THE FULLY CONNECTED RANDOM FIELD ISING MODEL

DEFINITION

SCALE J/N IN FULLY CONN FM TERM

$$H_h = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - \sum_i h_i s_i$$

HAMILTONIAN

$$P(\{h_i\}) = \prod_i p(h_i)$$

JOINT pdf

iid

$$p(h_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{h_i^2}{2\sigma^2}}$$

GAUSSIAN FIELDS

$$[h_i] = 0 \qquad [h_i^2] = \sigma^2$$

ISING SPINS

$s_i = \pm 1$

THE FERROMAGNETIC TERM

$$\frac{1}{2N}\sum_{i\neq j} s_i s_j = \frac{1}{2N}\sum_{ij} s_i s_j - \frac{1}{2N}\sum_i s_i^2$$

$$= \underbrace{\frac{1}{2N}\Big(\sum_i s_i\Big)\Big(\sum_j s_j\Big)}_{\Theta(N)} - \underbrace{\frac{1}{2}}_{\Theta(1)} \sim \frac{1}{2N}\Big(\sum_i s_i\Big)\Big(\sum_j s_j\Big)$$

CALCULATION OF THE DISORDER AV. FREE-ENERGY DENSITY WITH THE REPLICA TRICK

$$-\beta f_h^{(N)} = \frac{1}{N} \ln Z_h^{(N)}$$

(N) NOTATION TO INDICATE FINITE N EXPRESSIONS

$$-\beta [f_h^{(N)}] = \frac{1}{N} [\ln Z_h^{(N)}]$$

THERMODYN LIMIT

$$-\beta [f_h] = \lim_{N\to\infty} \frac{1}{N} [\ln Z_h^{(N)}]$$

REPLICA TRICK

$$\ln Z_h^{(N)} = \lim_{n\to 0} \frac{\left(Z_h^{(N)}\right)^n - 1}{n}$$

THEN, THE DISORDER AVERAGED FREE-ENERGY IN THE THERMODYNAMIC LIMIT IS

$$-\beta [f_h] = \lim_{N\to\infty} \frac{1}{N} \lim_{n\to 0} \frac{\left[\left(Z_h^{(N)}\right)^n\right] - 1}{n}$$

① WE START BY EXPRESSING $\left[\left(Z_h^{(N)}\right)^n\right]$

$$\left[\left(Z_h^{(N)}\right)^n\right] = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dh_i\, p(h_i)$$

$$\underbrace{\sum_{\{s_i^1=\pm1\}} e^{-\beta H_h[\{s_i^1\}]} \cdots \sum_{\{s_i^n=\pm1\}} e^{-\beta H_h[\{s_i^n\}]}}_{\text{THE } n \text{ FACTORS } Z_h^{(N)}}$$

$$= \sum_{\{s_i^a=\pm1\}} e^{\frac{\beta J}{2N}\sum_{a=1}^{n}\sum_{ij} s_i^a s_j^a} \quad \leftarrow \text{INDEP OF } \{h_i\}$$

$$\int \prod_{i=1}^{N} dh_i\, p(h_i)\; e^{\underbrace{\beta \sum_{a=1}^{n}\sum_{i=1}^{N} h_i s_i^a}_{= \beta \cdot \sum_{i=1}^{N} h_i \left(\sum_{a=1}^{n} s_i^a\right)}}$$

② AVERAGING OVER THE $\{h_i\}$ N GAUSSIAN INTEGRALS

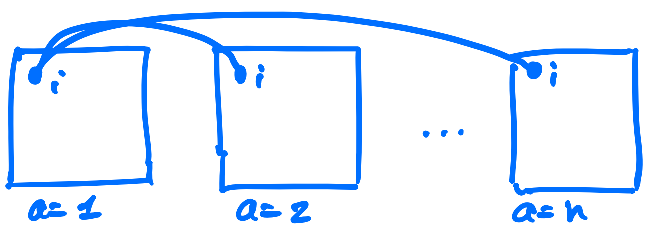
$$= \sum_{\{s_i^a=\pm1\}} e^{\frac{\beta J}{2N}\sum_{a=1}^{n}\sum_{ij} s_i^a s_j^a}\; e^{\frac{(\beta\sigma)^2}{2}\sum_{i=1}^{N}\left(\sum_{a=1}^{n} s_i^a\right)^2}$$

WHERE WE USED $\underbrace{\int \frac{du}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{u^2}{2\sigma^2}}\, e^{u\cdot x}\, e^{-\frac{x^2\sigma^2}{2}}}_{1}\; e^{\frac{x^2\sigma^2}{2}}$

IMPORTANT $\Longrightarrow$ WE INTRODUCED REPLICA-REPLICA INTER. WITH THE AVERAGE OVER DISORDER.

THE TERM $\left(\sum_{a=1} s_i^a\right)^2 = \sum_{a,b} \underbrace{s_i^a s_i^b}$ INTERACTION BETWEEN a & b

LOCAL IN REAL SPACE: i WITH i



n REPLICAS

③ WE USE NOW A GAUSSIAN DECOUPLING OR HUBBARD-STRATONOVICH TRANSFORMATION TO DECOUPLE DE (i,j) INTERACTION IN THE FM TERM

$$\int \frac{dz_a}{\sqrt{2\pi}}\, e^{-\frac{z_a^2}{2} + z_a\sqrt{\frac{\beta J}{N}}\left(\sum_i s_i^a\right)} = e^{\frac{\beta J}{2N}\left(\sum_i s_i^a\right)^2}$$

ONE PER EACH a INDEX. THUS

$$[(Z_{R}^{(N)})^n] =$$

$$\sum_{\{s_i^a=\pm 1\}} \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}} \, e^{-\sum_{a=1}^{n} \frac{z_a^2}{2} + \sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a \left(\sum_{i=1}^{N} s_i^a\right)} \, e^{\frac{(\beta\sigma)^2}{2} \sum_{i=1}^{N} \left(\sum_{a=1}^{n} s_i^a\right)^2}$$

AND WE NOTE THAT THE REAL SPACE INDICES $i = 1, \dots N$ ARE NOW DECOUPLED IN THE EXPONENTIAL WHILE THE a INDICES ARE COUPLED IN LAST TERM

$$[(Z_{R}^{(N)})^n] = \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}} \, e^{-\sum_{a=1}^{n} \frac{z_a^2}{2}}$$

$$\sum_{\{s_i^a=\pm 1\}} e^{\sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a \left(\sum_{i=1}^{N} s_i^a\right) + \frac{(\beta\sigma)^2}{2} \sum_{i=1}^{N} \left(\sum_{a=1}^{n} s_i^a\right)^2}$$

TAKING $\sum_{i=1}^{N}$ FROM EXP. BELOW AS $\prod_{i=1}^{N}$

$$= \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}} \, e^{-\frac{1}{2} \sum_{a=1}^{n} z_a^2} \sum_{\{s_i^a=\pm 1\}} \prod_{i=1}^{N} e^{\sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a s_i^a + \frac{(\beta\sigma)^2}{2} \left(\sum_{a=1}^{n} s_i^a\right)^2}$$

④ WE NOW NOTE THAT THE i INDEX IS FACTORIZED $\Rightarrow$

THE LAST LINE CAN BE SIMPLIFIED TO

$$\left(\sum_{\{s^a = \pm 1\}} e^{\sqrt{\frac{\beta J}{N}}\, z_a s^a + \frac{(\beta J)^2}{2} s^a s^b} \right)^N = \zeta^N$$
$$= e^{N \ln \zeta}$$

WITH

$$\boxed{\zeta = \sum_{\{s^a = \pm 1\}} e^{\sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a s^a + \frac{(\beta J)^2}{2} \sum_{ab} s^a s^b}}$$

THE PARTITION SUM OF A FM CURIE WEISS MODEL FOR n SPINS WITH

FM COUPLING STRENGTH $(\beta J)_{CW} = (\beta J)^2$

AND APPLIED LOCAL FIELDS $h^a_{CW} = \sqrt{\frac{\beta J}{N}}\, z_a$

THE DIFFERENCE IS THAT $a = 1, \dots, n$ INSTEAD OF $1, \dots, N$

RECALL THAT WHAT WE HAVE SO FAR IS

$$\boxed{\mathbb{E}\left[\left(z_a^{(N)}\right)^n\right] = \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{a=1}^{n} z_a^2}\; e^{N \ln \zeta}}$$

⑤ THE MEANING OF z_a

WE NOTE THAT

$$\frac{\partial \ln \mathcal{J}}{\partial z_a} = \frac{1}{\mathcal{J}} \sum_{\{s^a = \pm 1\}} e^{\sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a s^a + \frac{(\beta J)^2}{2} \sum_{ab} s^a s^b} \sqrt{\frac{\beta J}{N}}\, s^a$$

$$\boxed{\frac{\partial \ln \mathcal{J}}{\partial z_a} = \sqrt{\frac{\beta J}{N}} \langle s^a \rangle_{\mathcal{J}}} \qquad (1)$$

AND WE'LL SEE BELOW THAT THE LHS IS RELATED TO THE SADDLE-POINT VALUE OF z_a

⑥

WE SHOULD NOW COMPUTE THE PARTITION SUM $\mathcal{Z}$.

NOTE THAT $\mathcal{Z}(\{z_a\})$

THE

DIFFICULTY LIES IN THAT THE SUM RUNS OVER ISING VARIABLES. A WAY TO DEAL WITH IT IS TO PERFORM ANOTHER GAUSSIAN DECOUPLING (HUBBARD - STRATO) TO DECOUPLE THE (a,b) INDICES

HUBBARD-STRATO ON s^a :

$$e^{\frac{(\beta\sigma)^2}{2}\sum_{ab} s^a s^b} = \int \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2} + w(\beta\sigma)\sum_a s^a}$$

THEN,

$$\zeta = \sum_{\{s^a=\pm 1\}} e^{\sqrt{\frac{\beta J}{N}} \sum_{a=1}^{n} z_a s^a}$$

$$\int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} e^{w(\beta\sigma)\sum_{a=1}^{n} s^a}$$

$$= \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \sum_{\{s^a=\pm 1\}} e^{\sum_{a=1}^{n}\left(\sqrt{\frac{\beta J}{N}} z_a + w\beta\sigma\right) s^a}$$

$$= \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \prod_{a=1}^{n} \left[2\mathrm{ch}\left(\sqrt{\frac{\beta J}{N}} z_a + w\beta\sigma\right)\right]$$

$$= \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2} + \sum_{a=1}^{n} \ln\left[2\mathrm{ch}\left(\sqrt{\frac{\beta J}{N}} z_a + w\beta\sigma\right)\right]}$$

7 GOING BACK TO $\left[\left(\mathcal{Z}_h^{(N)}\right)^n\right]$:

$$\left[\left(\mathcal{Z}_h^{(N)}\right)^n\right] = \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{a=1}^{n} z_a^2 + N \ln \zeta}$$

$$= \int \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{a=1}^{n} z_a^2}$$

$$e^{N \ln \int \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2} + \sum_{a=1}^{n} \ln\left[2\operatorname{ch}\left(\sqrt{\frac{\beta J}{\sqrt{N}}}\, z_a + w\beta\sigma\right)\right]}}$$

WE SEE THAT THE SECOND EXPON. FACTOR HAS AN N IN FRONT WHILE THE FIRST ONE DOES NOT. MOREOVER, THERE IS A $\frac{1}{\sqrt{N}}$ WITHIN THE ch THAT WOULD BE BETTER TO ELIMINATE.

THUS RE-SCALE z_a $\boxed{\bar{z}_a = \frac{z_a}{\sqrt{N}}}$ & $dz_a = d\bar{z}_a \cdot \sqrt{N}$

$$\left[\left(Z_h^{(N)}\right)^n\right] =$$

$$= \int \prod_{a=1}^{n} \left(\frac{d\bar{z}_a \sqrt{N}}{\sqrt{2\pi}}\right) e^{-\frac{1}{2} N \sum_{a=1}^{n} \bar{z}_a^2}$$

$$e^{N \ln \underbrace{\int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \prod_{a=1}^{n} 2\mathrm{ch}\left(\sqrt{\beta J}\, \bar{z}_a + w\beta\sigma\right)}_{\mathcal{Z}(\{\bar{z}_a\})}}$$

NOW, WE DO HAVE N IN FRONT OF ALL TERMS IN THE EXPONENTIAL & CAN EVALUATE THE INT OVER THE n $\bar{z}_a$ BY SADDLE-POINT

THE SADDLE-POINT EQS ARE

$$\bar{z}_a^{SP} = \left.\frac{\partial \ln \mathcal{Z}}{\partial \bar{z}_a}\right|_{\bar{z}_a^{SP}} \qquad a = 1, \dots, n$$

AND FROM (1) THE rhs IS

$$\frac{\partial \ln \mathcal{Z}}{\partial \bar{z}_a} = \frac{\partial \ln \mathcal{Z}}{\partial z_a} \sqrt{N} = \sqrt{\frac{\beta J}{\cancel{N}}} \langle s^a \rangle_{\mathcal{Z}} \cancel{\sqrt{N}}$$

THUS

$$\boxed{\bar{z}_a^{SP} = \sqrt{\beta J} \langle s^a \rangle_{\mathcal{Z}}}$$

REPLICA SYMMETRIC ANSATZ FOR A VECTOR

$$\boxed{\bar{z}_a = z \qquad \forall a}$$

AT THE SADDLE-POINT LEVEL :

$$\left[\left(Z_h^{(N)}\right)^n\right] =$$

$$e^{-\frac{1}{2} N n \left(\bar{z}^{SP}\right)^2 + N \ln \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \left[2\mathrm{ch}\left(\sqrt{\beta J}\, \bar{z}^{SP} + w\beta\sigma\right)\right]^n}$$

EXPAND THE POWER NOW USING $\boxed{n \simeq 0}$

$$\left[2\mathrm{ch}\left(\sqrt{\beta J}\, \bar{z} + \beta w \sigma\right)\right]^n \simeq$$

$$1 + n \ln\left[2\mathrm{ch}\left(\sqrt{\beta J}\, \bar{z} + \beta w \sigma\right)\right] + \underbrace{\mathcal{O}(n^2)}_{\text{DROP}}$$

SO THAT THE EXPONENTIAL BECOMES

$$\left[\left(Z_h^{(N)}\right)^n\right] =$$

$$= e^{-\frac{1}{2} N n (\bar{z}^{SP})^2}$$

$$e^{N \ln \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \left\{ 1 + n \ln\left[2\,\mathrm{ch}\left(\sqrt{\beta J}\,\bar{z}^{SP} + \beta w \sigma\right)\right]\right\}}$$

$$\simeq e^{-\frac{N}{2} n (\bar{z}^{SP})^2}$$

$$e^{N \ln \left\{ 1 + n \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \ln\left[2\,\mathrm{ch}\left(\sqrt{\beta J}\,\bar{z}^{SP} + \beta w \sigma\right)\right]\right\}}$$

$$\simeq e^{-\frac{N}{2} n (\bar{z}^{SP})^2}$$

EXPANDING $\ln(1+\varepsilon)$

$$e^{N n \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \ln\left[2\,\mathrm{ch}\left(\sqrt{\beta J}\,\bar{z}^{SP} + \beta w \sigma\right)\right]}$$

NOW WE REALLY HAVE AN EXPRESSION IN THE EXPONENTIAL WHICH IS PROP TO (Nn)

WE GO BACK TO $-\beta\,[f_n]$:

$$\boxed{-\beta\,[f_n] = \lim_{N\to\infty}\ \lim_{n\to 0}\ \frac{\left[\left(Z_n^{(N)}\right)^n\right]-1}{n\,N}}$$

AND CONTINUE EXPANDING FOR $n\to 0$

$$= -\frac{\left(\overline{z}^{SP}\right)^2}{2} + \int \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2}}\, \ln\left[2\,ch\left(\sqrt{\beta J}\,\overline{z}^{SP} + \beta w \sigma\right)\right]$$

WITH $\overline{z}^{SP}$ DERIVED FROM

$$\boxed{\overline{z}^{SP} = \sqrt{\beta J} \int \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2}}\, th\left(\sqrt{\beta J}\,\overline{z}^{SP} + \beta w \sigma\right)}$$

WHEN RELATING $\overline{z}^{SP}$ TO AVERAGED $\langle s^a \rangle$ WE USED $N\to\infty$

SOLUTIONS

$\bar{z}^{SP} = 0$ SOLVES THIS EQ $\forall (\beta J, \beta \sigma)$

$\bar{z}^{SP} \neq 0$ AT A CRITICAL $(\beta J, \beta\sigma \simeq 0)_c$

$$\bar{z}^{SP} \sim \sqrt{\beta J} \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \left(\sqrt{\beta J}\, \bar{z}^{SP} + \beta w \sigma \right)$$

ZERO AVER.

$$\bar{z}^{SP} \sim \beta J \, \bar{z}^{SP} \implies$$

$$(\beta J)_c = 1 \qquad \beta\sigma = 0$$