPAND IN $n \to 0$. TAKE CARE OF TWO TERMS IN THE EXP. THE GRAY ONE AND THE LAST ONE, BOTH UNDER GAUSSIAN INT. OVER z

$$\Rightarrow 1 + n \left\{ -\frac{(\beta J)^2 q}{2} + \ln\left[2\mathrm{ch}\left(z(\beta J)^2 q\right)\right] \right\}$$

- THE TERMS WHICH DO NOT DEPEND ON z, INTEGRATED OVER z, YIELD THE SAME (INT. NORMALIZED)

- THE LAST ONE DEPENDS ON z SO WE HAVE TO KEEP IT IN NON TRIVIAL FORM

CALL $\langle\langle \cdots \rangle\rangle$ THE GAUSSIAN AVERAGE OVER z

$$\zeta(q) = 1 - n\frac{(\beta J)^2 q}{2}$$

$$+ n \left\langle\left\langle \ln\left[2\,\mathrm{ch}\left((\beta J)^2 q z\right)\right]\right\rangle\right\rangle$$

BUT IN $\bar{f}(q)$ THERE'S

$$\ln \zeta(q) = \ln(1+nA) \simeq nA \qquad \text{FOR } n\to 0$$

EVERYTHING TOGETHER :

$$-\beta\bar{f}(q) = \frac{1}{4}(\beta J)^2 q^2 n + n\left[-\frac{(\beta J)^2 q}{2}\right.$$

$$\left. + \left\langle\left\langle \ln\left[2\,\mathrm{ch}\left((\beta J)^2 q z\right)\right]\right\rangle\right\rangle\right]$$

BUT RECALL THERE WAS A FACTOR

$$e^{\frac{(\beta J)^2}{4} n N}$$

(INDEP. OF q)

IN FRONT OF THE INTEGRAL OVER $\{q_{ab}\}$

$\longrightarrow$ TRANSFORMED IN q

$$[Z^n] = \int dq \; e^{\frac{(\beta J)^2}{4} n N (1-q)^2} \; e^{+N n \langle\langle \ln\left[2\,\mathrm{ch}\left((\beta J)^2 q z\right)\right] \rangle\rangle}$$

THUS,

$$-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \frac{[Z^n] - 1}{nN}$$

$$= \lim_{n\to 0} \frac{e^{A(q_{sp})\, nN} - 1}{nN}$$

$$\boxed{-\beta f = A(q_{sp})}$$

NOW SADDLE-POINT WRT q

IT IS CONVENIENT TO REDEFINE z IN THE GAUSSIAN INTEGRATION

$$\langle\langle \dots \rangle\rangle = \int \frac{dz}{\sqrt{2\pi \left(q(\beta J)^2\right)^{-1}}} \, e^{-(\beta J)^2 \frac{q z^2}{2}} \dots$$

TO SIMPLIFY THE EXPRESSIONS IN $\frac{d}{dq} \dots$ TO BE USED TO GET THE S.P. VALUE OF q.

$$q z^2 = u^2 \implies z = u/\sqrt{q}\,, \quad dz = du/\sqrt{q}$$

$$\langle\langle \dots \rangle\rangle = \int \frac{du}{\cancel{\sqrt{q}}} \frac{1}{\sqrt{2\pi\left((\beta J)^2 \cancel{q}\right)^{-1}}} \, e^{-\frac{(\beta J)^2 u^2}{2}}$$

$$= \int \frac{du}{\sqrt{2\pi (\beta J)^2}} \, e^{-\frac{(\beta J)^2 u^2}{2}}$$

IT ALSO MODIFIES THE ARGUMENT $\longrightarrow$

$$\ln\left[2\,\mathrm{ch}\left((\beta J)^2 q^{1/2} u\right)\right]$$

THEN, FINALLY

$$\frac{dA(q)}{dq} = 0$$

$$= \frac{\cancel{(\beta J)^2}}{\cancel{4}}\,\cancel{2}(q-1) +$$

$$+ \langle\langle\, \mathrm{th}\left((\beta J)^2 q^{1/2} u\right) u \,\rangle\rangle\, \cancel{(\beta J)^2}\,\cancel{\tfrac{1}{2}}\,\frac{1}{\sqrt{q}}$$

NOT SUCH A SIMPLE EQ EITHER!

ASSUME $\exists\, T_c$ s.t. $q = 0$ AT $T > T_c$ AND

$q \simeq 0$ AT $T \lesssim T_c$

- is $q=0$ a sol ?

$$0 \overset{?}{=} -1 + \langle\langle (\beta J)^2 \cancel{q^{1/2}} u^2 \rangle\rangle \frac{1}{\cancel{q^{1/2}}}$$

$$0 \overset{?}{=} -1 + \cancel{(\beta J)^2} \underbrace{\langle\langle u^2 \rangle\rangle}_{\cancel{(\beta J)^{-2}}}$$

yes !

- WHAT is T_c ? LOOK AT EQ $\theta(q)$

$$0 = \frac{\cancel{(\beta J)^2}}{\cancel{4}\,\cancel{q}} \cancel{q}(q-1) +$$

$$+ \langle\langle th\left((\beta J)^2 q^{1/2} u\right) u \rangle\rangle \cancel{(\beta J)^2} \cancel{\frac{1}{2}} \frac{1}{\sqrt{q}}$$

$$th\left((\beta J)^2 q^{1/2} u\right) = (\beta J)^2 q^{1/2} u - (\beta J)^6 q^{3/2} \frac{u^3}{3}$$

$$q - \frac{(\beta J)^6}{3} q^{3/2} \langle\langle u^4 \rangle\rangle \frac{1}{q^{1/2}} = \Theta(q)$$

$$q \left[\frac{(\beta J)^6}{\cancel{3}} \cancel{3} \langle\langle u^2 \rangle\rangle^2 \right] = \left(\frac{1}{(\beta J)^2} \right)^2$$

$$q \quad (\beta J)^2 \quad \Longrightarrow \quad \boxed{\beta_c J = 1}$$ CRITICAL