Advanced Statistical Physics

Leticia F. Cugliandolo

Sorbonne Université

Institut Universitaire de France

leticia@lpthe.jussieu.fr
www.lpthe.jussieu.fr/~leticia

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Disorder

THIS YEAR I WILL CENTER THE DISCUSSION AROUND DISORDERED SYST

PLAN

DEFINITION & EXAMPLES

- PHYSICAL
- BEYOND COMP. SCIENCE
  BIOPHYSICS
  OTHER

- STATICS Q & A
- DYNAMICS Q & A

FINITE d VS. INFINITE d

LATTICES GRAPHS

Randomness

Impurities

No material is perfect and totally free of impurities

(vacancies, substitutions, amorphous structures, etc.)

~~First distinction~~

— ~~Weak randomness : phase diagram respected, criticality may change~~

— ~~Strong randomness : phases modified~~

Second distinction

— Annealed : fluctuating (easier)

— Quenched : frozen, static (harder)

$$\tau_0 \ll t_{\rm obs} \ll \tau_{eq}^{qd}$$

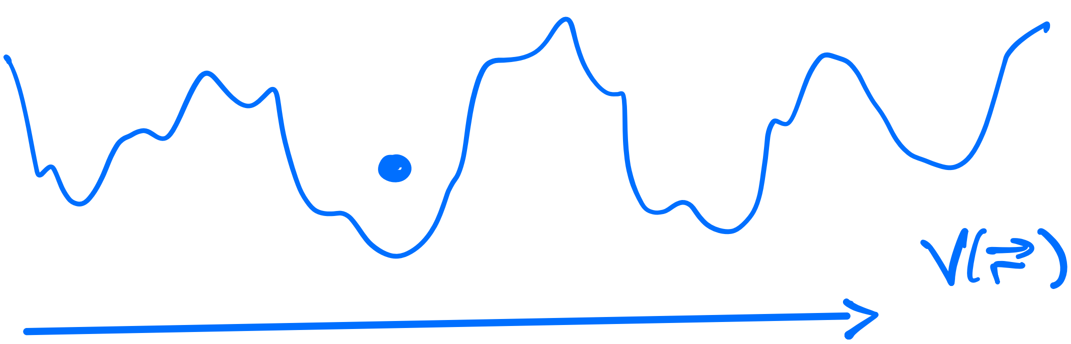
PARTICLE IN A RANDOM POTENTIAL

A MECHANICAL PROBLEM

$V(\vec{r})$ FROM SOME $P(V(\vec{r}))$

eg $\dfrac{e^{-V^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$ AT EACH $\vec{r}$ INDEP. OF OTHERS

OR WITH SOME SPATIAL CORR.



QUESTION: HOW DOES THE PART MOVE THERE IF FOLLOWING LANGEVIN DYN?

DIFFUSION IN RANDOM MEDIA

THE QUESTIONS CAN BE: HOW DOES THE PARTICLE BEHAVE?

e.g. UNDER LANGEVIN DYNAMICS

$$\gamma \dot{x}(t) = -\frac{dV(x)}{dx(t)} + \xi(t)$$

$$\langle \xi(t)\, \xi(t') \rangle = 2 k_B T \gamma\, \delta(t-t')$$

$$\langle \xi(t) \rangle = 0$$

- NORMAL OR ANOMALOUS DIFFUSION? SAY YOU FIX V TAKEN FROM THE pdf AND YOU MEASURE THE MEAN SQUARE DISPLACEMENT.

NO LEFT-RIGHT SYMM BROKEN

SO $\langle x(t) \rangle = 0 \quad \forall t$

BUT THE 2nd MOMENT DOESN'T VANISH AND

$$\langle x^2(t) \rangle \sim 2D\, t^{\alpha}$$

IS $\alpha = 1$ OR $\alpha \neq 1$?

IF $\alpha = 1$ HOW DOES D DEPEND ON THE PROPS OF $p(r)$?

WHAT ABOUT

$$\left[\langle x^2(t) \rangle \right] \quad ?$$

AVERAGE OVER DISORDER

JP BOUCHAUD, D. DEAN, A. GEORGES, P. LE DOUSSAL 80's–90's

THE POTENTIAL CAN BE FULLY STATIC – INDEP OF TIME

$\Rightarrow$ QUENCHED

OR, IT CAN DEPEND SOMEHOW ON TIME ITSELF $\Rightarrow$

ANNEALED

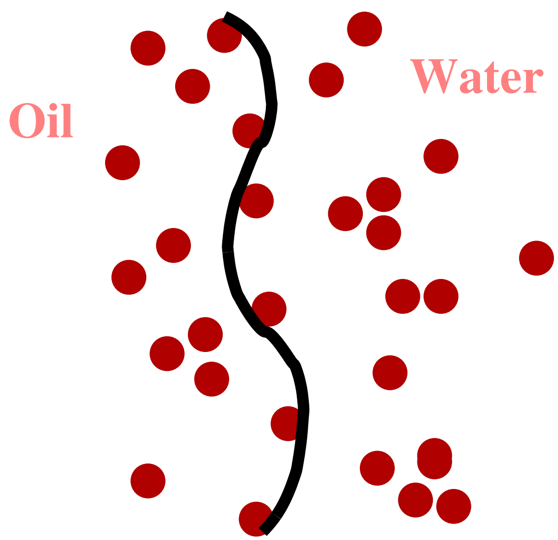
METALLURGICAL LANGUAGE

AND ASK SAME QUESTIONS ABOUT THE PARTICLE'S MOTION

Pinning by impurities

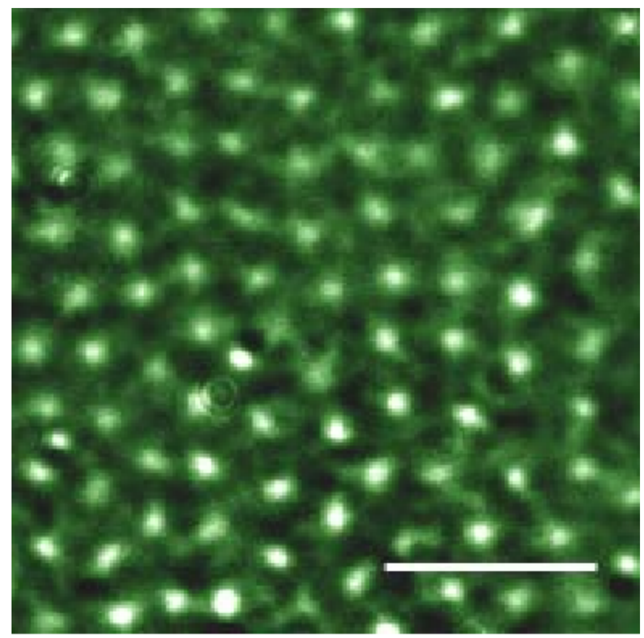
Competition between elasticity and quenched randomness

d-dimensional elastic manifold in a transverse N-dimensional **quenched random potential**.



Interface between two phases ; vortex line in type-II supercond ; stretched polymer.

Distorted Abrikosov lattice



Goa et al. 01

ONE CAN ASK ABOUT THE CONFORMATION OF THE LINE IN EQUILIBRIUM W/ A THERMAL BATH AND UNDER A GIVEN RANDOM POTENTIAL $V(x,y)$ DRAWN FROM A $P(V)$

IS IT ALMOST STRAIGHT WITH ELASTICITY DOMINATING OVER RANDOM POTENTIAL ?

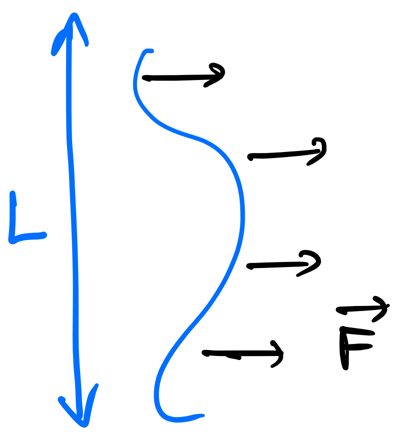
IS IT ROUGH – VERY CONTORTED ?

PHASE TRANSITION ?

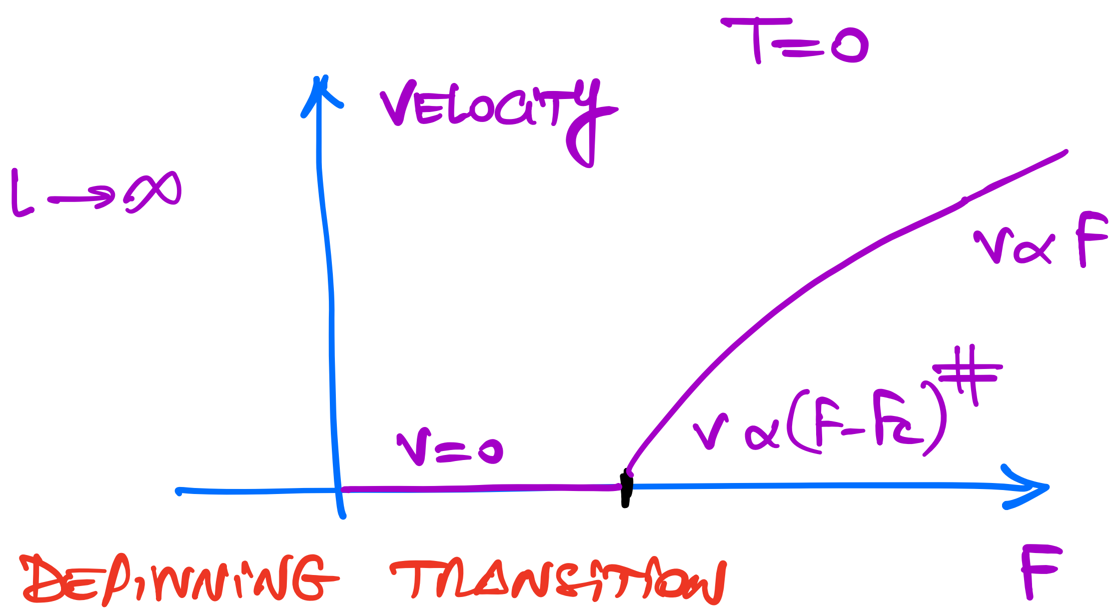
T. GIAMARCHI, P. LE DOUSSAL
A. ROSSO, C. FOINI
D. BONAMY

PROBLEM RELATED TO SOLIDS WITH QUENCHED RANDOMNESS

ONE CAN THINK ABOUT PULLING FROM THE STRING IN THE RIGHT DIRECTION WITH A FORCE THAT DOESN'T DEPEND ON WHERE ON THE VERT DIRECTION ONE IS.



DOES THE LINE GET PINNED OR DOES IT MOVE?



DEPINNING TRANSITION

T. GIAMARCHI, K. WIESE, P. LE DOUSSAL 00's

MACROSCOPIC SYSTEM —

MANY d.o.f.

GLOBAL BEHAVIOUR?

PHASE DIAGRAMS — STATIC
— DYNAMIC

ORGANIZATION OF THE GLOBAL BEH OF MACROSC SYSTEMS WHICH CAN CHANGE ABRUPTLY FOR SPECIAL VALUES OF THE CONTROL PARAM.

$\Rightarrow$ PHASE TRANSITIONS & CRITICAL PHENOMENA

(WE'LL REVISIT PHASE TRANS IN THE CONTEXT OF THESE STUDIES)

Quenched disorder

Variables frozen in time-scales over which other variables fluctuate.

Time scales

$$\boxed{\tau_0 \ll t_{\mathrm{obs}} \ll \tau_{\mathrm{eq}}^{qd}}$$

τ_{eq}^{qd} could be the diffusion time-scale for magnetic impurities the magnetic moments of which will be the variables of a **magnetic system** ;

or the flipping time of impurities that create random fields acting on other magnetic variables.

Weak disorder (modifies the critical properties but not the phases) *vs.* **strong** disorder (that modifies both).

e.g. **random ferromagnets** *vs.* **spin-glasses**.

First distinction

— Weak randomness : phase diagram respected, criticality may change

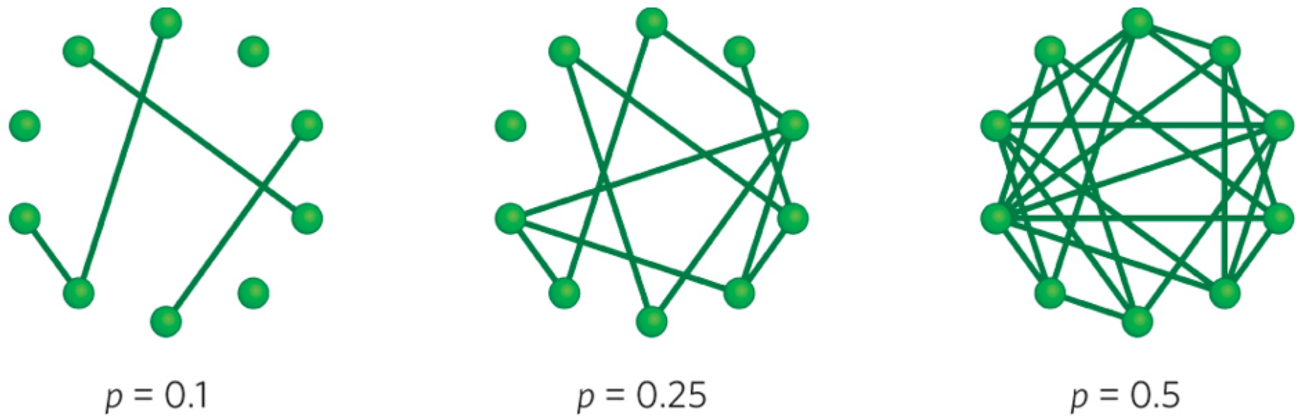
— Strong randomness : phases modified

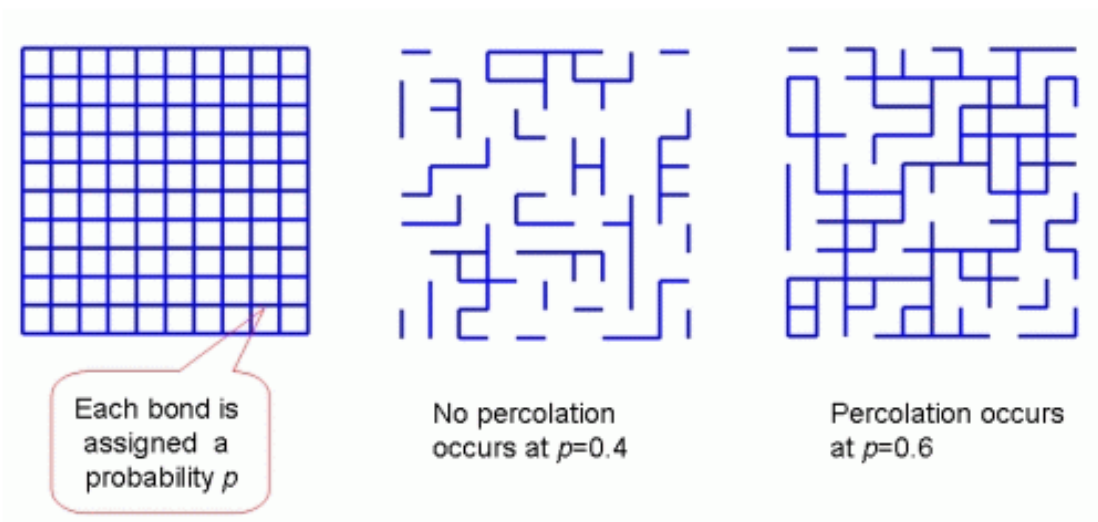
WEAK VS STRONG

DISORDER

Geometrical problems

Random graphs & Percolation





- DEFINE THEM MATHEMATICALLY
- NOTE: GEOMETRIC PROBLEMS
  NO COST / ENERGY
  GIVEN TO THE CONFIGURATIONS

- QUESTIONS ON RANDOM GRAPHS
  e.g. DIST OF CLUSTER SIZES

- QUESTIONS ON DILUTE LATTICES
  e.g. PERCOLATION TRANSITION
  p_c
  ABOVE WHICH ONE CAN GO
  FROM ONE END TO THE OTHER
  OF THE SYST W/ PROB = 1.

RANDOM GRAPHS & HYPER-GRAPHS

ARE ALSO IMPORTANT DEFINING THE SPACE WHERE A PHYSICAL PROBLEM TAKES PLACE

THE PERCOLATION PROBLEM IS AN EXAMPLE (WITH NO ENERGY FUNCTION DEFINED)

OTHERS WITH ENERGY FUNCTIONS ARE ALSO IMPORTANT

$\Rightarrow$ EX. BELOW

Spin-glasses

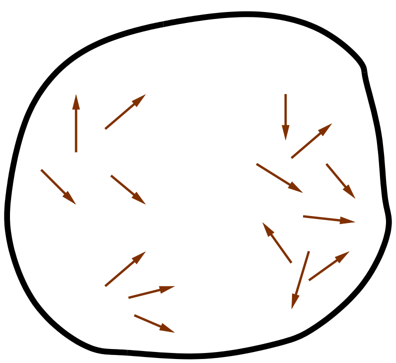
Magnetic impurities (spins) randomly placed in an inert host

$\vec{r}_i$ are random and time-independent since

the impurities do not move during experimental time-scales $\Rightarrow$

quenched randomness

Magnetic impurities in a metal host



spins can flip but not move

RKKY potential

$$V(r_{ij}) \propto \frac{\cos 2k_F r_{ij}}{r_{ij}^3} s_i s_j$$

very rapid oscillations about 0

positive & negative

slow power law decay.

Spin-glasses

Models on a lattice with random couplings

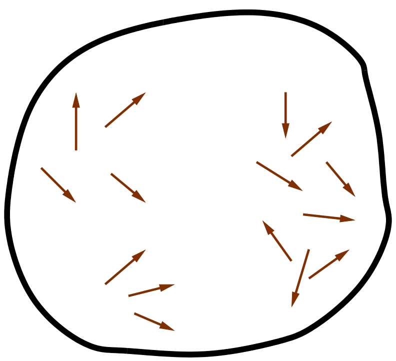
Ising (or Heisenberg) spins $s_i = \pm 1$ sitting on a lattice

J_{ij} are random and time-independent since

the impurities do not move during experimental time-scales $\Rightarrow$

quenched randomness

Magnetic impurities in a metal host



spins can flip but not move

Edwards-Anderson model

$$H_J[\{s_i\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

J_{ij} drawn from a pdf with

zero mean & finite variance

CONNECTION W/ RANDOM GRAPH
ON DILUTE LATTICES

THESE COULD BE THE SPACES WHERE THESE VARIABLES — THE SPINS — LIVE.

GIVE ANOTHER SOURCE OF QUENCHED RANDOMNESS, APART FROM J_{ij} signs AND VALUES.

e.g. SPIN GLASS MODELS ON RANDOM GRAPHS

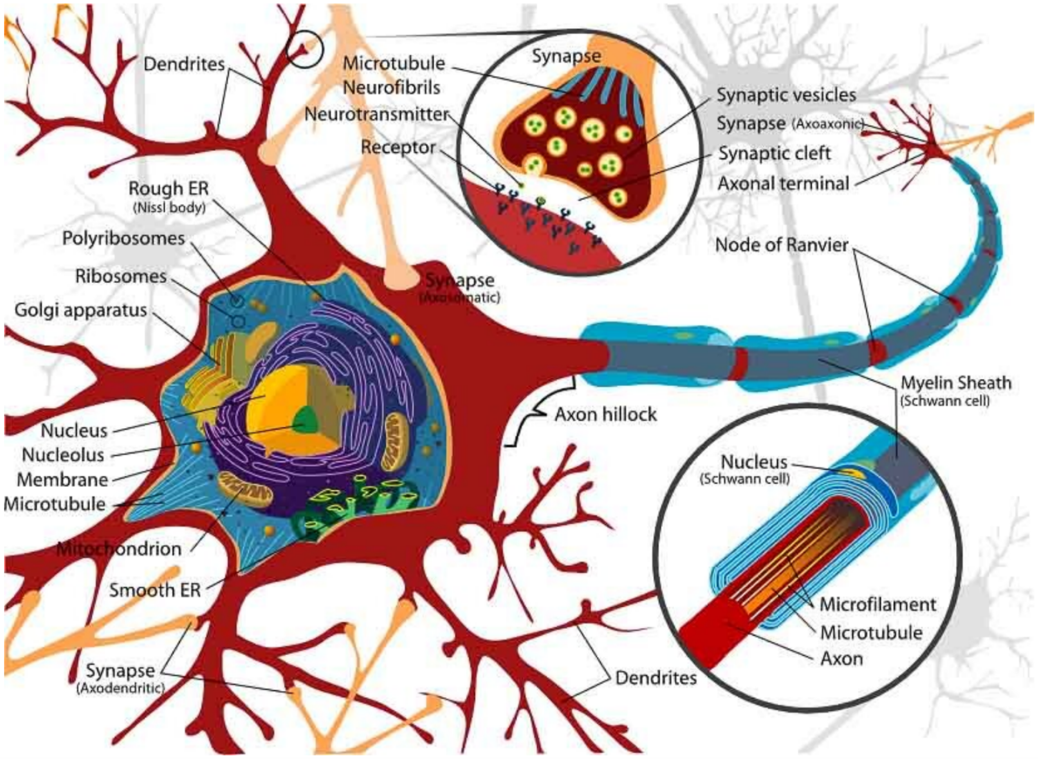
WE'LL SEE APPLICATIONS IN COMPUTER SCIENCE BELOW

MANY PEOPLE IN FRANCE & ITALY

ENS, LPTHE, LPTMC, LPTMS (ORSAY), IPHT (SACLAY)

Neural Networks

Real neural network



Neurons connected by synapsis on a random graph

Figures from AI, Deep Learning, and Neural Networks explained, A. Castrounis

Neural networks

Models on graphs with random couplings

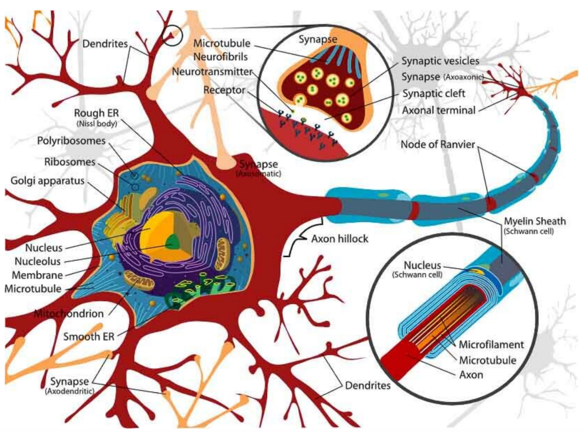
The neurons are Ising spins $s_i = \pm 1$ on a graph

J_{ij} are random and time-independent since

the synapsis do not change during experimental time-scales $\Rightarrow$

quenched randomness

The neural net



spins can flip but not move

Hopfield model

$$H_J[\{s_i\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

memory stored in the synapsis

$$J_{ij} = 1/N_p \sum_{\mu=1}^{N_p} \xi_i^\mu \xi_j^\mu$$

the patterns ξ_i^μ

are drawn from a pdf with

zero mean & finite variance

QUESTIONS

IN THE CONTEXT OF "ASSOCIATIVE NEURAL NETS" THE PATTERNS HAVE BEEN LEARNT & STORED IN J_{ij}

HEBB'S RULE

HOW MANY PATTERNS $\{\xi_i^\mu \; ; \; i=1,\dots,N \; \mu=1,\dots,N_p\}$ CAN THE NETWORK STORE?

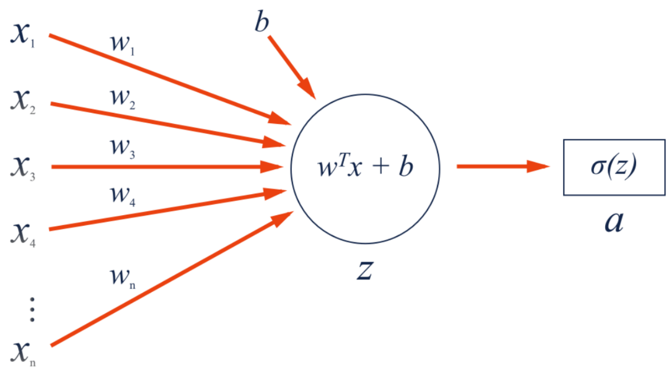
CAPACITY $\alpha = \dfrac{N_p}{N} \longrightarrow \alpha_c$

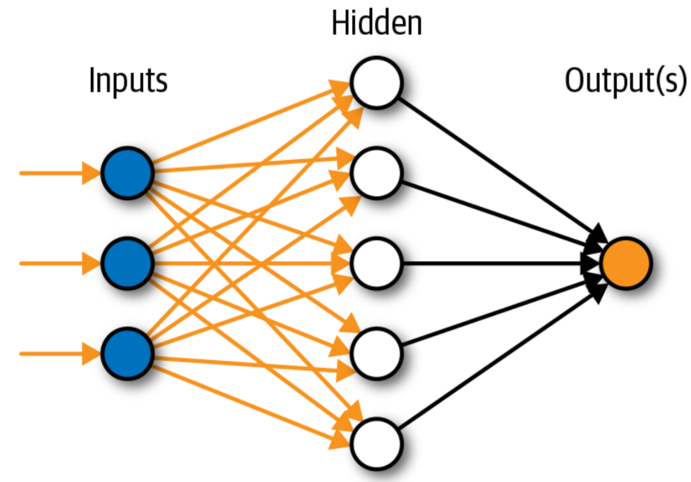
IF WE START THE NEURAL ACTIVITY FROM $S_i(t=0)$ DOES THE PROPOSED DYNAMICS APPROACH $S_i(t\to\infty) = \xi_i^\mu$

RETRIEVAL

Neural Networks

Sketch & artificial network





The connections in w^T may have a random component

The state of the neuron up (firing), down (quiescent) is a result of the calculation

In the artificial network on chooses the geometry (number of nodes in internal layer, number of hidden layers, connections between layers)

Figures from AI, Deep Learning, and Neural Networks explained, A. Castrounis

CLASSIFICATION TASK

CATS & DOGS' PICTURES → CLASSIFY THEM.

TRAINING

PRESENT PICTURES OF CATS & DOGS

- PIXELS AS INPUTS TO THE LEFT LAYER OF NEURONS
- OUTPUT OF THE RIGHT MOST SINGLE NEURON : UP (DOG), DOWN (CAT)
- LET THE J_{ij}'S ACCOMMODATE, ARCHITECTURE AND VALUES

FUNCTIONING

PRESENT A NEW PICTURE → OUTPUT ?

Optimization problems

Constrained satisfaction problems

Problems involving variables which must satisfy some constraints

e.g. equalities, inequalities or both

studied in computer science to

compute their complexity or develop algorithms to most efficiently solve them

Typically, N variables, which have to satisfy M constraints.

e.g. the variables could be the weights of a neural network, and each constraint imposes that the network satisfies the correct input-output relation on one of M training examples (e.g. distinguishing images of cats from dogs).

Statistical physics approach

thermodynamic limit $N \to \infty$ and $M \to \infty$ with $\alpha = M/N$ finite

Optimization problems

K-Satisfiability

The problem is to determine whether the variables of a given Boolean formula F can be assigned in such a way to make the formula evaluate to TRUE (satisfied)

Example. Call the variable x

We use x for the evaluation x = TRUE and $\overline{x}$ for the requirement x = FALSE

Take the formula $F = C_1$: x_1 OR $\overline{x}_2$ made by a single clause C_1

it is satisfiable because one can find the values x_1 = TRUE (and x_2 free) or x_2 = FALSE (and x_1 free), which make C_1 : x_1 OR $\overline{x}_2$ TRUE

This formula is so simple that 3 out of 4 possible configurations of the two variables solve it. This example belongs to the $k = 2$ class of satisfiability problems since the clause is made by two literals (involving different variables) only. It has $M = 1$ clauses and $N = 2$ variables.

Optimization problems

K-Satisfiability

Harder to decide formulæ are made of M clauses involving k literals required to take the true value (x) or the false value ($\overline{x}$) each, these taken from a pool of N variables. An example in $k = 3$-SAT is

$$F = \begin{cases} C_1 & : & x_1 & \text{OR} & \overline{x}_2 & \text{OR} & x_3 \\ C_2 & : & \overline{x}_5 & \text{OR} & \overline{x}_7 & \text{OR} & x_9 \\ C_3 & : & x_1 & \text{OR} & \overline{x}_4 & \text{OR} & x_7 \\ C_4 & : & x_2 & \text{OR} & \overline{x}_5 & \text{OR} & x_8 \end{cases}$$

All clauses have to be satisfied simultaneously so the formula has to be read

$$\text{F} : C_1 \text{ AND } C_2 \text{ AND } C_3 \text{ AND } C_4$$

When $\alpha \equiv M/N \gg 1$ the problems typically become unsolvable while many solutions exist for $\alpha \ll 1$. A sharp threshold at α_c for $N, M \to \infty$

Optimization problems

Random K-Satisfiability

An instance of the problem, i.e. a formula F, is chosen at random with the following procedure :

First one takes k variables out of the N available ones.
Second one decides to require x_i or $\overline{x}_i$ for each of them with probability 1/2
Third one creates a clause taking the OR of these k literals.
Forth one returns the variables to the pool and the outlined three steps are repeated M times.
The M resulting clauses form the final formula.

Change of focus from worse case (most difficult formula) to typical case (just one such constructed formula)

Optimization problems

Random K-Satisfiability as a physical model

Boolean variables $\Rightarrow$ Ising spins

x_i evaluated to TRUE (FALSE) corresponds to $s_i = 1\ (-1)$

The requirement that a formula be evaluated TRUE by an assignment of variables (i.e. a configuration of spins) $\Rightarrow$ ground state of an adequately chosen energy function = cost function

In the simplest setting, each clause will contribute zero (when satisfied) or one (when unsatisfied) to this cost function.

There are several equivalent ways to reach this goal. The fact that the variables are linked together through the clauses suggests to define k-uplet interactions between them.

Optimization problems

Random K-Satisfiability as a physical model

A way to represent a clause in an energy function, for instance,

$$C_1 \; : \; x_1 \;\; \text{OR} \;\; \overline{x}_2 \;\; \text{OR} \;\; x_3$$

k=3

as an interaction between spins. In this case

$$(1 - s_1)(1 + s_2)(1 - s_3)/8$$

(1)

This term vanishes if $s_1 = 1$ or $s_2 = -1$ or $s_3 = 1$ and does not contribute to the total energy, that is written as a sum of terms of this kind.

It is then simple to see that the total energy can be rewritten in a way that resembles strongly physical spin models,

$$2^K H_J[\{s_i\}] = M + \sum_{R=1}^{K} (-1)^R \sum_{i_1 < \cdots < i_R} J_{i_1 \ldots i_R} s_{i_1} \ldots s_{i_R}$$

$J_{i_1 \ldots i_R} = \sum_{a=1}^{M} J_{a i_1} \ldots J_{a i_R}$ and $J_{ai} = \pm 1$ according to x_i or $\overline{x}_i$ in clause a and zero otherwise

THE PRODUCT OF THE THREE FACTORS IN (1) YIELDS TERMS

- WITH NO S_i's
- WITH ONE S_i
- WITH TWO S_i, S_j

  ...

- WITH k, $S_{i_1}, S_{i_2} \dots S_{i_k}$

THE TERM HERE READS $(k=3)$

$$1 - S_1 + S_2 - S_3 - S_1 S_2 + S_1 S_3 - S_2 S_3 - S_1 S_2 S_3$$

AND WE SEE THE 1 AND THE $\mp$ SIGNS, EXPLICITLY.

Optimization problems

K-Satisfiability & complexity theory

Special interest in computational complexity theory

K-Sat for K≥ 3 is in the NP complexity class

- No algorithm (as yet) has been found that can find an assignment for the variables x_i in polynomial time
- one can verify in polynomial time whether an assignment satisfies the given formula

K-SAT is an NP-complete problem

- all other problems in the NP complexity class can be formally reduced to the K-SAT problem

ONCE YOU THINK IN TERMS OF AGENTS IN INTERACTION

→ BUILD

- ECONOPHYSICS MODEL FOR MARKETS
  eg. WORK BY JP BOUCHAUD ET AL
- ECOLOGY MODELS FOR COMPETITION BETWEEN SPECIES. eg. MAY'S
  eg. WORK BY A. ALTIERI ET AL

MAY MODEL 1972

n POPULATIONS WITH $N_i(t)$ INDIVIDUALS EACH. $i = 1, \dots, n$

- SAY THAT THERE IS AN EQUIL. STATE SUCH THAT $N_i(t) = N_i^*$ FOR EACH i

- STUDY THE STABILITY PROP. BY LINEARIZING THE DYN. CLOSE TO N_i^*:

$$N_i(t) = N_i^* + \delta N_i(t)$$

WITH $\delta N_i(t) \ll N_i^*$

- $\delta N_i(t)$ WILL FOLLOW

$$\frac{d\,\delta N_i(t)}{dt} = A_{ij}\,\delta N_j(t)$$

- PROPOSE THAT A_{ij} IS A SYMM (OR NOT) MATRIX W/ ELEMENTS TAKEN FROM A pdf $p(A_{ij})$

RANDOM MATRIX

- NECESSARY & SUFF COND. FOR N_i^* TO BE STABLE

ALL λ_i EIGENVALUES OF $\mathbb{A}$ SHOULD HAVE A POSITIVE REAL PART

$$\mathbb{R}(\lambda_i) > 0 \quad \forall i$$

- DYNAMICS ? HOW DO $\delta N_i(t)$ DEPEND ON TIME ?

- DO YOU KNOW OF OTHER EXAMPLES?
- WHICH ONE HAVE YOU LIKED BEST?

NOTE SOME OF THE PROBLEMS I PRESENTED ARE

FINITE DIMENSIONAL

AND OTHERS

∞ DIMENSIONAL IN THE SENSE OF **NO** DISTANCE

END LECTURE HERE

Randomness

Properties

— Spatial inhomogeneity

— Frustration

(spectrum pushed up, degeneracy of ground state)

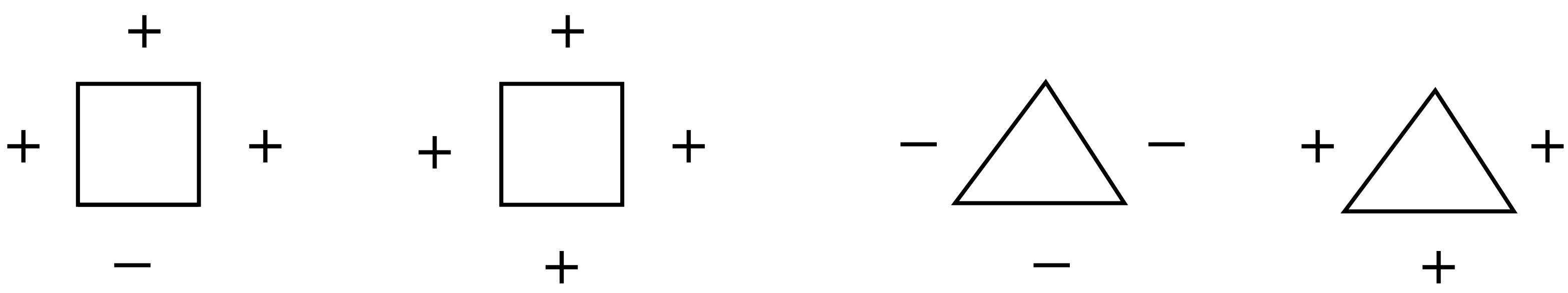
— Probability distribution of couplings, fields, etc.

— Self-averageness

Frustration

Properties

$$\boxed{H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad \textbf{Ising model}}$$



Disordered **Geometric**

$$\boxed{E_{gs}^{frust} > E_{gs}^{FM} \qquad \text{and} \qquad S_{gs}^{frust} > S_{gs}^{FM}}$$

Frustration enhances the **ground-state** energy and entropy

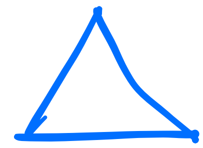
One can expect to have **metastable states** too

One cannot satisfy all couplings simultaneously if $\boxed{\prod_{loop} J_{ij} < 0}$

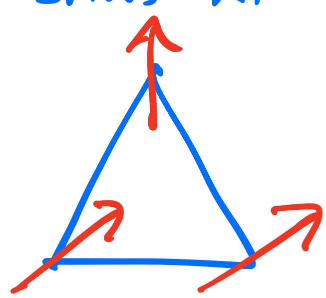
NOTE THAT WE TOOK $S_i = \pm 1$ IN THE EXAMPLE ABOVE

EXERCISE 1

TAKE A TRIANG LATTICE



& PLACE PLANAR SPINS AT THE VERTICES



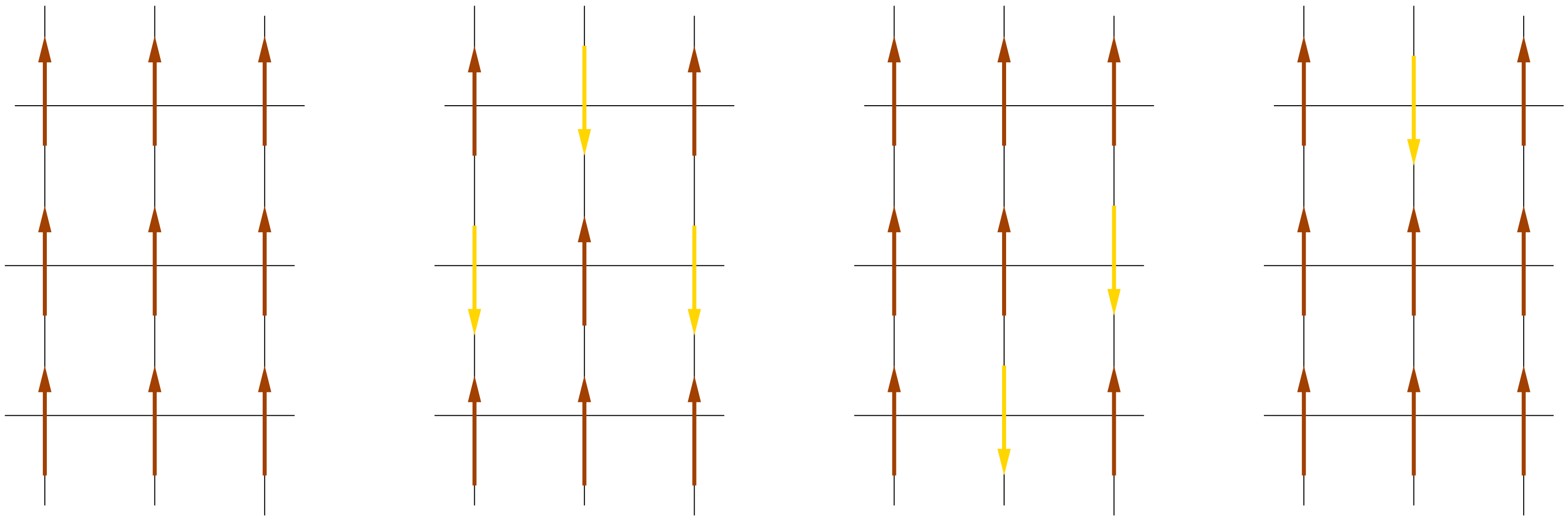
TAKE AF INTERACTIONS ON NEAREST NEIGHBOURS $-J \, \vec{S}_i \cdot \vec{S}_j$

WITH $J<0$

WHICH CONF MINIMIZES THE PLAQ. ENERGY ?

Heterogeneity

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$, contrary to what happens in a ferromagnetic sample, for instance.



Homogeneous **Heterogeneous**

$h_i = 4J \;\; \forall\, i$. $h_j = -2J$ $h_k = 0$ $h_l = 2J$.

Each sample is *a priori* different but,

do they all have a different thermodynamic and dynamic behavior ?

MICROCANONICAL VS. CANONICAL DESCRIPTION

CLOSED VS. OPEN SYSTEM

$\Delta E = 0$ $\qquad\qquad$ $\Delta E \neq 0$

- DEPENDS ON SITUATION
- EQUIVALENCE OF ENSEMBLES $\Rightarrow$

$$\text{MICRO} \equiv \text{CANONICAL}$$

$$N \to \infty$$

NOT OBVIOUS FOR LONG-RANGE INTERACTING SYST

DAUXOIS, RUFFO, MUKAMEL, ETC

SEE NOTES IF INTERESTED

IGNORE THIS ISSUE $\longrightarrow$ CANONICAL

CANONICAL EQUILIBRIUM

$$Z = \sum_{conf} e^{-\beta H(conf)}$$

$$-\beta f = \frac{\ln Z}{N}$$

BUT HERE THERE ARE ∞-PARAM.

J_{ij} OR V SO?

$$Z_J = \sum_{conf} e^{-\beta H_J(conf)}$$

$-\beta f_J$ MAY DEPEND ON THE REALIZATION OF DISORDER

IT WOULD BE TOO SPECIFIC

Self-averageness

The disorder-induced free-energy density distribution approaches a Gaussian with vanishing dispersion in the thermodynamic limit :

$$\boxed{\lim_{N\to\infty} f_N(\beta, J) = f_\infty(\beta) \qquad \text{independently of disorder}}$$

— Experiments : all *typical* samples behave in the same way.

— Theory : one can perform a (hard) average of disorder, $[\dots]$,

$$\boxed{-\beta N f_\infty(\beta) = \lim_{N\to\infty} [\ln \mathcal{Z}_N(\beta, J)]}$$

From here, we see that, e.g., the energy density is self-averaging

Replica theory

$$\boxed{-\beta f_\infty(\beta) = \lim_{N\to\infty} \lim_{n\to 0} \frac{[\mathcal{Z}_N^n(\beta, J)] - 1}{Nn}}$$

TO BE DISCUSSED LATER

Self-averageness

The question

Given two samples with different quenched randomness

(e.g. different interaction strengths J_{ij}s or random fields h_i)

but drawn from the same (kind of) distribution

is their behaviour going to be totally different ?

Which quantities are expected to be the same and which not ?

Self-averageness

Observables & distributions

Given a quantity A_J, which depends on the quenched randomness J, it is distributed according to

$$P(A) = \int dJ\, p(J)\, \delta(A - A_J)$$

This pdf is expected to be narrower and narrower (more peaked) as $N \to \infty$

Therefore, one will observe $A_{\rm typ} = A\, s.t.\, \max_A P(A)$

However, it is difficult to calculate $A_{\rm typ}$, what about calculating $[A] = \int dA P(A) A$?

Self-averageness

EX 2

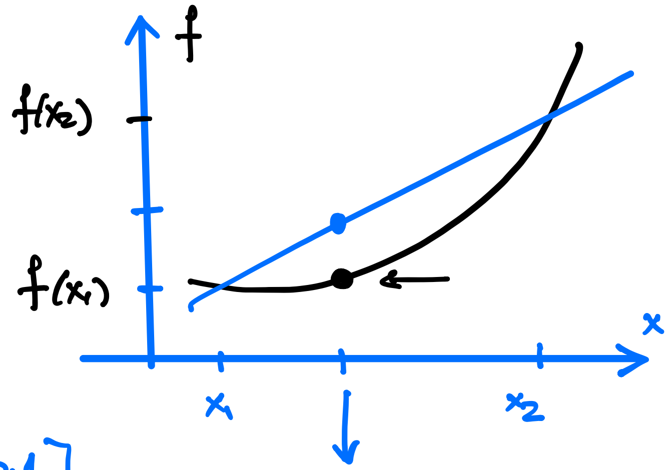
Warm-up exercise

Exercise 5.1 This exercise provides a useful example of the distinction between *typical* and *average* values of random variables. Consider a random variable z that takes only two values $z_1 = e^{\alpha\sqrt{N}}$ and $z_2 = e^{\beta N}$, with α and β two positive and finite numbers with α unconstrained and $\beta > 1$. The probabilities of the two events are $p_1 = 1 - e^{-N}$ and $p_2 = e^{-N}$. First, confirm that these probabilities are normalised. Second, compute the average $\langle z \rangle$, where the angular brackets indicate average with the probabilities p_1, p_2, and evaluate it in the limit $N \to \infty$. Third, calculate the most probable value taken by z, that we call z_{typ}, for typical (indeed, if we were to draw the variable we would typically get this value). Compare and conclude. Now, let us study the behaviour of the quantity $\ln z$ that is also a random variable. Compute its average. By which value of z is it determined? Does $\langle \ln z \rangle = (\ln z)_{\text{typ}}$ in the large N limit? Is $\langle \ln z \rangle = \ln \langle z \rangle$? The last result demonstrates the difference between what are called *quenched* and *annealed* averages. Which value is larger? Does the comparison comply with Jenssen's inequality? (See App. 5.A for its definition.)

A function is convex function iff $\forall x_1, x_2$ and $t \in [0, 1]$:

$$f(tx_1 + (1-t)x - 2) \leq tf(x_1) + (1-t)f(x_2) \ .$$

JENSEN'S INEQUALITY



CONVEX FUNCTION

$t \in [0,1]$

$t x_1 + (1-t)\, x_2$

$$f(t x_1 + (1-t)\, x_2) \leq t f(x_1) + (1-t) f(x_2)$$

SECANT IS ABOVE FUNCTION ITSELF, IF THE FUNCTION IS CONVEX AS IN THE DRAWING

IN PROB THEORY, IF X IS A RANDOM VARIABLE AND φ A FUNCTION OF X $\Longrightarrow$

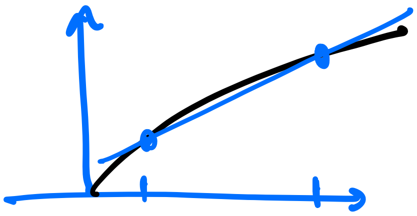
$$\mathbb{E}\left[\varphi(x)\right] \geq \varphi\left[\mathbb{E}(x)\right]$$

IF WE APPLY IT TO QUENCHED VS. ANNEALED

$-[\ln Z] \geq -\ln[Z]$

LOGARITHM IS CONCAVE

$\Rightarrow$ $-$ LOGARITHM IS CONVEX



$-\ln Z$ IS CONVEX

- SQ. BRACKETS MEAN AVER. OVER DISORDER HERE
- Z IS THE PART FUNCT.
- lhs IS THE QUENCHED AVER OF THE FREE-ENERGY
- rhs IS THE ANNEALED AVER OF THE FREE-ENERGY

$$F_{\text{QUENCHED}} \geq \underbrace{F_{\text{ANNEALED}}}_{\text{UNDER ESTIMATES THE TRUE FCT.}}$$

THE SOL. TO THE EX. IS BELOW

Self-averageness

Warm-up exercise

1

TYPICAL VS. AVERAGE

X RANDOM VARIABLE

$$\begin{cases} x = x_1 = e^{\alpha\sqrt{N}} \\ x = x_2 = e^{\beta N} \end{cases} \qquad \begin{matrix} \alpha, \beta > 0 \\ \beta > 1 \end{matrix} \qquad \begin{matrix} p_1 = 1 - e^{-N} \\ p_2 = e^{-N} \end{matrix}$$

NORM $\quad p_1 + p_2 = 1 - e^{-N} + e^{-N} = 1 \quad$ ok.

$$\begin{aligned} \langle x \rangle &= x_1 p_1 + x_2 p_2 \\ &= e^{\alpha\sqrt{N}} (1 - e^{-N}) + e^{\beta N} e^{-N} \\ &= e^{\alpha\sqrt{N}} - e^{\alpha\sqrt{N} - N} + e^{(\beta - 1) N} \end{aligned}$$

$$\lim_{N \to \infty} \langle x \rangle = e^{(\beta-1)N} \longrightarrow \infty$$

MOST PROB VALUE OF X IS THE ONE WITH MAX PROB

ie $\quad x_1 = e^{\alpha\sqrt{N}}$

WE NOTE $\quad \langle x \rangle = e^{(\beta-1)N}$, $\quad x_{typ} = e^{\alpha\sqrt{N}} \quad$ THEY ARE $\neq$.

$$\begin{aligned} \langle \ln x \rangle &= (\ln x_1)\, p_1 + (\ln x_2)\, p_2 \\ &= \alpha\sqrt{N} (1 - e^{-N}) + \beta N e^{-N} \end{aligned}$$

$$\lim_{N \to \infty} \langle \ln x \rangle = \alpha\sqrt{N} = \ln x_{typ}$$ "SELF-AVERAGENESS"

Self-averageness

Warm-up exercise

$$\langle \ln x \rangle \stackrel{?}{=} \ln \langle x \rangle$$

$$\lim_{N\to\infty} \langle \ln x \rangle = \alpha\sqrt{N}$$

$$\lim_{N\to\infty} \ln \langle x \rangle = \lim_{N\to\infty} \ln \left[e^{\alpha\sqrt{N}} - e^{\alpha\sqrt{N} - N} + e^{(\beta-1)N} \right]$$

$$= (\beta-1)N$$

$$\lim_{N\to\infty} \ln \langle x \rangle \neq \lim_{N\to\infty} \langle \ln x \rangle$$

$>$ ok

The logarithm is concave

Self-averageness

Example : the disordered Ising chain

$$H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1}$$ J_i distributed $p(J_i)$ with any pdf

Compute the partition function Z_J by introducing $\sigma_i = s_i s_{i+1}$

$$Z_J[\{s_i\}] = \sum_{s_i=\pm 1} e^{\beta \sum_i J_i s_i s_{i+1}} = \sum_{\sigma_i=\pm 1} e^{\beta \sum_i J_i \sigma_i} = \prod_{i=1}^{N} 2\coth \beta J_i$$

(boundary condition effects negligible for $N \to \infty$)

It is a **product** of N random numbers

The free-energy is $-\beta F_J[\{s_i\}] = \sum_{i=1}^{N} \ln \coth \beta J_i + N \ln 2$

It is a **sum** of N random numbers

Self-averageness

Example : the disordered Ising chain

$$H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1}$$ J_i distributed $p(J_i)$ with any pdf

The partition function & the free energy density are different objects

$$Z_J[\{s_i\}] = \prod_{i=1}^{N} 2\coth\beta J_i \qquad -\beta f_J[\{s_i\}] = \frac{1}{N}\sum_{i=1}^{N} \ln\coth\beta J_i + \ln 2$$

Take J_i to be *i.i.d* with zero mean $[J_i] = 0$ & finite variance $[J_i^2] = \sigma^2$ and use the **Central Limit Theorem** :

$X = \frac{1}{N}\sum_i x_i$ is Gaussian distributed with average $\langle X\rangle = \langle x_i\rangle$ and variance $\langle (X - \langle X\rangle)^2\rangle = \sigma^2/N$

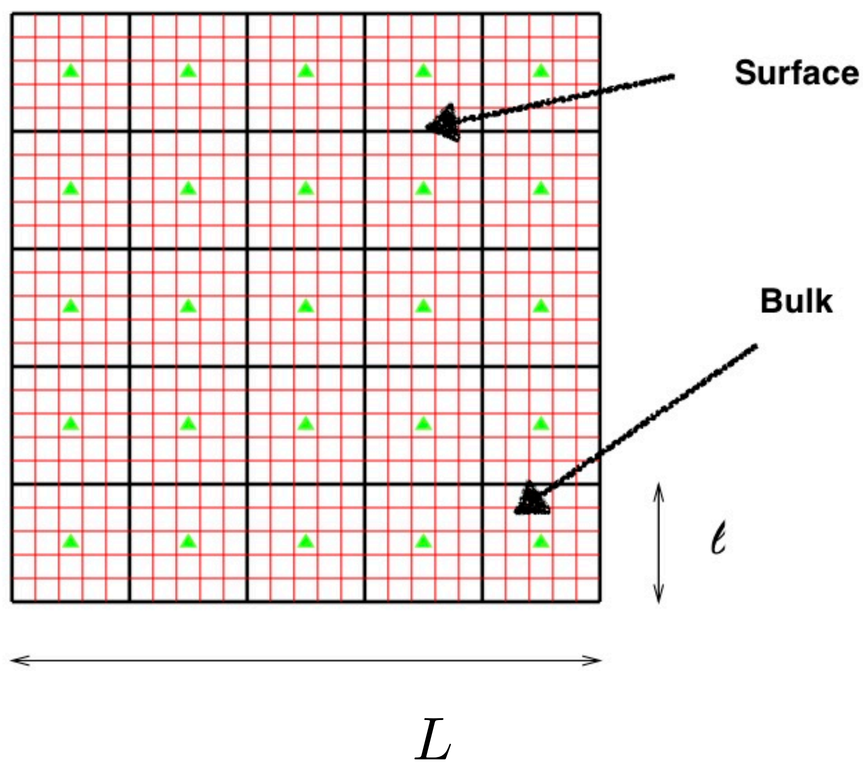
Therefore f_J is Gaussian distributed and its variance vanishes for $N \to \infty$

Moreover, $f_J^{\mathrm{typ}} = [f_J]$

Self-averageness

Systems with short-range interactions

Divide a, say, cubic system of volume $V = L^d$ in n sub-cubes, of volume $v = \ell^d$ with $V = nv$



Self-averageness

Systems with short-range interactions

For short-range interactions the total free-energy is the sum of two terms, a contribution from the bulk of the subsystems and a contribution from the interfaces between the subsystems :

$$-\beta F_J = \ln Z_J = \ln \sum_{\rm conf} e^{-\beta H_J({\rm conf})} \approx \ln \sum_{\rm conf} e^{-\beta H_J({\rm bulk})-\beta H_J({\rm surf})}$$
$$= \ln \sum_{\rm bulk} e^{-\beta H_J({\rm bulk})} + \ln \sum_{\rm surf} e^{-\beta H_J({\rm surf})} = -\beta F_J^{\rm bulk} - \beta F_J^{\rm surf}$$

where the $\approx$ indicates that we dropped the contributions of interactions between the bulk and the interfaces (surf)

Self-averageness

Systems with short-range interactions

If the interaction extends over a short distance l and the linear size of the boxes is $\ell \gg l$, we also assume that the surface energy is negligible with respect to the bulk one (same for possible entropic contributions) and

$$-\beta F_J \approx -\beta F_J^{\mathrm{bulk}} = \ln \sum_{\mathrm{bulk}} e^{-\beta H_J(\mathrm{bulk})}$$

The disorder dependent free-energy is a sum of $n = (L/\ell)^d$ independent random numbers, each one being the disorder dependent free-energy of the bulk of each subsystem :

$$-\beta F_J \approx \textstyle\sum_{k=1}^{n} \ln \sum_{\mathrm{bulk}_k} e^{-\beta H_J(\mathrm{bulk}_k)}$$

In the limit of a very large number of subsystems ($L \gg \ell$ or $n \gg 1$) the CLT $\Rightarrow$ the free-energy density is Gaussian distributed with

$$f_J^{\mathrm{typ}} = [\, f_J \,]$$

Self-averageness

Systems with short-range interactions

The dispersion about the typical value of the total free-energy vanishes in the large n limit, $\sigma_{F_J}/[\,F_J\,] \propto \sqrt{n}/n = n^{-1/2} \to 0$

The one of the free-energy density, or intensive free-energy, $f_J = F_J/N$, as well, $\sigma_{f_J}/[f_J] = O(n^{-1/2})$

In a sufficiently large system the typical free-energy density $f_J^{\rm typ}$ is then very close to the averaged $[\,f_J\,]$ and one can compute the latter to understand the static properties of typical systems.

Much easier to do analytically. More later.

Self-averageness

Failure and quenched vs. Annealed

Go back to the one dimensional disordered Ising chain and show that

the partition function and the spatial correlations

are not self-averaging.

The annealed free-energy is defined as $-\beta F^{\text{annealed}} = \ln[Z_J]$

The quenched free-energy is defined as $-\beta F^{\text{quenched}} = [\ln Z_J]$

Jenssen's inequality applied to the convex function $-\ln y$ implies

$$-\ln[Z_J] \leq -[\ln Z_J]$$

and for the free-energies one deduces

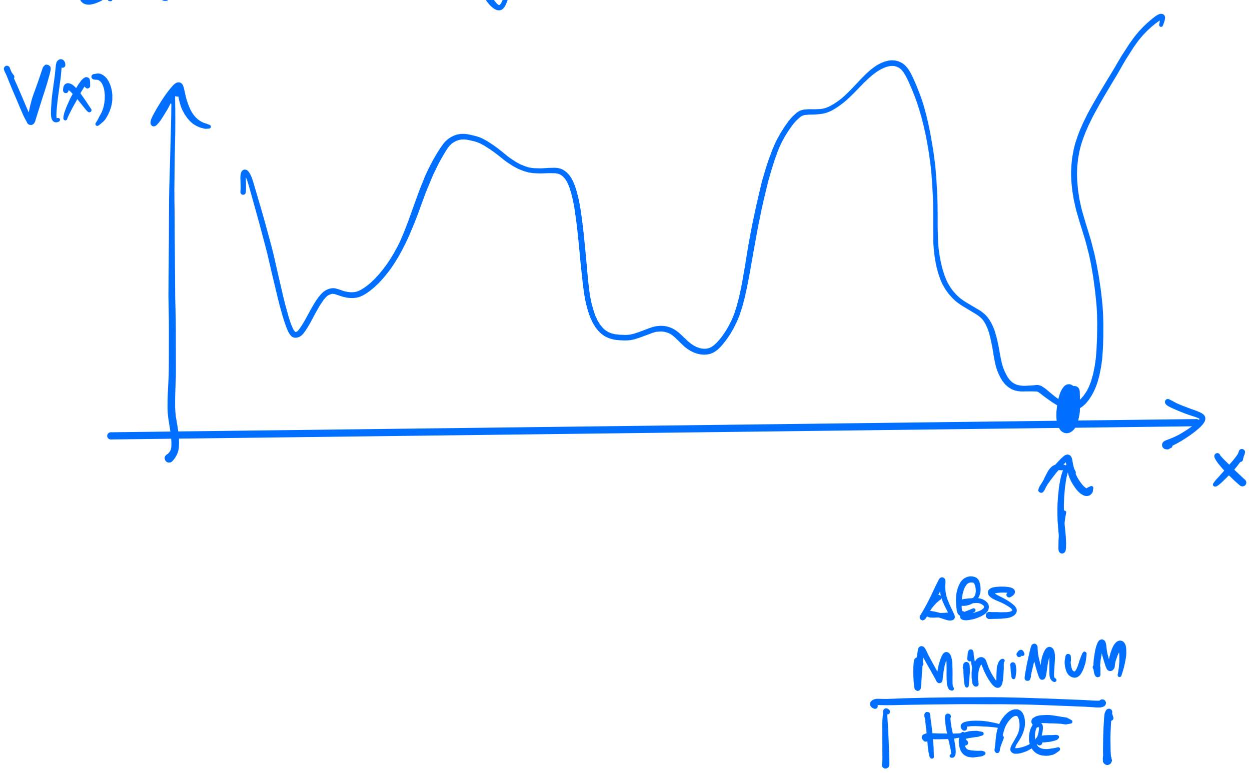
$$F^{\text{annealed}} = -\beta^{-1} \ln[Z_J] \leq -\beta^{-1}[\ln Z_J] = F^{\text{quenched}}$$

THE PLANTED ENSEMBLE

IMAGINE THAT YOU WANT A GIVEN CONFIGURATION TO BE AN EQUILIBRIUM ONE OF YOUR SYSTEM.

THEN, GENERATE THE DISORDER UNDER THIS CONDITION.

e.g. FOR A PARTICLE FOLLOWING LANGEVIN'S DYN



AND NOT ELSEWHERE.

THE REST OF THE POT MAY CHANGE BUT THE MINIMUM SHOULD BE THERE.

FOR A SPIN MODEL $\Rightarrow$

FIX J_{ij} SUCH THAT

$\{s_i^0\}$ IS AN EQUIL. STATE

USEFUL IN COMP. SCIENCE.

Methods

disordered systems

Statics

TAP Thouless-Anderson-Palmer Replica theory	fully-connected (complete graph) Gaussian approx. to field-theories
Cavity or Peierls approx.	dilute (random graph)
Bubbles & droplet arguments functional RG[1]	finite dimensions

Dynamics

Generating functional for classical field theories (MSRJD).

Schwinger-Keldysh closed-time path-integral for quantum dissipative models (the previous is recovered in the $\hbar \to 0$ limit).

Perturbation theory, renormalization group techniques, self-consistent approx.

Randomness

Properties

— Spatial inhomogeneity

Not all sites behave in the same way

— Frustration

Impossibility to satisfy all conditions imposed by the Hamiltonian

(spectrum pushed up, degeneracy of ground state)

— Annealed vs quenched

Couplings, fields, etc. fluctuate or are frozen

— Quenched disorder : static pdfs of couplings, fields, etc.

$f^{\text{annealed}} \leq f^{\text{quenched}}$

— Self-averageness

$\lim_{N\to\infty}[f^{\text{quenched}}] = \lim_{N\to\infty} f^{\text{typ}}$

— Complex free-energy landscapes

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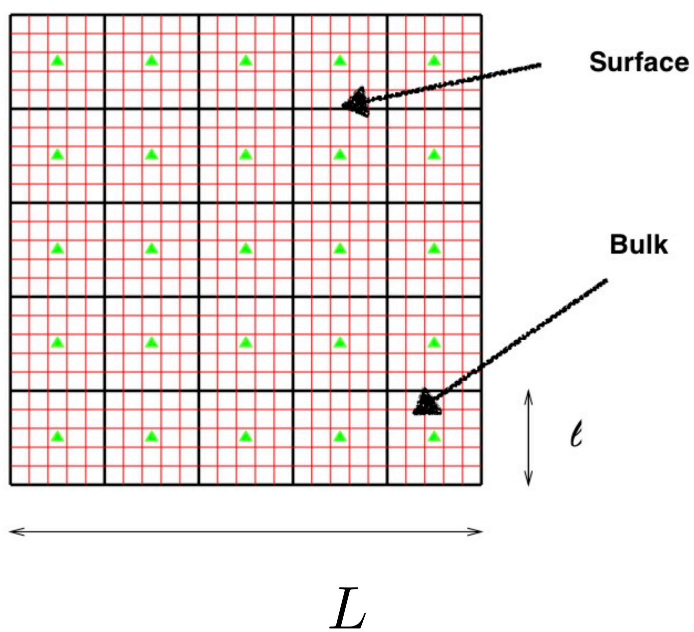
Divide a, say, cubic system of volume $V = L^d$ in n sub-cubes, of volume $v = \ell^d$ with $V = nv$



$$-\beta F_J \approx \sum_{k=1}^{L/\ell} \ln \sum_{\text{bulk}_k} e^{-\beta H_J(\text{bulk}_k)}$$

For $L \gg \ell$ the CLT

$\Rightarrow$ f_J is Gaussian distributed and

$$f_J^{\text{typ}} = [\, f_J \,]$$

THIS WAS THE END OF THE 2nd LECTURE

IN THE 3rd LECTURE WE WILL INTRODUCE & USE THE TAP APPROACH

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Low temperature phases

Phenomenology : homogeneity vs inhomogeneity

In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

Low temperature phases

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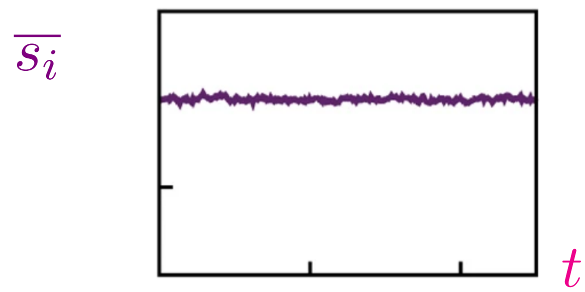
In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T) \; \forall i$ or $\langle s_i \rangle = -m(T) \; \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

If one were to follow the time evolution of each spin in one of the two equilibrium states at $T < T_c$, one would see $\overline{s_i}(t) = m(T) + \delta_i(t)$ with $\delta_i(t)$ small time-dependent fluctuation and the overline states for a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt' \, s_i(t')$



A SYSTEM WITH QUENCHED RANDOMNESS

IS A SYSTEM IN WHICH NOT ALL EXCHANGES ARE THE SAME

$$H_J[\{s_i\}] = -\sum_{ij} J_{ij}\, s_i s_j$$

$$H_J[\{s_i\}] = -\sum_{i_1<\dots<i_p} J_{i_1\dots i_p}\, s_{i_1}\cdots s_{i_p}$$

$$\left.\begin{array}{l} J\text{'s FROM A pdf} \\ \sum_{i\text{'s}} \text{ TO BE CHOSEN} \end{array}\right\} \text{DEF MODEL}$$

$$\text{VERY IMPORTANT} \left\{\begin{array}{l} \text{SUPPORT OF } P(J_{ij}) \\ \text{SCALING OF } J_{ij} \text{ WITH } N \end{array}\right.$$

Low temperature phases

Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each i, in each inhomogeneous and degenerate equilibrium state.

There may be many different ensembles $\{m_i(T)\}$ that are equilibrium states (degeneracy, similar to what we saw in the frustrated magnets for the ground states but here in the full low T phase)

There is also the up-down symmetry $\{m_i(T)\} \mapsto \{-m_i(T)\}$

Low temperature phases

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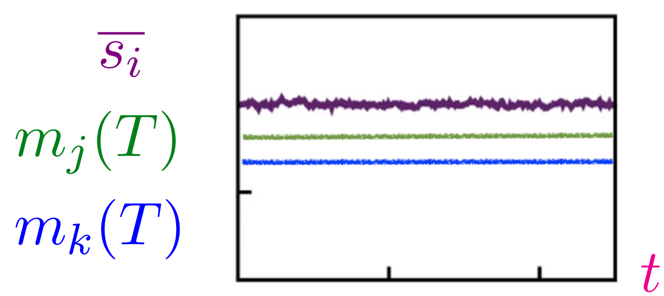

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Mean-field theory

Fully connected Ising models

General model

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

What is a local field ?

It is the field felt by a selected site

$$h_i = \frac{1}{2}\sum_{j(\neq i)} J_{ij} s_j$$

and we require it to be $\mathcal{O}(1)$

Mean-field theory

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$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

In the Curie-Weiss ferromagnetic case

$$J_{ij} = \frac{J}{N} \;\; \text{such that} \;\; h_i = \frac{J}{2N} \sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the two ferromagnetic $s_i = 1 \;\forall i$ or $s_i = -1 \;\forall i$ phases

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in the ferromagnetic $s_i = 1 \; \forall i$ or $s_i = -1 \; \forall i$ phases

In the Sherrington-Kirkpatrick disordered case

$$J_{ij} = \mathcal{O}(\frac{J}{\sqrt{N}}) \;\text{ such that }\; h_i \sim \frac{J}{2\sqrt{N}}\sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the PM or spin-glass phases $s_i = \pm 1 \; \forall i$

Mean-field theory

Fully connected Ising models

General model

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

In the Sherrington-Kirkpatrick disordered case

$$J_{ij} = \mathcal{O}(\frac{J}{\sqrt{N}}) \;\text{ such that }\; h_i \sim \frac{J}{2\sqrt{N}} \sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the PM or spin-glass phases, say, $s_i = \pm 1$ with equal probability

One can use a Gaussian pdf

$$P(J_{ij}) = (2\pi\sigma^2)^{-1/2} \exp[-J_{ij}^2/(2\sigma^2)] \;\text{ with }\; \sigma^2 = J^2/N$$

Mean-field theory

Fully connected Ising models

Even more general models (recall the K-sat problem)

$$H_J[\{s_i\}] = -\frac{1}{3!}\sum_{i\neq j\neq k} J_{ijk} s_i s_j s_k \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ijk}

In the $p=3$ Curie-Weiss ferromagnetic case

$$J_{ijk} = \frac{J}{N^{p-1}} \text{ such that } h_i \sim \frac{J}{\cancel{2}N^{p-1}} \sum_{jk(\neq i)} s_j s_k = \mathcal{O}(1)$$

in the two ferromagnetic $s_i = 1 \ \forall i$ or $s_i = -1 \ \forall i$ phases

In the $p=3$ disordered case

$$J_{ijk} = \mathcal{O}\left(\frac{J}{\sqrt{N^{p-1}}}\right) \text{ such that } h_i \sim \frac{J}{\cancel{2}\sqrt{N^{p-1}}} \sum_{j\neq k(\neq i)} s_j s_k = \mathcal{O}(1)$$

in the PM or spin-glass phases $s_i = \pm 1$ with equal probability

Randomness

Properties

— Spatial inhomogeneity

Not all sites behave in the same way, local order parameters $\{m_i\}$

— Frustration

Impossibility to satisfy all conditions imposed by the Hamiltonian

(spectrum pushed up, degeneracy of ground state)

— Annealed vs quenched

Couplings, fields, etc. fluctuate or are frozen

— Quenched disorder : static pdfs of couplings

Gaussian pdf of J_{ij} with $\sigma^2 = J^2/N$

— Self-averageness

$\lim_{N\to\infty}[f^{\text{quenched}}] = \lim_{N\to\infty} f^{\text{typ}}$

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$\lim_{N\to\infty}[f^{\rm quenched}] = \lim_{N\to\infty} f^{\rm typ}$

— Complex free-energy landscapes : beyond Ginzburg-Landau

Mean-field theory for PM-FM

Fully connected Curie-Weiss Ising model

Normalize J by the size of the system N to have $\mathcal{O}(1)$ local fields

$$\boxed{H = -\frac{J}{2N} \sum_{i \neq j} s_i s_j - h \sum_i s_i}$$

The partition function reads $\mathcal{Z} = \int_{-1}^{1} du \, e^{-\beta N \mathbf{f}(u)}$ with $Nu = \sum_i s_i$

$$\boxed{\mathbf{f}(u) = -\frac{J}{2} u^2 - hu + T \left[\frac{1+u}{2} \ln \frac{1+u}{2} + \frac{1-u}{2} \ln \frac{1-u}{2} \right]}$$

Energy terms and entropic contribution stemming from $\mathcal{N}(\{s_i\})$ yielding the same u value.

Use the **saddle-point**, $\lim_{N \to \infty} f_N(\beta J, \beta h) = \mathbf{f}(u_{sp})$, with

$$\boxed{u_{sp} = \tanh\left(\beta J u_{sp} + \beta h\right) = \langle u \rangle = m}$$

PROOF

MANY WAYS. SIMPLEST

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

FACTORIZED JOINT pdf

$$p_i(s_i) = \frac{1+m}{2}\,\delta_{s_i,1} + \frac{1-m}{2}\,\delta_{s_i,-1}$$

CHECK NORMALIZATION

$$\sum_{s_i=\pm 1} p_i(s_i) = \frac{1+m}{2} + \frac{1-m}{2} = 1$$

WHAT IS m ?

$$\sum_{\{s_i=\pm 1\}} s_i\, p_i(s_i) = \frac{1+m}{2} + \frac{1-m}{2}(-1) = m$$

$$m = \langle s_i \rangle$$

NB THIS IS FOR A UNIFORM $J_{ij} = J$ & m

AVERAGED THERMODYN FREE ENERGY

$$F(m) = U(m) - T\,S(m)$$

$$U(m) \equiv \langle H \rangle$$

AVER OVER $P(\{s_i\})$

$$U(m) = -\frac{1}{2} J \sum_{\substack{\langle ij \rangle \\ i \neq j}} \langle s_i s_j \rangle$$

$$= -\frac{J}{2} \sum_i \sum_{\partial_i} m^2$$

$$= -\frac{J}{2} z N m^2$$

$$S(m) \equiv -k_B \langle \ln P(\{s_i\}) \rangle$$

NB $P < 1 \Rightarrow \ln P < 0$

$\Rightarrow \quad S > 0 \quad$ ✓

$$S(m) = -k_B \sum_{\{s_i=\pm 1\}} \prod_k p_k(s_k) \ln \prod_j p_j(s_j)$$

$$= -k_B \sum_{\{s_i=\pm 1\}} \prod_k p_k(s_k) \underbrace{\sum_j \ln p_j(s_j)}_{\text{LOOK AT ONE TERM}}$$

$$\sum_{\{s_1=\pm 1\}} \cdots \sum_{\{s_j=\pm 1\}} \cdots \sum_{\{s_N=\pm 1\}} p_1(s_1) \cdots p_j(s_j) \cdots p_N(s_N) \ln p_j(s_j)$$

ALL SUMS $= 1$ APART FROM THE j's ONE

$$= -\left[p_j(1) \ln p_j(1) + p_j(-1) \ln p_j(-1)\right]$$

$$= -\left[\frac{1+m}{2} \ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

AND SIMILARLY FOR ALL SITES, THEN

$$S(m) = -k_B N \left[\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

THUS $f(m) = F(m)/N$ IS

$$f(m) = -\frac{J}{2} z m^2$$

$$+ k_B T \left[\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

WE NOTE THAT IF

$$\left. \begin{array}{l} z = N-1 \\ J \to J/N \end{array} \right\} \text{FULLY CONN. CASE}$$

FIRST TERM IS $\mathcal{O}(1)$

$$\frac{\partial f(m)}{\partial m} = 0 = -Jzm + \frac{k_B T}{2} \ln\left(\frac{1+m}{1-m}\right)$$

$$\Rightarrow$$

$$e^{\frac{2Jzm}{k_B T}} = \frac{1+m}{1-m}$$

$$\Rightarrow (1-m)\, e^{2\beta Jzm} = 1-m$$

$$\Rightarrow \frac{e^{2\beta Jzm}-1}{e^{2\beta Jzm}+1} = m$$

$m = \mathrm{th}(\beta Jzm)$ THE MF EQ.

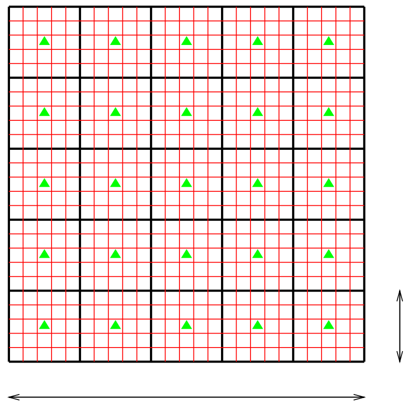
Ginzburg-Landau for PM-FM

Continuous scalar statistical field theory with local aspects

Coarse-grain the spin

$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$

Set $h = 0$



The partition function is $\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\beta V \mathbf{f}(\phi)}$ with V the volume and

$$\boxed{\mathbf{f}(\phi) = \int d^d r \ \left\{ \frac{1}{2}[\nabla\phi(\vec{r})]^2 + \frac{T-J}{2}\phi^2(\vec{r}) + \frac{\lambda}{4}\phi^4(\vec{r}) \right\}}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments.

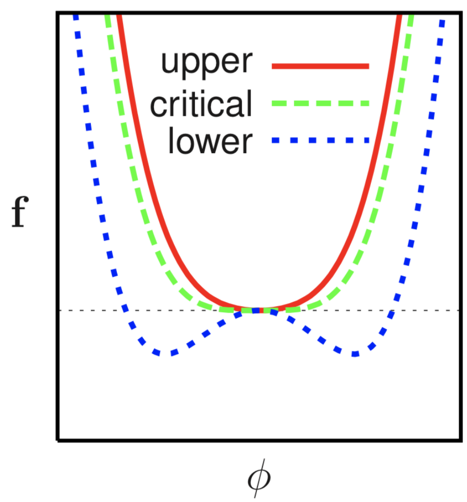
Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = m$

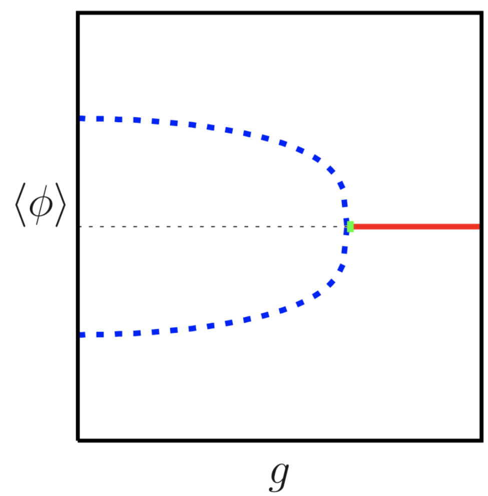
The free-energy density is $\lim_{V\to\infty} f_V(\beta, J) = \mathbf{f}(\phi_{sp})$

2nd order phase-transition

Continuous scalar statistical field theory

bi-valued equilibrium states related by symmetry





Ginzburg-Landau free-energy

Scalar order parameter

$g = \beta J$

Features

- **Spontaneous symmetry breaking** below T_c
- Two equilibrium states related by **symmetry** $\phi \to -\phi$
- The state is chosen by a **pinning field**
- If the partition sum is performed over the whole phase space $\langle \phi \rangle = 0$ (a consequence of the symmetry of the action)
- **Restricted statistical averages**, running over *half* phase space, yield $\langle \phi \rangle \neq 0$
- Under a magnetic field the free-energy landscape is tilted and one of the minima becomes a **metastable state**
- The barrier in the **free-energy landscape** between the two states diverges with the size of the system implying **ergodicity breaking**

Features

- The function(al)s $\mathbf{f}(u)$ ($\mathbf{f}(\phi(\vec{r}))$) are **large deviation function(al)s** determining the probability of finding an equilibrium system with u or $\phi(\vec{r})$
- The system spends $t_\pm \simeq e^{N\tau}$ close to each minima and it makes rapid transitions between the two

These results were not fully accepted as realistic at the time

Recall. the discussion on phase transitions & ergodicity breaking

- With $p > 2$-uplet interactions one finds **first order phase transitions** (relevant for glasses & K-sat like problems)

MFT for disordered spin models

Fully connected SG : Sherrington-Kirkpatrick model

$k_B = 1$

$$\boxed{H = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j - \sum_i h_i s_i}$$

with J_{ij} i.i.d. Gaussian variables, $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2/N = \mathcal{O}(1/N)$.

One finds the naive free-energy landscape

$$\boxed{N\mathbf{f}(\{m_i\}) = -\frac{1}{2} \sum_{i \neq j} J_{ij} m_i m_j + T \sum_{i=1}^{N} \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2}}$$

and the (naive) TAP equations

$$\boxed{m_{i\,sp} = \tanh(\beta \sum_{j(\neq i)} J_{ij} m_{j\,sp} + \beta h_i)}$$

that determine the restricted averages $m_i = \langle s_i \rangle = m_{i\,sp}$.

MFT for disordered spin models

Fully connected SG : A simple proof

$k_B = 1$

The more traditional one assumes independence of the spins,

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

with $p_i(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1} = \frac{1+s_i m_i}{2}$

and uses this form to express $\langle H\rangle - T\langle S\rangle$ with $S = \ln \mathcal{N}(\{s_i\})$

The energetic contribution is straightforward to evaluate

The entropic contribution is the one we already computed for the Curie-Weiss model, taking care of keeping the indices i

A more powerful proof expresses $\mathbf{f}$ as the Legendre transform of $-\beta F(h_i)$ with $m_i = N^{-1}\partial[-\beta F(h_i)]/\partial h_i$ and takes care of a "problem" to be solved in the next slides

Georges & Yedidia 91

I WILL START THE 4th LECTURE, AFTER JORGE'S 2nd AND 3rd LECTURES, HERE

- YOU'VE SEEN THE FULLY CONNECTED FM MODEL

  $\longrightarrow$ IDEA OF A STATE

- A BIT ON MANY STATES FOR MF DISORDERED SYST WITH TAP

LET'S DERIVE THE TAP EQS.

EXERCISE ON THE BLACKBOARD

PROVE NAIVE TAP EQS. USING THE FACTORIZED pdf

$$\mathcal{P}^{\alpha}(\{s_i\}) = \prod_i p_i^{\alpha}(s_i)$$

GENERALIZING THE ONE FOR THE FM-MODEL

$$p_i^{\alpha}(s_i) = \frac{1+m_i^{\alpha}}{2}\,\delta_{s_i,1} + \frac{1-m_i^{\alpha}}{2}\,\delta_{s_i,-1}$$

$$= \frac{1+m_i^{\alpha} s_i}{2}$$

WHAT IS α? IT LABELS "PURE STATES"

$$\mathcal{P}(\{s_i\}) = \frac{e^{-\beta H}}{Z}$$

$$= \sum_{\alpha} \underbrace{w_{\alpha}}_{\text{WEIGHT OF PURE STATE } \alpha} \mathcal{P}^{\alpha}(\{s_i\})$$

cfr FM $T<T_c$ $\alpha = 1,2$ $w_{\alpha} = 1/2$

THE VALUES TAKEN BY m_i^α DIFFER IN
$\neq$ PURE STATES

e.g. IN FM $T<T_c$ $\quad m_i^\alpha = |m| \quad$ FOR $\alpha=1$

$$m_i^\alpha = -|m| \quad \text{FOR } \alpha=2$$

IN A DISORDERED CASE $\{m_i^\alpha\}$ ALL $\neq 0$
AND

$$\frac{1}{N}\sum_{i=1}^{N} m_i^\alpha \xrightarrow[N\to\infty]{} 0 \qquad \forall\alpha \quad T<T_c$$

IN p. 81 AND THE FOLLOWING SOLUTION
FOR FM MODEL
$\Longrightarrow$ EXTEND TO DISORDERED ONE.

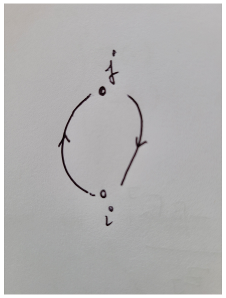
MFT for disordered spin models

Missing : the Onsager reaction term

These equations are not completely correct.

The *Onsager reaction term* is missing.

This term represents the reaction of the spin i to itself



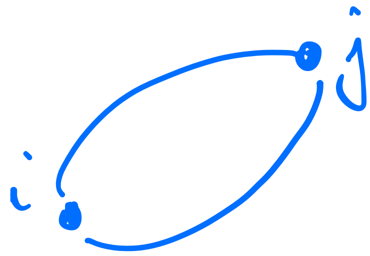
The magnetisation in i produces a field $h'_{j(i)} = J_{ji} m_i = J_{ij} m_i$ on spin j

This field induces a magnetisation $m'_{j(i)} = \chi_{jj} h'_{j(i)} = \chi_{jj} J_{ij} m_i$ on the spin j.

This magnetisation produces a field $h'_{i(j)} = J_{ij} m'_{j(i)} = J_{ij} \chi_{jj} J_{ij} m_i$ on site i.

The equilibrium fluctuation-dissipation relation between susceptibilities and connected correlations implies $\chi_{jj} = \beta \langle (s_j - \langle s_j \rangle)^2 \rangle = \beta(1 - m_j^2)$ and one then has $h'_{i(j)} = \beta(1 - m_j^2) J_{ij}^2 m_i$

PROOF



$$h'_{j(i)} = J_{ji}\, m_i = J_{ij}\, m_i$$

$$m'_j = \chi_{jj}\, h'_{j(i)} = \chi_{jj}\, J_{ij}\, m_i$$

$$= \beta\,(1 - m_j^2)\, J_{ij}\, m_i$$ SEE FDT PROOF BELOW

$$h'_{i(j)} = J_{ij}\, m'_j$$
$$= J_{ij}^2\, \beta\,(1 - m_j^2)\, m_i$$

AND WE WANT TO SUBTRACT IT

EQUILIBRIUM FDT

DO IT AS
AN EXCERCISE
TOO

ADD AN INFINITESIMAL FIELD

$$H(\{s_i\}) \longrightarrow H(\{s_i\}) - \sum_k h_k s_k$$

$$= H_h(\{s_i\})$$

AND SEE HOW THE MAGN OF A GIVEN SPIN s_j IS MODIFIED BY h_j

$$\chi_{jj} = \left.\frac{\partial \langle s_j \rangle_h}{\partial h_j}\right|_{\vec{h}=0}$$

IN THE LIMIT $\vec{h} \to 0$

THE LINEAR SUSCEPTIBILITY

$$\chi_{ij} = \frac{\partial \langle s_i \rangle}{\partial h_j}\Bigg|_{h_j=0}$$

$$= \frac{\partial}{\partial h_j}\left\{ \frac{1}{Z(h)} \sum_{\{s_i\}} s_j \, e^{-\beta H_h(\{s_i\})} \right\}\Bigg|_{h\to 0}$$

$$= -\frac{1}{Z^2(h)} \frac{\partial Z(h)}{\partial h_j} \sum_{\{s_i\}} s_j \, e^{-\beta H_h(\{s_i\})}\Bigg|_{h\to 0}$$

$$+ \frac{1}{Z(h)} \sum_{\{s_j\}} s_j (-\beta) \frac{\partial H_h(\{s_i\})}{\partial h_j}\Bigg|_{h\to 0}$$

WE HAVE TO COMPUTE

$$\frac{\partial Z(h)}{\partial h_j} = \frac{\partial}{\partial h_j} \sum_{\{s_i\}} e^{-\beta H_h(\{s_i\})}$$

$$-\beta H_h(\{s_i\})$$

$$= \sum_{\{s_i\}} -\beta \frac{\partial H_h(\{s_i\})}{\partial h_j} e^{-\beta H_h(\{s_i\})}$$

AND HERE WE ALSO NEED

$$\frac{\partial H_h(\{s_i\})}{\partial h_j} = -s_j$$

PUTTING ALL TOGETHER

$$\chi_{jj} = \frac{-1}{Z^2(h)} \sum_{\{s_i\}} \beta s_j e^{-\beta H_h(\{s_i\})} \Big|_{h\to 0}$$

$$\times \sum_{\{s_i\}} s_j e^{-\beta H_h(\{s_i\})} \Big|_{h\to 0}$$

$$+ \frac{(-\beta)}{Z(h)} \sum_{\{s_i\}} s_j (-s_j) e^{-\beta H_h(\{s_i\})} \Big|_{h\to 0}$$

$$\chi_{jj} = -\beta \langle s_j \rangle \langle s_j \rangle + \beta \langle s_j^2 \rangle$$

$$\chi_{jj} = \beta \left(\langle s_j^2 \rangle - \langle s_j \rangle \langle s_j \rangle \right)$$

FOR ISING SPINS $s_j^2 = 1$

$$\boxed{\chi_{jj} = \beta \left(1 - \langle s_j \rangle^2 \right)}$$

FDT
EQUILIBRIUM
MODEL IND.

WHICH is $\beta (1 - m_j^2)$

MFT for disordered spin models

The Onsager reaction term

The idea of Onsager – or *cavity method* – is that one has to study the ordering of the spin i in the absence of its own effect on the rest of the system.

The total field produced by the sum of $h'_{i(j)} = \beta(1 - m_j^2) J_{ij}^2 m_i$ over all the spins j with which it can connect, has to be subtracted from the mean-field created by the other spins in the sample, *i.e. the total local field* should be

$$h_i^{\rm loc} = \sum_{j(\neq i)} J_{ij} m_j - \beta m_i \sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2)$$

recall that $J_{ij} = \mathcal{O}(1/\sqrt{N})$. Finally, the TAP equations read

$$m_i = \tanh \left\{ \sum_{j(\neq i)} \left[\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2) \right] \right\}$$

MFT for disordered spin models

Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

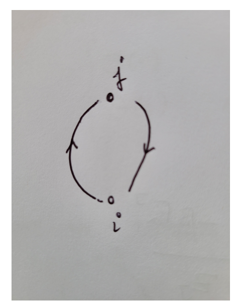
$$m_i = \tanh \left\{ \sum_{j(\neq i)} \left[\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2) \right] \right\}$$

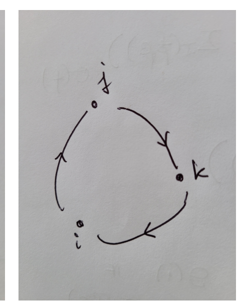
The first term in the rhs $\sum_{j(\neq i)} J_{ij} m_j \simeq \frac{1}{\sqrt{N}} \sqrt{N} = \mathcal{O}(1)$ because of the central limit theorem.

The second term $\sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2) \simeq \frac{1}{N} N = \mathcal{O}(1)$ because all terms in the sum are positive definite ($m_j \leq 1 \; \forall j$)

Exercise

Check that in the Curie Weiss model $J_{ij} = J/N$ there is no need of Onsager terms





MFT for disordered spin models

Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

$$m_i = \tanh \left\{ \sum_{j(\neq i)} [\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2)] \right\}$$

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Recall that $m_i = \langle s_i \rangle$

MFT for disordered spin models

Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

$$m_i = \tanh \left\{ \sum_{j(\neq i)} \left[\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2) \right] \right\}$$

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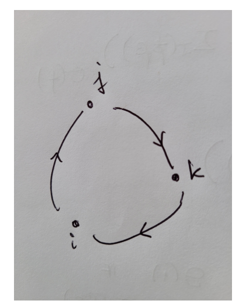
Exercise

Check that higher order loops are negligible,

since sub-leading in powers of N



EXERCISE FIELD ON j CREATED BY m_i

$$h'_{j(i)} = J_{ji}\, m_i = J_{ij}\, m_i$$

MAGN. INDUCED

$$m'_j = \chi_{jj}\, h'_{j(i)} = \beta(1-m_j^2)\, J_{ij}\, m_i$$

THIS CREATES A FIELD ON k

$$h'_{k(j)} = J_{kj}\, m'_j = J_{kj}\, J_{ij}\, m_i\, \beta(1-m_j^2)$$

$$= J_{ij}\, J_{jk}\, m_i\, \beta(1-m_j^2)$$

AND THIS A MAGN ON k

$$m'_k = \chi_{kk}\, h'_{k(j)}$$

$$= \beta(1-m_k^2)\, J_{ij}\, J_{jk}\, m_i\, \beta(1-m_j^2)$$

$$= \beta^2 (1-m_k^2)(1-m_j^2)\, J_{ij} J_{jk}\, m_i$$

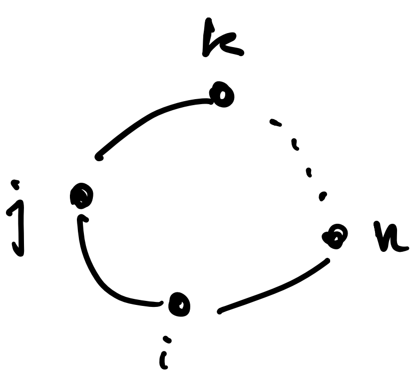
WHICH, BACK ON i

$$h'_{i(k)} = J_{ik}\, m'_k$$

$$= J_{ij} J_{jk} J_{ki}\, \beta^2 (1-m_j^2)(1-m_k^2)\, m_i$$

GENERIC LOOP

$$h'_i = \underbrace{J_{ij} J_{jk} \cdots J_{ni}}_{n \text{ FACTORS}}\, \beta^{n-1} \underbrace{(1-m_j^2)\cdots(1-m_n^2)}_{n-1 \text{ FACTORS}}\, m_i$$



ORDER WITH N POWERS FOR SK MODEL

CLAIM

$$J_{ij} = \Theta\left(1/\sqrt{N}\right) \implies$$

$$J_{ij} J_{jk} \cdots J_{ni} = \Theta\left(N^{-n/2}\right)$$

ALL OTHER FACTORS ARE $\Theta(1)$ AND POSITIVE

$$h'_i = \Theta\left(N^{-n/2}\right)$$

THERE WILL BE SUMS OVER INTERNAL INDICES TO CONSIDER ALL LOOPS OF LENGTH n

$$h'_i = \Theta\left(N^{-n/2}\right) \times \underbrace{\sum_j \sum_k \cdots \sum_n}_{n-1 \text{ SUMS}} (\pm 1)\, m_i$$

$$h_i' = \Theta(N^{-n/2}) \underbrace{\Theta(N^{1/2}) \ldots \Theta(N^{1/2})} m_i$$

EACH SUM OF ± 1
IS $\Theta(N^{1/2})$
& THERE ARE $n-1$
OF THEM

$$h_i' = \Theta\left(N^{-n/2} \quad N^{(n-1)/2}\right) m_i$$

$$\boxed{h_i' = \Theta(N^{-1/2})}$$

IT'S ONLY THE "SELF" LOOPS
THAT CONTRIB $\Theta(1)$ IN
SK MODEL

THE HOPFIELD MODEL

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \, s_i s_j$$

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{N_p} \xi_i^{\mu} \xi_j^{\mu}$$ HEBB RULE

$$P(\{\xi_i^{\mu}\})$$ GAUSSIAN $$\frac{e^{-\frac{(\xi_i^{\mu})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

OR BINARY $\xi_i^{\mu} = \pm 1$ $p = 1/2$ EASIER

$$N \to \infty$$
$$N_p \to \infty$$
$$\alpha = N_p / N$$ FINITE

$$[J_{ij}] = \frac{1}{N} \sum_{\mu=1}^{N_p} [\xi_i^{\mu} \xi_j^{\mu}] = 0$$

$$[J_{ij}^2] = \frac{1}{N^2}\sum_{\mu\nu}\left[\xi_i^\mu \xi_j^\mu \xi_i^\nu \xi_j^\nu\right] \qquad i \neq j$$

$$= \frac{1}{N^2}\sum_{\mu\nu}\delta^{\mu\nu}\left[(\xi_i^\mu)^2\right]\delta^{\mu\nu}\left[(\xi_j^\mu)^2\right]$$

$$= \frac{1}{N^2}\sum_{\mu} 1 = \frac{Np}{N^2} = \frac{\alpha N}{N^2} = \frac{\alpha}{N}$$

NOTE SIMILAR SCALING TO THE SK MODEL

$$\boxed{[J_{ij}^2] = \frac{\alpha}{N}}$$

MOREOVER, THERE ARE CORRELATIONS BETWEEN NESTED THREE J_{ij}'s

$$[J_{ij} J_{jk} J_{ki}] =$$

$$= \frac{1}{N^3}\sum_{\mu\nu\rho}\left[\xi_i^\mu \xi_j^\mu \xi_j^\nu \xi_k^\nu \xi_k^\rho \xi_i^\rho\right]$$

$i \neq j$ $\quad$ $j \neq k$ $\quad$ $i \neq k$

$$= \frac{1}{N^3} \sum_{\mu\nu\rho} \delta^{\mu\nu} \delta^{\nu\rho} \delta^{\mu\rho}$$

$$= \frac{1}{N^3} \sum_{\mu\nu} \delta^{\mu\nu} \delta^{\nu\mu}$$

$$= \frac{1}{N^3} \sum_{\mu} 1 = \frac{N_p}{N^3} = \frac{\alpha N}{N^3} = \frac{\alpha}{N^2}$$

$$\boxed{[J_{ij} J_{jk} J_{ki}] = \frac{\alpha}{N^2}}$$

NB NO SUM OVER jk HERE

WHICH WILL BECOME IMPORTANT SINCE THERE'S NO FREE sgn LEFT

NB FOR SK EACH $J_{ij} \propto \Theta(N^{-1/2})$ THE PRODUCT OF THREE

$$J_{ij} J_{jk} J_{ki} = \Theta(N^{-3/2})$$

BUT THEY ARE INDEPENDENT $\Rightarrow$ TWO sgns

TAP EQS FOR HOPFIELD MODEL

$$m_i = \tanh\left(\beta \sum_{j(\neq i)} J_{ij} m_j - \beta h_i'\right)$$

$$= \tanh\Big[\beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 \sum_{j(\neq i)} J_{ij}^2 (1-m_j^2)\, m_i$$

$$- \beta \sum_{\substack{j(\neq i,k) \\ k(\neq i,j)}} J_{ij} J_{jk} J_{ki}\, \beta^2 (1-m_j^2)(1-m_k^2)\, m_i$$

$$- \quad \cdots \Big]$$

$\downarrow$ all h.o.t.

APPROX FOR $N \to \infty$

REPLACE J's WITH AVERAGES

KIND OF SELF-AVERAGING HYPOTHESIS

1st ONSAGER TERM

$$\sum_{j(\neq i)} J_{ij}^{2} (1-m_j^2) \approx \frac{\alpha}{N}\sum_{j(\neq i)} (1-m_j^2) = \alpha(1-q)$$

SQUARED

$$Nq = \sum_{j(\neq i)} m_j^2$$

WE REPLACED HERE J_{ij}^2 BY $[J_{ij}^2]$

FOR SK CAN DO THE SAME (LIKE $\alpha=1$)
SINCE J_{ij}^2 SQUARED

2nd ONSAGER TERM

$$\sum_{j(\neq i,k)} \sum_{k\neq(i,j)} J_{ij} J_{jk} J_{ki} (1-m_j^2)(1-m_k^2)$$

$$\approx \underbrace{\sum_{j(\neq i,k)} \sum_{k(\neq i,j)}} \frac{\alpha}{N^2} \underbrace{(1-m_j^2)(1-m_k^2)} \to N^2(1-q)^2$$

$$\approx \quad \alpha(1-q)^2 \Longrightarrow \text{FINITE CONTRIBUTION}$$

EACH CORRECTION TERM $\Longrightarrow$ SERIES
is FINITE

$$-\beta\alpha \sum_{n=1}^{\infty} \left[\beta(1-q)\right]^n$$

RECALL GEOMETRIC SERIES

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

$$\Longrightarrow \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1 = \frac{1-(1-x)}{1-x}$$

$$= \frac{x}{1-x}$$

THUS

$$-\beta\alpha \sum_{n=1}^{\infty} \left(\beta(1-q)\right)^n = -\beta\alpha \cdot \frac{\beta(1-q)}{1-\beta(1-q)}$$

FINALLY

$$\boxed{m_i = \mathrm{th}\left[\beta \sum_{j(\neq i)} J_{ij}\, m_j - \frac{\alpha \beta^2 (1-q)}{1-\beta(1-q)}\, m_i\right]}$$

TAP EQS. FOR HOPFIELD MODEL

COMMENTS

- CAVITY METHOD
  e.g. MÉZARD- PARISI 80's–90's
  SHAMIR & SOMPOLINSKY 2000

- LEGENDRE TRANSFORMS
  GEORGES YEDIDIA '91
  BASIS FOR DMFT (QUANTUM)
  BIROLI – LFC EARLY '00

ALLOW ONE TO PROVE THAT THERE ARE NO OTHER CORRECTIONS

SUMMARY OF 3rd LECTURE

- WORKED AT FIXED REALIZATION $\{J_{ij}\}$
- DERIVED $f(\{m_i\}) = U(\{m_i\}) - k_B\, s(\{m_i\})$

$$U = \langle H \rangle \qquad s = -k_B \langle \ln P \rangle$$

$$P(\{s_i\}) = \prod_k P_k(s_k)$$

$$P_k(s_k) = \frac{1+m_k}{2}\,\delta_{s_k,1} + \frac{1-m_k}{2}\,\delta_{s_k,-1}$$

- $\dfrac{\partial f(\{m_i\})}{\partial m_k} = 0 \Rightarrow m_i = \tanh\left(h_i^{loc} m_i\right)$

$$h_i^{loc} = \sum_{j(\neq i)} J_{ij} m_j + h_i^{ext}$$

SK

- ONSAGER REACTION TERM

$$h_i^{loc} = h_i^{loc} - \beta \sum_{j(\neq i)} J_{ij}^2 (1-m_j^2)$$

SK

HOW TO SOLVE THESE EQS IN PRACTICE?

ITERATION.

cfr CURIE-WEISS EQ.

SOLUTION OF TAP EQS

ITERATIONS CURIE-WEISS
BOLTHAUSEN

ROLE OF DELAY

BOLTHAUSEN'S ARGUMENT

ALTMEIER & MOORE

STILL WORK TO BE DONE

BLUME-CAPEL $\qquad s_i = 1, 0, -1$

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

$$= \prod_i \left(a_i \, \delta_{s_i,1} + b_i \, \delta_{s_i,-1} + c_i \, \delta_{s_i,0} \right)$$

NORM $\qquad 1 = \sum_{s_i=1,0,-1} p_i(s_i) = a_i + b_i + c_i \qquad (1)$

$$\langle s_i \rangle = m_i = \sum_{s_i=1,0,-1} s_i \, p_i(s_i) = a_i - b_i \qquad (2)$$

$$\langle s_i^2 \rangle = q_i = \sum_{s_i=1,0,-1} s_i^2 \, p_i(s_i) = a_i + b_i \qquad (3)$$

$(2)+(3) \Rightarrow \quad 2a_i = m_i + q_i \quad \Rightarrow \quad a_i = \dfrac{m_i + q_i}{2}$

$-(2)+(3) \Rightarrow \quad 2b_i = q_i - m_i \quad\quad b_i = \dfrac{q_i - m_i}{2}$

$(1) \Rightarrow \quad c_i = 1 - a_i - b_i = 1 - \frac{1}{2}(\cancel{m_i} + q_i) - \frac{1}{2}(q_i - \cancel{m_i})$

$$c_i = 1 - q_i$$

$$p_i(s_i) = \frac{q_i + m_i}{2} \, \delta_{s_i,1} + \frac{q_i - m_i}{2} \, \delta_{s_i,-1} + (1 - q_i) \, \delta_{s_i,0}$$

$$F(\{m_i, q_i\}) = U(\{m_i, q_i\}) - T\, S(\{m_i\})$$

$$U(\{m_i, q_i\}) = \langle H(\{m_i, q_i\}) \rangle_P$$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} \langle s_i \rangle \langle s_j \rangle - \Delta \sum_i \langle s_i^2 \rangle$$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} m_i m_j - \Delta \sum_i q_i$$

$$S(\{m_i, q_i\}) = -k_B \langle P(\{s_i\}) \rangle_P$$

$$= -k_B \sum_{\{s_i\}} P(\{s_i\}) \ln P(\{s_i\})$$

$$= -k_B \sum_{\{s_i\}} \prod_\ell P_\ell(s_\ell) \sum_k \ln P_k(s_k)$$

$$= -k_B \sum_k \sum_{s_k = 1, 0, -1} P_k(s_k) \ln P_k(s_k)$$

$$= -k_B \sum_k \Big[\frac{q_k + m_k}{2} \ln \frac{q_k + m_k}{2} +$$

$$+ \frac{q_k - m_k}{2} \ln \frac{q_k - m_k}{2} +$$

$$+ (1-q_k) \ln (1-q_k) \Big]$$

PUTTING TOGETHER THESE RESULTS

$$F(\{m_i, q_i\}) = -\frac{J}{2} \sum_{\langle ij \rangle} m_i m_j - \Delta \sum_i q_i$$

$$+ k_B T \sum_i \Big[\frac{q_i+m_i}{2} \ln \frac{q_i+m_i}{2} + \frac{q_i-m_i}{2} \ln \frac{q_i-m_i}{2}$$

$$+ (1-q_i) \ln (1-q_i) \Big]$$

CHECK YOKOTA

$$\frac{1}{2} \Big\{ (q_i+m_i) \ln (q_i+m_i) + (q_i-m_i) \ln (q_i-m_i)$$

$$+ 2 (1-q_i) \ln \big[2 (1-q_i) \big] - 2 \ln 2 \Big\}$$

$$= \frac{q_i+m_i}{2} \ln \left(\frac{q_i+m_i}{2} \right) + \cancel{\frac{q_i+m_i}{2}} \ln 2 +$$

$$+ \frac{q_i-m_i}{2} \ln \left(\frac{q_i-m_i}{2} \right) + \cancel{\frac{q_i-m_i}{2}} \ln 2 +$$

$$+ \left(\cancel{1}-\cancel{q_i}\right) \ln 2 + (1-q_i) \ln (1-q_i) - \cancel{\ln 2}$$

OK, IT'S THE SAME.

FDT

$$\left.\frac{\partial m_i^{(h)}}{\partial h_j}\right|_{h\to 0} = \beta \left\langle \left(s_i - \langle s_i \rangle\right)\left(s_j - \langle s_j \rangle\right)\right\rangle$$

$$\frac{\partial m_i^h}{\partial h_i} = \beta \left(\langle s_i^2 \rangle - \langle s_i \rangle^2\right) \qquad i=j$$

$$= \beta \left(q_i - m_i^2\right)$$

EXERCISE FIELD ON j CREATED BY m_i

$$h'_{j(i)} = J_{ji}\, m_i = J_{ij}\, m_i$$

MAGN. INDUCED $\qquad q_j = \langle s_j^2 \rangle$

$$m'_j = \chi_{jj}\, h'_{j(i)} = \beta\,(q_j - m_j^2)\, J_{ij}\, m_i$$

THIS CREATES A FIELD ON k

$$h'_{k(j)} = J_{kj}\, m'_j = J_{kj}\, J_{ij}\, m_i\, \beta\,(q_j - m_j^2)$$

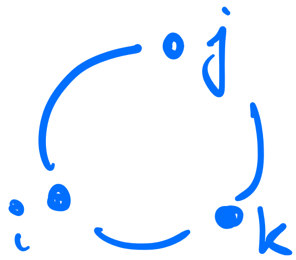
$$= J_{ij}\, J_{jk}\, m_i\, \beta\,(q_j - m_j^2)$$

AND THIS A MAGN ON k

$$m'_k = \chi_{kk}\, h'_{k(j)}$$

$$= \beta\,(q_k - m_k^2)\, J_{ij}\, J_{jk}\, m_i\, \beta\,(q_j - m_j^2)$$

$$m_k' = \beta^2 m_i J_{ij} J_{jk} (q_k - m_k^2)(q_j - m_j^2)$$



<u>EXERCISE</u> BLUME-CAPEL

EXPLAIN

<u>EXERCISE</u> HOPFIELD MODEL

EXPLAIN

<u>CAVITY METHOD</u>

NEXT

TAP EQS $\Longrightarrow$ T_c FOR SK

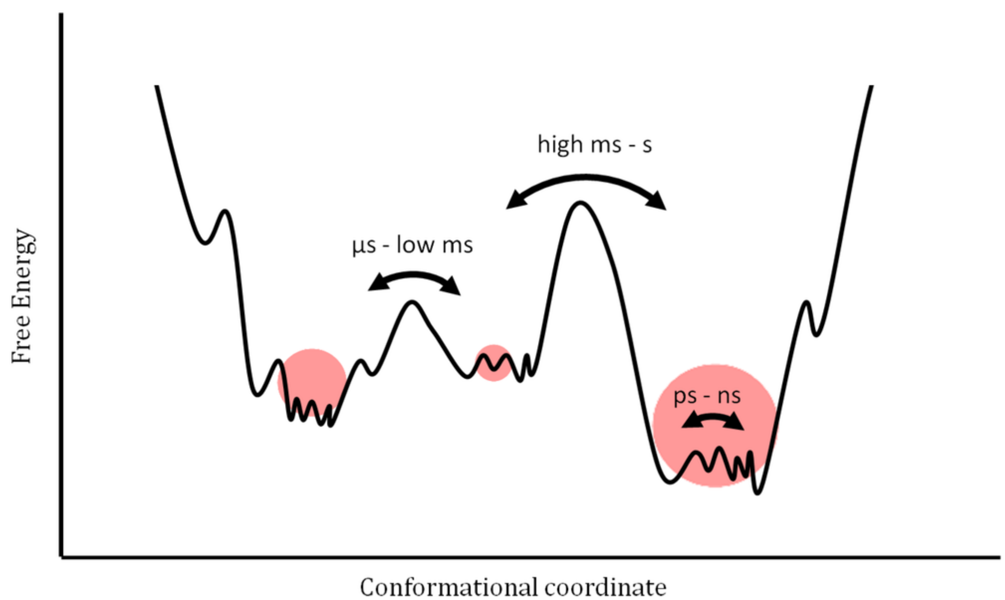
Landscape

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$F_J^{\rm tap}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2)$$

$$+ T\sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right]$$



At low temperatures $\{m_i\}$

Summary

Local & global order parameters

$$m_i \equiv \langle s_i \rangle$$

$= 0$ at $T \geq T_c$

$\neq 0$ at $T < T_c$

Magnetization

$m = \frac{1}{N} \sum_i m_i = 0$ at all temperatures

Edwards-Anderson order parameter

$$q_{\mathrm{EA}}^{\alpha} \equiv \frac{1}{N} \sum_i (m_i^{\alpha})^2 = \frac{1}{N} \sum_i \langle s_i \rangle_{\alpha}^2$$

$= 0$ at $T \geq T_c$

$\neq 0$ at $T < T_c$

CAVITY METHOD

TAKE A SYSTEM WITH N SPINS

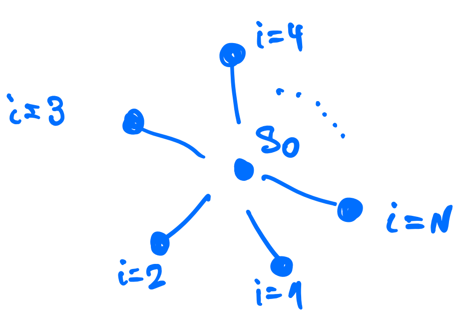
$$S_i \qquad i = 1, \dots, N$$

IN EQUIL. WITH A BATH AT $k_B T$

$$P_N(\{S_i\}) = \frac{e^{-\beta H_N[\{S_i\}]}}{Z_N}$$

$$Z_N = \sum_{\{S_j = \pm 1\}} e^{-\beta H_N[\{S_j\}]}$$

ADD ONE <u>CAVITY</u> SPIN S_0



& CONNECT IT TO THE OTHER N SPINS

$$H_{N+1}[s_0, \{s_i\}] = H_N[\{s_i\}] - h_0 s_0$$

$$= -\frac{1}{2} \sum_{k=0}^{N} \sum_{\substack{l=0 \\ (k \neq l)}}^{N} J_{kl} s_k s_l$$

WITH THE LOCAL FIELD ACTING ON S_0

$$h_0 = \sum_{j=1}^{N} J_{0j} s_j$$

EQUILIBRIUM OF THE SYST OF N+1 SPINS

$$P_{N+1}[s_0, \{s_i\}] = \frac{1}{Z_{N+1}} e^{-\beta H_{N+1}[s_0, \{s_i\}]}$$

$$Z_{N+1} = \sum_{\{s_0 = \pm 1\}} \sum_{\{s_i = \pm 1\}} e^{-\beta H_{N+1}[s_0, \{s_i\}]}$$

JOINT PROB. OF (s_0, h_0)

$$P_{N+1}(s_0, h_0) = \sum_{\{s_i = \pm 1\}} \delta\Big(h_0 - \sum_j J_{0j}\, s_j\Big)\, P_{N+1}[s_0, \{s_i\}]$$

$$P_{N+1}[s_0, \{s_i\}] = \frac{e^{-\beta H_{N+1}[s_0, \{s_i\}]}}{Z_{N+1}}$$

AS ABOVE

h_0 LOCAL FIELD CREATED BY N SPINS $\{s_j\}$ IN EQUIL WITHOUT s_0

WE DEFINE $P_N(h_0)$

$$P_N(h_0) \equiv \sum_{\{s_i = \pm 1\}} \delta\Big(h_0 - \sum_{j=1}^{N} J_{0j}\, s_j\Big)\, P_N[\{s_j\}]$$

$$P_N[\{s_j\}] = \frac{e^{-\beta H_N[\{s_j\}]}}{Z_N}$$

GOING BACK TO $P_{N+1}(s_0, h_0)$ ABOVE AND USING THIS DEFINITION

$$P_{N+1}(h_0, s_0) = P_N(h_0)\, e^{\beta h_0 s_0}\, \frac{Z_N}{Z_{N+1}}$$

COMES FROM $H_{N+1} = H_N - h_0 s_0$

THE RATIO OF PART FUNCTIONS IS

$$\frac{Z_{N+1}}{Z_N} = \frac{\sum_{\{s_j=\pm1\}} \sum_{s_0=\pm1} e^{-\beta H_N[\{s_j\}] + \beta h_0 s_0}}{\sum_{\{s_i=\pm1\}} e^{-\beta H_N[\{s_i\}]}}$$

$$= \frac{\sum_{\{s_j=\pm1\}} e^{-\beta H_N[\{s_j\}]}\, 2\operatorname{ch} \beta h_0}{\sum_{\{s_i=\pm1\}} e^{-\beta H_N[\{s_i\}]}}$$

RECALL THAT $h_0 = \sum_{j=1}^{N} J_{0j} S_j$ DEPENDS ON THE $\{S_i\}$ IN N SPIN SYSTEM

$$= \langle 2\operatorname{ch} \beta h_0 \rangle_N$$

AVERAGE OVER N-SPIN SYSTEM

THEN

$$P_{N+1}(h_0, S_0) = \frac{P_N(h_0)\, e^{\beta h_0 S_0}}{\langle 2\operatorname{ch} \beta h_0 \rangle_N}$$

NOTE THAT S_0 IS ONLY IN EXP. FACTOR

AVERAGE OF THE CAVITY SPIN S_0 IN N+1 - SYSTEM

TWO WAYS OF WRITING

$$\langle s_0 \rangle_{N+1} = \sum_{s_0=\pm 1} \sum_{\{s_i=\pm 1\}} s_0 \, P_{N+1}[s_0, \{s_i\}]$$

$$= \sum_{s_0=\pm 1} \int dh_0 \, P_{N+1}(h_0, s_0) \, s_0$$

$$= \sum_{s_0=\pm 1} s_0 \frac{\int dh_0 \, P_N(h_0) \, e^{\beta h_0 s_0}}{\langle 2 \mathrm{ch}\, \beta h_0 \rangle_N}$$

$$= \frac{\int dh_0 \, P_N(h_0) \quad 2 \mathrm{sh}\, \beta h_0}{\langle 2 \mathrm{ch}\, \beta h_0 \rangle_N}$$

$$\boxed{\langle s_0 \rangle_{N+1} = \frac{\langle \mathrm{sh}\, \beta h_0 \rangle_N}{\langle \mathrm{ch}\, \beta h_0 \rangle_N}}$$

AVERAGE OF THE LOCAL FIELD h_0 IN THE N+1 SPIN SYSTEM

$$\langle h_0 \rangle_{N+1} = \sum_{s_0=\pm 1} \sum_{\{s_i=\pm 1\}} h_0 \, P_{N+1}[s_0, \{s_i\}]$$

$$= \sum_{s_0=\pm 1} \frac{\int dh_0 \, P_N(h_0) \, e^{\beta h_0 s_0} \, h_0}{\langle 2\mathrm{ch}\, \beta h_0 \rangle_N}$$

$$= \frac{\int dh_0 \, P_N(h_0) \, h_0 \, 2\mathrm{ch}\, \beta h_0}{\langle 2\mathrm{ch}\, \beta h_0 \rangle_N}$$

$$\boxed{\langle h_0 \rangle_{N+1} = \frac{\langle h_0 \, \mathrm{ch}\, \beta h_0 \rangle_N}{\langle \mathrm{ch}\, \beta h_0 \rangle_N}}$$

STATISTICS OF LOCAL FIELD h_0

IN THE SYSTEM WITH N SPINS

WE NEED IT TO COMPUTE THE AVERAGES ABOVE $\langle S_0 \rangle_{N+1}$ & $\langle h_0 \rangle_{N+1}$

$$\langle h_0 \rangle_N = \sum_{j=1}^{N} J_{0j} \langle s_j \rangle_N$$

$$\langle (\delta h_0)^2 \rangle_N = \sum_{ij} J_{0i} J_{0j} \langle \delta s_i \, \delta s_j \rangle_N$$

$$\delta s_i = s_i - \langle s_i \rangle_N$$

NOW USE ORDER OF MAGN OF J_{0i}'s
DEPEND ON MODEL AT HAND.

SK - MODEL $\quad J_{0i} \sim \frac{1}{\sqrt{N}} \quad [J_{0i}] = 0$

NB J_{0i}, J_{0j} AND $\underbrace{\langle \delta s_i \, \delta s_j \rangle_N}_{\text{DON'T SEE SO}}$ INDEP

$\langle \delta s_i \, \delta s_j \rangle_N \simeq 1/\sqrt{N}$ ASSUME

CONTRIB $i \neq j$ $\quad \underbrace{\sum_{i \neq j}}_{\sqrt{N^2}} \underbrace{J_{0i}}_{\frac{1}{\sqrt{N}}} \underbrace{J_{0j}}_{\frac{1}{\sqrt{N}}} \langle \delta s_i \, \delta s_j \rangle$

EXPECTED TO DECREASE WITH A POWER OF N

WOULD BE $\Theta(1)$

SQ. ROOT BECAUSE OF $\pm$ signs.

KEEP only $i=j$ CONTRIB TO $\langle (\delta h_0)^2 \rangle_N$

$$\langle (\delta h_0)^2 \rangle_N \sim \sum_i J_{0i} J_{0i} \langle \delta s_i \, \delta s_i \rangle_N$$

$$= \sum_i \frac{J^2}{N} \left(\langle s_i^2 \rangle_N - \langle s_i \rangle_N^2 \right)$$

$$= J^2 \left(1 - \frac{1}{N} \sum_i \langle s_i \rangle_N^2 \right)$$

$$\boxed{\langle (\delta h_0)^2 \rangle_N = J^2 (1 - q_N)}$$

ASSUME **GAUSSIAN STATISTICS**

$$P_N(h_0) = \frac{1}{\sqrt{2\pi\langle(\delta h_0)^2\rangle_N}}\, e^{-\frac{\left(h_0 - \langle h_0\rangle_N\right)^2}{2\langle(\delta h_0)^2\rangle_N}}$$

WITH AVERAGE AND DISPERSION GIVEN ABOVE

TAP EQS

WE HAD DERIVED

$$\langle S_0\rangle_{N+1} = \frac{\langle \mathrm{sh}\,\beta h_0\rangle_N}{\langle \mathrm{ch}\,\beta h_0\rangle_N}$$

WE CAN NOW USE $P_N(h_0)$ TO EVALUATE THE AVERAGES

$$\langle S_0\rangle_{N+1} = \int dh_0\, P_N(h_0)\, \mathrm{sh}\,\beta h_0$$

$$\int d\rho_0 \, P_N(\rho_0) \, \operatorname{ch} \beta\rho_0$$

$$\frac{\displaystyle\int \frac{dz}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{(z-\langle z\rangle)^2}{2\sigma^2}} \, \frac{e^{\beta z} - e^{-\beta z}}{2}}{\displaystyle\int \frac{du}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{(z-\langle z\rangle)^2}{2\sigma^2}} \, \frac{e^{\beta z} + e^{-\beta z}}{2}} = A$$

$\Rightarrow$ ONE TERM

$$e^{\beta\langle z\rangle} \int \frac{dz}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{(z-\langle z\rangle)^2}{2\sigma^2} + \beta(z-\langle z\rangle)}$$

$$= e^{\beta\langle z\rangle} \, e^{\frac{\sigma^2\beta^2}{2}}$$

COMPLETING THE SQUARE IN THE EXP.

AND THE SAME STRUCT IN ALL TERMS IN NUM & DEN.

THUS,

$$A = \frac{e^{\beta\langle z\rangle} - e^{-\beta\langle z\rangle}}{e^{\beta\langle z\rangle} + e^{-\beta\langle z\rangle}}$$

$$A = \text{th}\ \beta\langle z\rangle$$

BACK TO PROBLEM

$$\boxed{\langle S_0\rangle_{N+1} = \text{th}\ \beta\langle h_0\rangle_N}$$

THE OTHER UNKNOWN

$$\langle h_0\rangle_{N+1} = \frac{\langle h_0\,\text{ch}\,\beta h_0\rangle_N}{\langle \text{ch}\,\beta h_0\rangle_N}$$

$$= \frac{\frac{\partial}{\partial\beta}\int Dh_0\, e^{-\frac{(h_0-\langle h_0\rangle_N)}{2\langle(\delta h_0)^2\rangle_N}}\left[e^{\beta h_0} - e^{-\beta h_0}\right]}{\int Dh_0\, e^{-\frac{(h_0-\langle h_0\rangle_N)^2}{2\langle(\delta h_0)^2\rangle}}\left[e^{\beta h_0} + e^{-\beta h_0}\right]}$$

$$Dh_0 = \frac{dh_0}{\sqrt{2\pi\langle(\delta h_0)^2\rangle}}$$

EXERCISE

THERE WILL BE A COMMON FACTOR

$$e^{\frac{\langle h_0\rangle_N^2}{2\langle(\delta h_0)^2\rangle_N}}$$

COMING FROM ALL TERMS $\Longrightarrow$ THEY CANCEL

$$= \frac{\frac{\partial}{\partial\beta}\left[e^{\left(\beta + \frac{\langle h_0\rangle_N}{\langle(\delta h_0)^2\rangle_N}\right)^2\cdot\frac{\langle(\delta h_0)^2\rangle_N}{2}} - (\beta\leftrightarrow-\beta)\right]}{\left[e^{\left(\beta + \frac{\langle h_0\rangle_N}{\langle(\delta h_0)^2\rangle_N}\right)^2\cdot\frac{\langle(\delta h_0)^2\rangle_N}{2}} + (\beta\leftrightarrow-\beta)\right]}$$

TAKING THE DERIVATIVE $\Rightarrow$ FACTORS IN NUMERATOR

$$\left(\beta + \frac{\langle h_0\rangle_N}{\langle(\delta h_0)^2\rangle_N}\right)\langle(\delta h_0)^2\rangle_N \; e^{\left(\beta + \frac{\langle h_0\rangle_N}{\langle(\delta h_0)^2\rangle_N}\right)^2 \cdot \frac{\langle(\delta h_0)^2\rangle_N}{2}}$$

$$+ (\beta \to -\beta) \quad =$$

$$= e^{\beta^2 \frac{\langle(\delta h_0)^2\rangle_N}{2} + \frac{\langle h_0\rangle_N^2}{2}} \langle(\delta h_0)^2\rangle_N$$

$$\left[\left(\beta\langle(\delta h_0)^2\rangle_N + \langle h_0\rangle_N\right) e^{\beta\langle h_0\rangle_N} + (\beta \to -\beta)\right]$$

$$\text{NUM} = e^{\beta^2 \frac{\langle(\delta h_0)^2\rangle_N}{2} + \frac{\langle h_0\rangle_N^2}{2}} \langle(\delta h_0)^2\rangle_N$$

$$\left[\beta\langle(\delta h_0)^2\rangle_N \; 2\,\mathrm{sh}(\beta\langle h_0\rangle_N) + \langle h_0\rangle_N \; 2\,\mathrm{ch}(\beta\langle h_0\rangle_N)\right]$$

DENOMINATOR

$$= e^{\beta^2 \frac{\langle(\delta h_0)^2\rangle_N}{2} + \frac{\langle h_0\rangle_N^2}{2}} \quad 2\,\mathrm{ch}\,\beta\langle h_0\rangle_N$$

$$\frac{\text{NUM}}{\text{DEN}} = \left(\begin{array}{c}\text{EXP. FACTORS}\\ \text{CANCEL}\end{array}\right) \cdot \text{WHAT REMAINS}$$

THUS,

$$\langle h_0 \rangle_{N+1} = \text{WHAT REMAINS}$$

$$= \beta \langle (\delta h_0)^2 \rangle_N \underbrace{\operatorname{th} \beta \langle h_0 \rangle_N}_{\langle S_0 \rangle_{N+1}} + \langle h_0 \rangle_N$$

$$\langle h_0 \rangle_{N+1} = \beta \langle (\delta h_0)^2 \rangle_N \langle S_0 \rangle_{N+1} + \langle h_0 \rangle_N$$

$$\langle h_0 \rangle_N = \langle h_0 \rangle_{N+1} - \beta J^2 (1 - q_N) \langle S_0 \rangle_{N+1}$$

$$\langle h_0 \rangle_{N+1}$$

$$= \langle h_0 \rangle_N + \beta \langle (\delta h_0)^2 \rangle_N \, \langle S_0 \rangle_{N+1}$$

NOW APPLY THIS TO

$$\langle S_0 \rangle_{N+1} = \mathrm{th}\, \beta \langle h_0 \rangle_N$$

$$= \mathrm{th}\, \beta \left(\langle h_0 \rangle_{N+1} - \beta J^2 (1-q) \langle S_0 \rangle_{N+1} \right)$$

$$= \mathrm{th}\, \beta \left(\sum_j J_{0j} \langle S_j \rangle_{N+1} - \beta J^2 (1-q) \langle S_0 \rangle_{N+1} \right)$$

$$\boxed{m_i = \mathrm{th}\, \beta \left(\sum_{j (\neq i)} J_{ij} m_j - \beta J^2 (1-q) m_i \right)}$$

THE TAP EQS W/ REACTION TERM

Today's Plan

- How do we know that the TAP equations are correct ?

  Phase transition in the SK model
- Back to landscapes
- Statistical averages
- Real replicas
- Replica method

MFT for disordered spin models

Phase transition

For large N one expects $J_{ij}^2 \simeq [J_{ij}^2] = J^2/N$ with $J = \mathcal{O}(1)$

Simplification $m_i = \tanh\left\{\beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 m_i \frac{J^2}{N} \sum_{j(\neq i)} (1 - m_j^2)\right\}$

A 2nd order phase transition $\Rightarrow m_i \simeq 0$ at $T \stackrel{<}{\sim} T_c$ then using $\tanh y \sim y$

The TAP equations become $m_i \sim \beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 J^2 m_i$

Diagonalize this eq. going to the basis of eigenvectors of the J_{ij} matrix

The eqs read $m_\lambda \sim \beta \Big(J_\lambda - \beta J^2\Big) m_\lambda$

The notation we use is such that

J_λ is an eigenvalue of the J_{ij} matrix associated to the eigenvector $\vec{v}_\lambda$

m_λ represents the projection of $\vec{m}$ on the eigenvector $\vec{v}_\lambda$, $m_\lambda = \vec{v}_\lambda \cdot \vec{m}$

with $\vec{m}$ the N-vector with components m_i, $\vec{m} = (m_1, \dots, m_N)$

MFT for disordered spin models

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2)m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta(J_\lambda - \beta J^2)\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec{v}_\lambda$)

$$\chi_\lambda \equiv \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta\ \left(1 - \beta J_\lambda + (\beta J)^2\right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix $J_{ij} = \mathcal{O}(1/\sqrt{N})$ are distributed with the Wigner semi-circle law and the largest eigenvalue is $J_\lambda^{\rm max} = 2J$

> The staggered susceptibility of staggered magnetization in the direction of the largest eigenvalue diverges at $\beta_c J = 1$ the correct value

MFT for disordered spin models

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta(J_\lambda - \beta J^2) \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec{v}_\lambda$)

$$\chi_\lambda \equiv \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta \, \left(1 - \beta J_\lambda + (\beta J)^2\right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix J_{ij} are distributed with the Wigner semi-circle law

For $J_{ij} = \mathcal{O}(1/\sqrt{N})$ the largest eigenvalue is $J_\lambda^{\rm max} = 2J$

The staggered susceptibility for the largest eigenvalue diverges at $\beta_c J = 1$

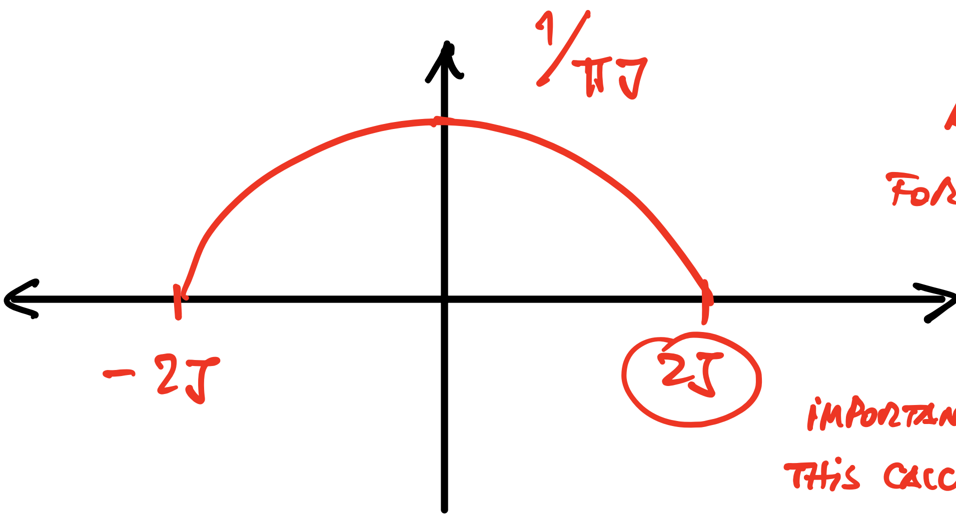
Without the reaction term the divergence is at the inexact value $T^* = 2T_c$

GOE ENSEMBLE

$J_{ij} = J_{ji} \in \mathbb{R}$ $\quad$ J $N \times N$ MATRIX

$[\rho(\lambda)]$ ALSO $\rho(\lambda)$ FOR $N \to \infty$

$$\rho(\lambda) = \frac{1}{2\pi J^2} \sqrt{(2J)^2 - \lambda^2}$$



ACTUALLY AN ELLIPSE FOR THIS PARAMET.

IMPORTANT FOR THIS CALCULATION

$$[J_{ij}^2] = \frac{J^2}{N}$$

WE WILL DERIVE IT LATER

SUMMARY

- WE DERIVED TAP EQS. WITH ONSAGER TERM WITH CAVITY IDEAS

- WE LOOKED AT SUSC $\frac{\partial m_x}{\partial h_x}$

AND FOUND $\chi_{\lambda_{MAX}} = \chi_{2J}$ DIVERGES

AT $T = J$ $\quad (k_B = 1)$

THANKS TO ONSAGER TERM OTHERWISE AT $T = 2J$

Today’s Plan

— How do we know that the TAP equations are correct ?

Phase transition in the SK model

— Back to landscapes

— Statistical averages

— Real replicas

— Replica method

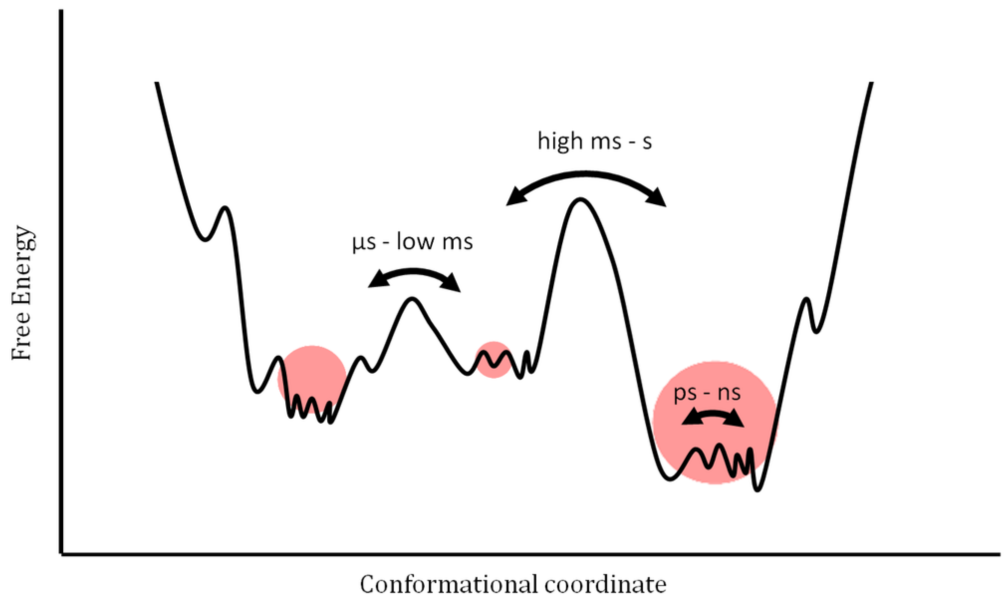
Landscape

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$\frac{\delta F_J^{\rm tap}(\{m_i\})}{\delta m_j} = 0$$

The stability of the solutions is determined by the Hessian $\frac{\delta^2 F_J^{\rm tap}(\{m_i\})}{\delta m_j \delta m_k}$



At low temperatures $\{m_i\}$

SK MODEL

$$f^{TAP} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\, m_i m_j - \sum_i h_i^{ext} m_i$$

$$-\frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2)$$

$$+T\sum_i \left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right]$$

EXTREMA $\qquad \dfrac{\partial f}{\partial m_i} = 0 \qquad \forall i$

STABILITY $\qquad \dfrac{\partial^2 f}{\partial m_i \partial m_j} = \mathcal{H}_{ij}$

Features

At fixed randomness

- There are N **local order parameters**, $m_i, \ i = 1, \dots, N$
- The saddle-points are **heterogeneous**: m_i differ from site to site
- At high temperatures only one trivial solution $\{m_i = 0\}$
- At low temperatures the TAP equations have **many solutions** $\{{m_i}^\alpha\}$, which are extrema of the TAP free-energy landscape, *i.e.* saddles of all types, $\alpha = 1, \dots, \mathcal{N}_J$
- For each solution $\{{m_i}^\alpha\}$, there is also $\{-{m_i}^\alpha\}$ but apart from this trivial doubling, the remaining solutions are not related by symmetry
- The TAP free-energy can take different values at different $\{{m_i}^\alpha\} \Rightarrow f^\alpha_{\mathrm{tap}}$

Features

All this is reshuffled for another realization of disorder

- Still N **local order parameters**, $m_i, \ i = 1, \ldots, N$
- The TAP equations have other solutions $\{m_i{}^\alpha\}$, extrema of the TAP free-energy landscape, $F_J^{\rm tap}$, labelled by $\alpha = 1, \ldots, \mathcal{N}_J$
- A **global order parameter** ? The simplest guess $\frac{1}{N}\sum_{i=1}^N m_i^\alpha$ cannot be since it is $= 0$ One expects as many positive as negative m_is and similarity in all respects. Another try

$$q_{\rm EA}^\alpha = \frac{1}{N}\sum_{i=1}^N (m_i^\alpha)^2$$

- “Typicality expected” (though see below for equilibrium states)

Features

Numbers of metastable states

- N **local order parameters**, $m_i, \ i = 1, \ldots, N$

- The TAP equations have many solutions $\{m_i{}^\alpha\}$, extrema of the TAP free-energy landscape, $\alpha = 1, \ldots, \mathcal{N}_J$

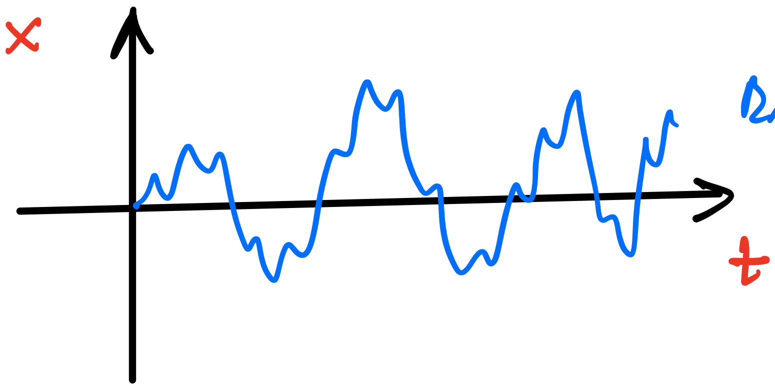
- One can count how many saddles of each kind exist and their **complexity**

$$\mathcal{N}_J = \sum_{\alpha} \prod_{i=1}^{N} \int_{-1}^{1} dm_i \ \delta(m_i - m_i^\alpha) \qquad\qquad \Sigma_J = \ln \mathcal{N}_J$$

- how many of these at each level of free-energy density, by inserting a delta-function $\delta(f_J^{\rm tap}(\{m_i^\alpha\}) - f) \Rightarrow \mathcal{N}_J(f)$

- How many with a given stability $\mathcal{N}_J(f, K)$ with K the number of positive eigenvalues of the Hessian, with adequate delta-functions

DISCUSS HERE KAC-RICE, BRAY-MOORE



RANDOM WALK

QUESTION: HOW MANY TIMES THE WALK CROSSES 0 IN A GIVEN TIME INTERVAL?

RICE

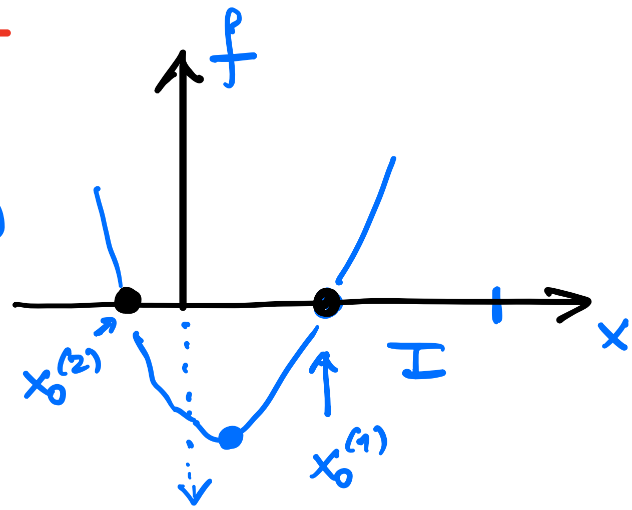
KAC

HOW MANY REAL ZEROES DOES A POLYNOME WITH RANDOM COEFF HAVE?

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

KAC–RICE FORMULA

TAKE A FUNCT $f(x)$



$$\mathcal{N} = \#\ \text{ZEROES} = \sum_{\alpha=1}^{2} \int dx\ \delta(x - x_0^{\alpha}) = 2$$

WITH x_0^{α} $\alpha = 1, \dots, \mathcal{N}$ THE ZEROES OF f: $f(x_0^{\alpha}) = 0$

BUT ONE DOES NOT NECESSARILY KNOW x_0^{α} EXPLICITLY NOR THEIR NUMBER $\mathcal{N}$, THAT'S WHAT WE WANT TO CALCULATE

SO THE TRICK IS THE FOLLOWING

TAKE ONE INTERVAL $[x_a, x_b]$ ON WHICH $f(x)$ IS MONOTONIC.

INTEGRATE

$$I = \int_{x_a}^{x_b} dx \; \delta(\underbrace{f(x)}_{y}) = \int_{y_a}^{y_b} \frac{dy}{|f'(x(y))|} \delta(y)$$

CHANGE OF VARIABLES

ONE WAY TO JUSTIFY THE ABSOLUTE VALUE IS THAT RESULT MUST BE ≥ 0 ON ANY INTERVAL

$y = f(x) \Rightarrow dy = f'(x)\, dx$

$x = f^{-1}(y)$

NOTE THAT ON THE ZERO OF $f(x)$

eg. $x_0^{(1)}$ ABOVE $\longrightarrow y = 0$

$$I = \int_{y_a = f(x_a)}^{y_b = f(x_b)} \frac{dy}{|f'(f^{-1}(y))|} \delta(y)$$

if $y_a < 0 < y_b$ $y=0$ is within the interval of integ

$$I = \frac{1}{|f'(f^{-1}(0))|} = \frac{1}{|f'(x_0)|}$$

otherwise just 0

This, in each interval in which $f(x)$ is monotonic and may have (or not, then $=0$) a zero.

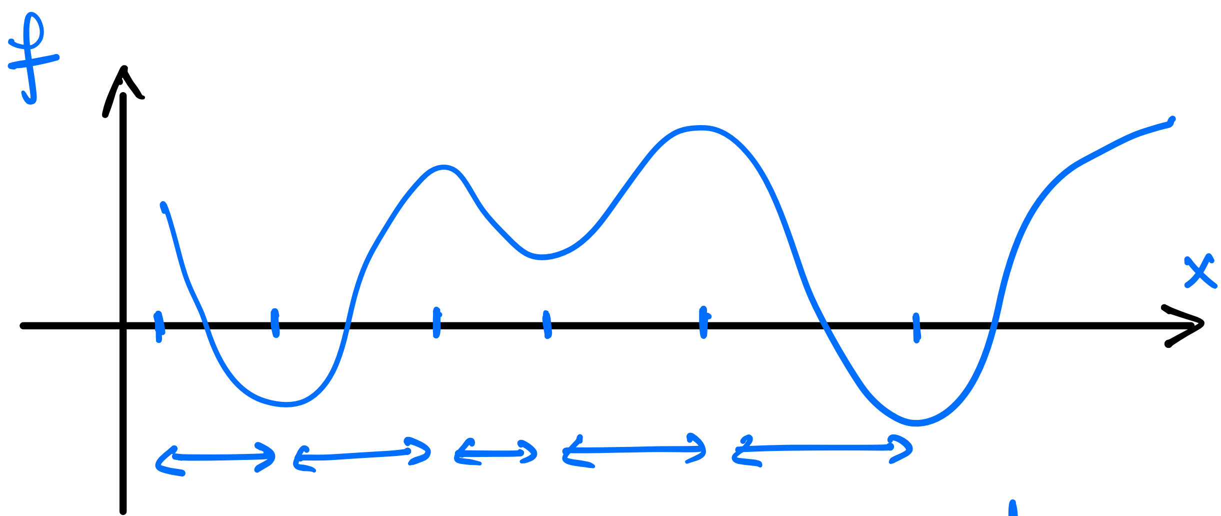
BUT WE WANT TO COUNT 1 AND NOT $\frac{1}{|f'(x_0)|}$

$\Rightarrow$

$$1 = \int_{x_a}^{x_b} dx \; \delta(f(x)) \, |f'(x)|$$

NOW EXTEND TO $-\infty, \infty$ THE INT OVER x

$$\boxed{\mathcal{N} = \int_{-\infty}^{\infty} dx \; \delta(f(x)) \, |f'(x)|}$$



etc.

$$\mathcal{N} = 1 \to 2 \to 2 \to 2 \longrightarrow 3 \longrightarrow 4$$

AND WE COUNT THEM ALL.

EXERCISE

APPLY KAC-RICE FORMULA TO CALCULATE THE AVERAGED # OF ZEROS OF

$$F(t) = \sum_k a_k f_k(t)$$

WHERE a_k ARE iid. FROM A GAUSSIAN

$[a_k] = 0 \qquad [a_k a_{k'}] = \sigma_k^2 \delta_{kk'}$

STEPS

1) STATISTICS OF $\{F(t), F'(t)\}$

GAUSSIAN WITH $[F(t)] = [F'(t)] = 0$

$$[F(t)\, F(t')] = \sum_k \sigma_k^2 f_k(t)\, f_k(t') \qquad \text{etc.}$$

NEEDED AT EQUAL TIMES t

$$\Delta^2 \equiv [F^2][F'^2] - [FF']^2$$

ALL AT TIME t

$$p(F(t),F'(t)) = \frac{1}{2\pi\Delta}$$

$$e^{-\frac{1}{2\Delta^2}\left(F^2[F'^2] - 2FF'[FF'] + F'^2\,[F^2]\right)}$$

ALL IS LOCAL IN TIME t

if $[FF']=0$ TO CHECK NORMALIZATION & FACTORS IN EXPONENTIAL

$$\Delta^2 = [F^2]\,[F'^2]$$

NORM SHOULD BE $\frac{1}{\sqrt{2\pi[F^2]}}\,\frac{1}{\sqrt{2\pi[F'^2]}} = \frac{1}{2\pi}\,\frac{1}{\Delta}$

CHECK EXPONENTIAL $\frac{-1}{2\,[F^2]\,[F'^2]}\left\{F^2\,[F'^2] + F'^2\,[F^2]\right\}$

$$= -\frac{1}{2}\left\{\frac{F^2}{[F^2]} + \frac{F'^2}{[F'^2]}\right\} \Longrightarrow \text{ok.}$$

2) APPLY

$$[\mathcal{N}(F(t))] = \int dF \int dF' \; \delta(F) \, |F'| \, p(F, F')$$

$$= 2 \int_0^\infty dF' \; F' \, \frac{1}{2\pi\Delta} \, e^{-\frac{[F^2] \, F'^2}{2\Delta^2}}$$

$$= \frac{\Delta}{[F^2]} \int_0^\infty \frac{du}{\pi} \, u \, e^{-\frac{u^2}{2}}$$

HAVING PERFORMED THE CHANGE OF VARIABLES

$$u = \frac{[F^2]^{1/2} F'}{\Delta} \quad \Longrightarrow \quad F' = \frac{u \Delta}{[F^2]^{1/2}} \qquad dF' = \frac{\Delta \, du}{[F^2]^{1/2}}$$

$$= \frac{\Delta}{[F^2]} \, \frac{1}{\pi}$$

THIS IS THE RESULT.

BACK TO LANDSCAPES

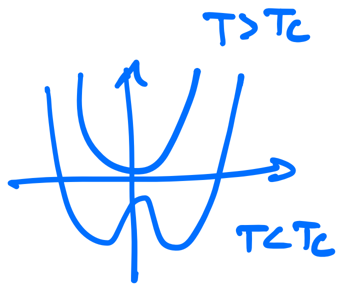
$\mathcal{N}(\{J_{ij}\}, k_B T, f, k)$

DISORDER REALIZ

TEMP.

FREE-ENERGY LEVEL

STABILITY

eg. CURIE-WEISS $\quad f(m) =$ 

$\mathcal{N} = 1 \quad T > T_c$

$\mathcal{N} = 3 \quad T < T_c$

TWO MINIMA, SAME f

ONE MAX, HIGHER f

COMPLEXITY

$$\Sigma = \frac{1}{N} \ln \mathcal{N}$$

IT MAY BE EXPECTED $\mathcal{N} \propto e^{N\Sigma} \quad T < T_c$

$\Rightarrow$ FINITE COMPLEXITY

FOR SOME OF THE DISORDERED MODELS WE STUDY

KAC-RICE MULTIDIMENSIONAL

$$\mathcal{N} = \int \prod_{i=1}^{N} dm_i \, \delta\left(\frac{\partial f^{TAP}}{\partial m_i}\right) \left| \det \frac{\partial^2 f^{TAP}}{\partial m_i \partial m_j} \right|$$

TAP EQS.

DETERM. OF HESSIAN

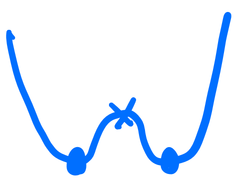
INDEX k_α # OF NEGATIVE EIGENVALUES OF $\mathcal{H}$ HESSIAN

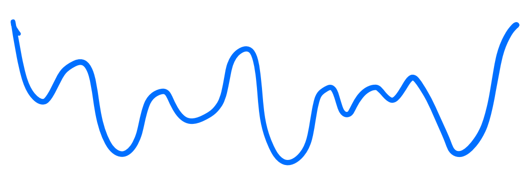
MORSE THEORY $\displaystyle\sum_\alpha (-1)^{k_\alpha} = -1$

FOR A FUNCTION GOING TO ∞ AT THE BOUNDARIES (OR GROWING ENOUGH)

eg DOUBLE WELL: $(-1)^0 + (-1)^1 + (-1)^0$

MIN MAX MIN.





CHECK IT!

ONE CAN ADD FURTHER CONSTRAINTS IN THE INTEGRAL TO IMPOSE FREE-ENERGY LEVEL,

$$\delta(f - f^{TAP}(\{m_i\}))$$

OR INDEX $\delta(k - k^{TAP}(\{m_i\}))$

THE CALCULATIONS BECOME HARDER.

ANNEALED VS. QUENCHED COMPLEXITY

$$\Sigma^{ANNEALED} = \frac{1}{N} \ln [\mathcal{N}]$$

$$\Sigma^{QUENCHED} = \frac{1}{N} [\ln \mathcal{N}]$$

JENSEN'S INEQ, $\ln$ IS CONCAVE $\Longrightarrow$

$$\ln [\mathcal{N}] \geq [\ln \mathcal{N}]$$

$$\boxed{\Sigma^{ANNEALED} \geq \Sigma^{QUENCHED}}$$

NOTE THAT THEY CAN BE EQUAL

eg. HIGH T WHEN BOTH VANISH

BUT THEY CAN BE DIFFERENT

AS ALREADY DISCUSSED

typ. TEND TO BE EQUAL TO QUENCHED

BECAUSE OF SUMS OF LARGE NUMBERS

BUT ANNEALED IS EASIER TO COMPUTE

Today's Plan

— How do we know that the TAP equations are correct ?

  Phase transition in the SK model

— Back to landscapes

— Statistical averages

— Real replicas

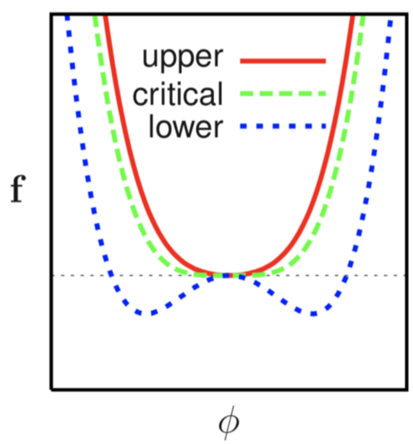
— Replica method

Statistical averages

At fixed interactions

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

In the **FM case**, each state ($\langle \phi \rangle = \pm \phi_0$) has weigth $w_\pm = 1/2$ and the sum is $\langle O \rangle = \frac{1}{2}\langle O \rangle_+ + \frac{1}{2}\langle O \rangle_-$ with $\langle O \rangle_\pm$ the average in each of the states. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to only one of them.



FM case

The dashed blue line with

two minima $\pm|\phi_0|$

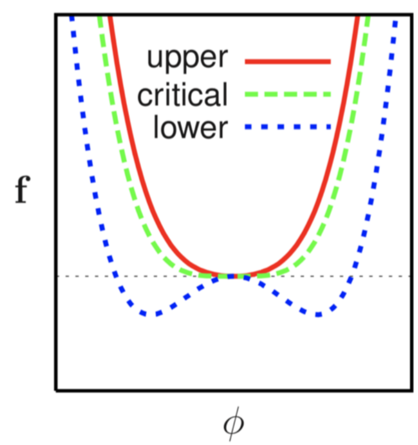
If we have many more ?

Statistical averages

At fixed interactions

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

In the **FM case**, each state ($\langle \phi \rangle = \pm \phi_0$) has weigth $w_\pm = 1/2$ and the sum is $\langle O \rangle = \frac{1}{2} \langle O \rangle_+ + \frac{1}{2} \langle O \rangle_-$ with $\langle O \rangle_\pm$ the average in each of the states. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to only one of them.



FM case $f_+ = f_-$

$$w_\pm = \frac{e^{-\beta N f_\pm}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \simeq \frac{1}{2}$$

$$w_0 = \frac{e^{-\beta N f_0}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \ll w_\pm$$

Statistical averages

At fixed randomness

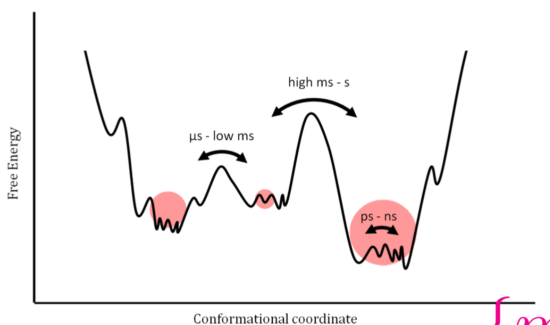
The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

For systems with **quenched randomness** $\boxed{w_\alpha^J = \dfrac{e^{-\beta N \mathbf{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathbf{f}_\gamma^J}}}$

where we added a super-script to the weight w

^{J} indicates that the weights depend on the the disorder realization

and α is a label that identifies the TAP solution



$\{m_i\}$

One can sum over all saddles irrespectively of their stability. Higher lying ones will be exponentially suppressed or will dominate depending on $\Sigma_J(f, K)$

Statistical averages

At fixed randomness

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

For systems with **quenched randomness** $\boxed{w_\alpha^J = \frac{e^{-\beta N \mathbf{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathbf{f}_\gamma^J}}}$

The sum over α, in the case in which there are an exponential in N number of TAP solutions, can be replaced by an integral over $\mathbf{f}$

$$\boxed{\langle O \rangle = \mathcal{Z}^{-1}(\beta, J) \int d\mathbf{f} \; e^{-\beta[N\mathbf{f} - T \ln \mathcal{N}_J(\mathbf{f},\beta)]} \; O(\mathbf{f}, \beta)}$$

$\mathcal{N}_J$ is the number of solutions to the TAP eqs. with free-energy density $\mathbf{f}$.

For $N \to \infty$ the integral is dominated by the saddle point

$$\frac{1}{T} = \frac{1}{N} \frac{\partial \ln \mathcal{N}_J(\mathbf{f}, \beta)}{\partial \mathbf{f}}\bigg|_{\mathbf{f}_{sp}} = \frac{1}{N} \frac{\partial \Sigma_J(\mathbf{f}, \beta)}{\partial \mathbf{f}}\bigg|_{\mathbf{f}_{sp}} \qquad \textbf{complexity}$$

Statistical averages

Consequences

The **equilibrium free-energy** f is given by the saddle-point evaluation of the partition sum:

$$\boxed{f = \mathbf{f}_{sp} - \frac{T}{N} \ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta)}$$

The rhs is the **Landau free-energy** of the problem, with $\mathbf{f}_{sp}$ playing the role of the energy and $N^{-1} \ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta)$ of the entropy

The contribution of the complexity or configurational entropy contribution is negative and in some cases higher lying extrema (metastable states) can dominate the partition sum with respect to lower lying ones if $\ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta) \propto N$

This feature is proposed to describe **super-cooled liquids**.

A global observable

Effect of multi-states

What is the expression of the global order parameter once one takes into account the multi-states ?

$$q \equiv \frac{1}{N}\sum_i \langle s_i\rangle^2 = \frac{1}{N}\sum_i \Big(\sum_\alpha w^J_\alpha m^\alpha_i\Big)^2 = \frac{1}{N}\sum_i \sum_\alpha w^J_\alpha m^\alpha_i \sum_\beta w^J_\beta m^\beta_i$$

note that this is different from $q_{\mathrm{EA}} = \frac{1}{N}\sum_i (m^\alpha_i)^2$

Defining now $\boxed{q_{\alpha\beta} \equiv \frac{1}{N}\sum_i m^\alpha_i m^\beta_i}$ an overlap between different states

and $\boxed{P_J(q') \equiv \sum_{\alpha\beta} w^J_\alpha w^J_\beta \, \delta(q' - q_{\alpha\beta})}$

we obtain $\boxed{q \equiv \frac{1}{N}\sum_i \langle s_i\rangle^2 = \int dq' \, P_J(q') \, q'}$

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Real replicas

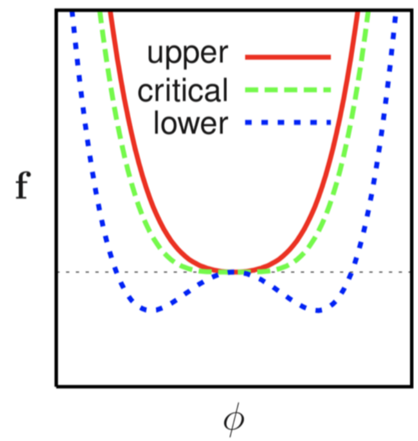
Overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **PM system** the overlap will typically vanish as, say, $N^{-1/2}$



Many repetitions

for a system with $N \gg 1$

$$\boxed{P(q_{s\sigma}) = \delta(q_{s\sigma})}$$

Real replicas

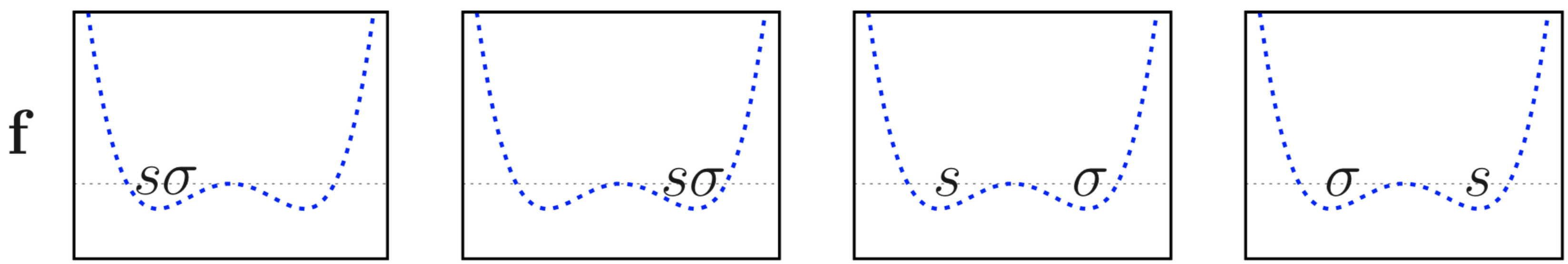
Overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **FM system** there are four possibilities



$q_{s\sigma} = m^2$ $\qquad m^2 \qquad -m^2 \qquad -m^2$

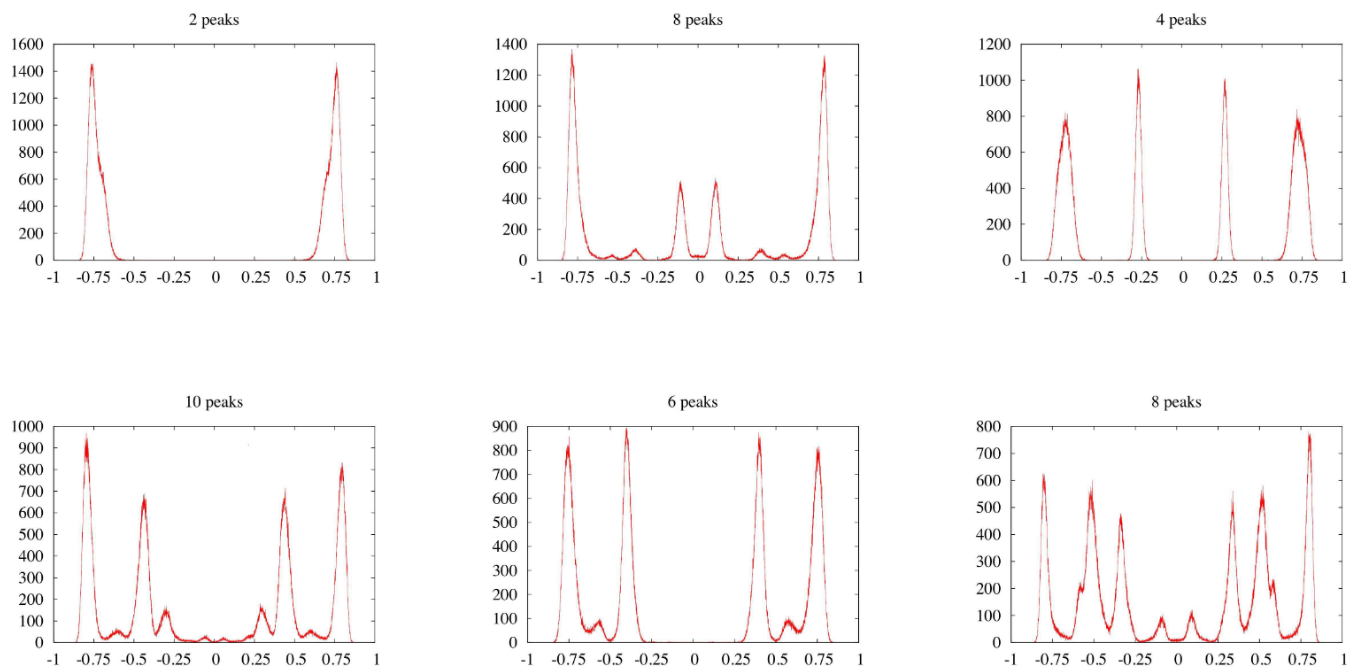
Many repetitions $\boxed{P(q_{s\sigma}) = \frac{1}{2}\delta(q_{s\sigma} - m^2) + \frac{1}{2}\delta(q_{s\sigma} + m^2)}$

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad\qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad\qquad P_J(q_{s\sigma})$$



Finite size corrections in the Sherrington-Kirkpatrick model

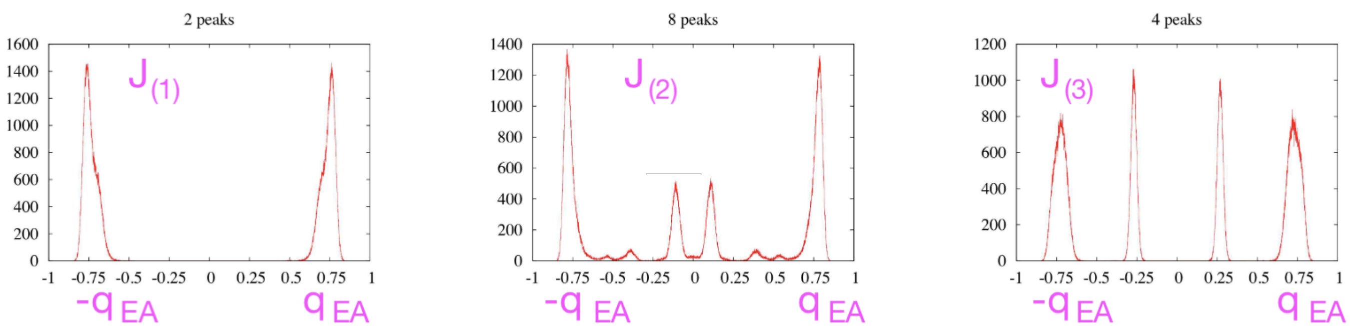
Aspelmeier, Billoire, Marinari & Moore (2007)

Real replicas

Ooverlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

Each sample has peaks at $q_{s\sigma} = \pm q_{\mathrm{EA}} \simeq \pm 0.75$:

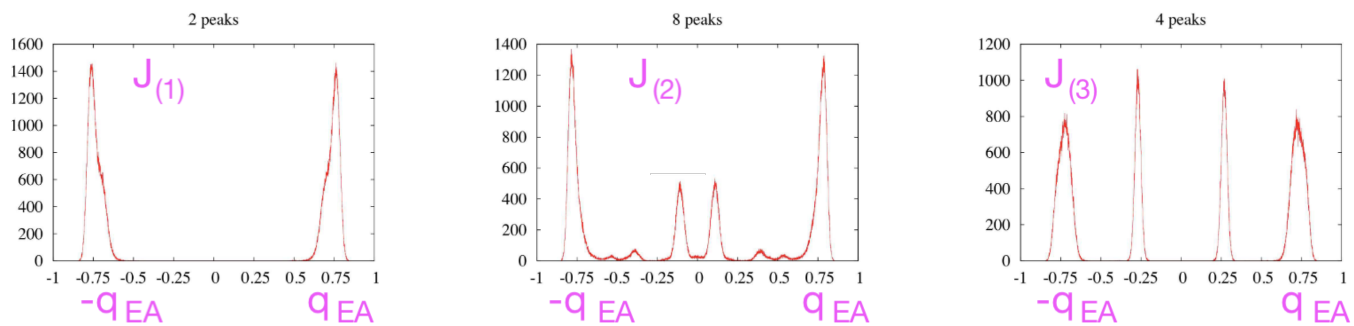
two configurations in the same (or the spin-reversed) state

Real replicas

Overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij}s_is_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i\sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

$0.75 \simeq q_{\rm EA} < 1$ and the width of the peaks at $q_{s\sigma} = \pm q_{\rm EA}$:

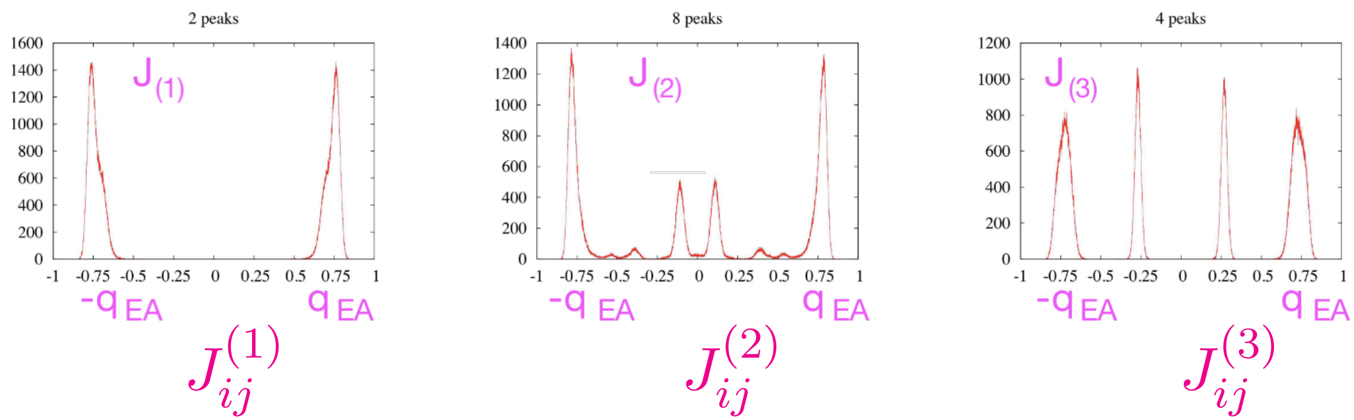
due to $0 < T < T_c$ and finite N, respectively

Real replicas

Overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i \neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



Data in each panel for a different realization of the random couplings

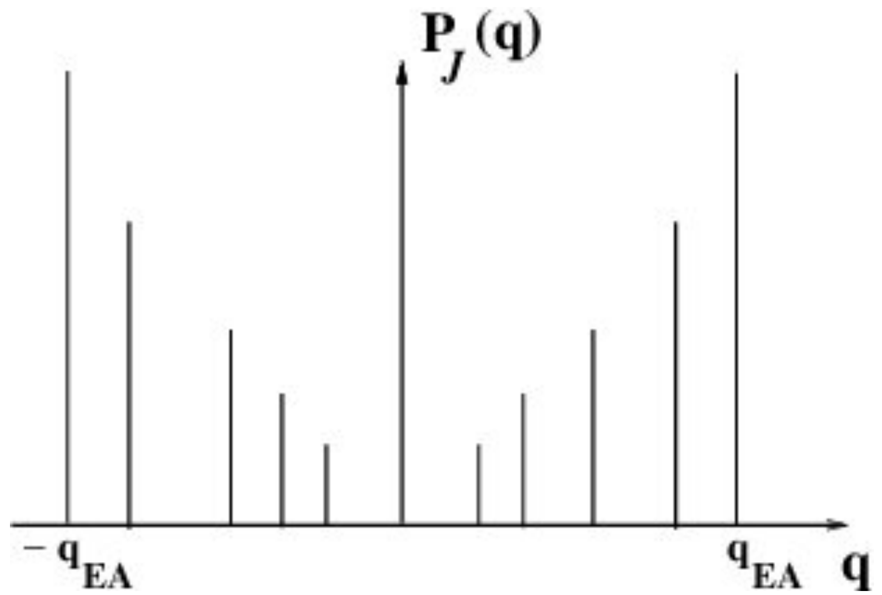
Most samples also have peaks at $|q_{s\sigma}| < q_{\rm EA}$:

replicas $\{s_i\}$ and $\{\sigma_i\}$ falling in different states

Real replicas

Overlaps between replicas at fixed randomness

SK model with $N \to \infty$ at $T < T_c$

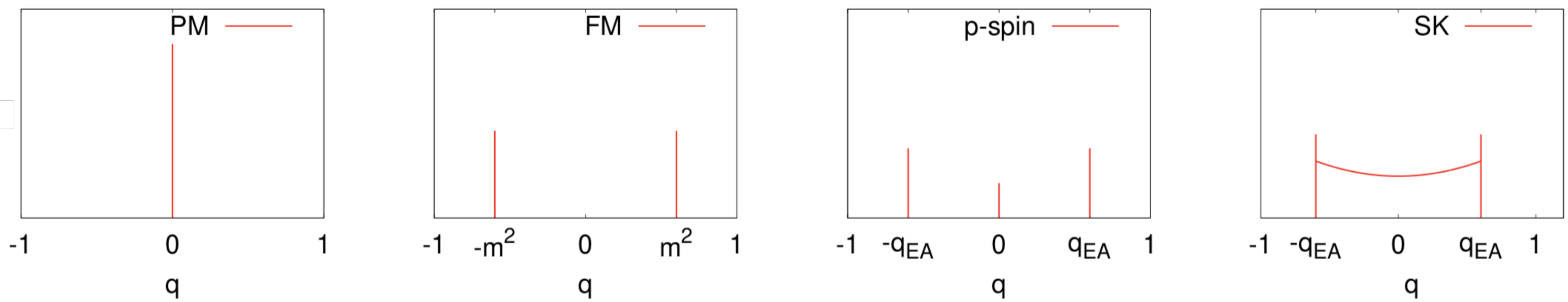


What happens if one averages $P_J(q)$ over disorder

Real replicas

Disordered averaged pdf of overlaps $P(q) = [P_J(q)]$

Parisi 79-82 prescription for the replica symmetry breaking Ansatz yields



High temperature FM Structural glasses Spin-glasses

Thermodynamic quantities, in particular the equilibrium free-energy density are expressed as functions of $P(q)$.

The equilibrium free-energy density predicted by the replica theory was confirmed by **Guerra & Talagrand 00-04** indepedent mathematical-physics mthods.

Today's Plan

— How do we know that the TAP equations are correct ?

Phase transition in the SK model

— Back to landscapes

— Statistical averages

— Real replicas

— Replica method

HERE: REPLICA METHOD

→ START WITH $\boxed{\text{OSC RANDOM FREQ.}}$ THEN $\boxed{\text{RFIM}}$

Typical vs. averaged

TAP vs. Replicas

Precursors

Look at an integer parameter n

and its $n \to 0$ limit

In 1972 Fortuin and Kasteleyn studied the **Potts model** with n components :

$n = 2$ Ising

$n = 1$ percolation

$n = 0$ random resistors

Use the identify $x^n = \exp(n \ln x)$ and expand around $n = 0$:

$$\lim_{n \to 0} x^n = 1 + n \ln x + \mathcal{O}(n^2)$$

Replica method

A sketch

$$\boxed{-\beta[f_J] = \lim_{N\to\infty} \frac{[\ln Z_N(\beta,J)]}{N} = \lim_{N\to\infty} \lim_{n\to 0} \frac{[Z_N^n(\beta,J)]-1}{Nn}}$$

Z_N^n partition function of n independent copies of the system : **replicas**.

Gaussian average over disorder : coupling between replicas

$$\boxed{\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \;\Rightarrow\; \sum_{i\neq j} \left(\sum_a s_i^a s_j^a\right)^2}$$

Quadratic decoupling with the Hubbard-Stratonovich trick

$$\boxed{Q_{ab} \sum_i s_i^a s_i^b + \frac{1}{2} Q_{ab}^2}$$

Q_{ab} is a 0×0 matrix but it admits an interpretation in terms of **overlaps**

The elements of Q_{ab} can evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

Replica method

In more detail

Z_N^n partition function of n independent copies of the system : **replicas**.

$$Z_N^n(\beta, J) = \underbrace{\sum_{\{s_i^{(1)}=\pm 1\}} \cdots \sum_{\{s_i^{(n)}=\pm 1\}}}_{\text{notation } \mathrm{Tr}_{\{s_i^a\}}} e^{-\beta \sum_{a=1}^n \sum_{i \neq j} J_{ij} s_i^a s_j^a}$$

One can exchange the order of the trace and the average over disorder

$$[Z_N^n(\beta, J)] = \mathrm{Tr}_{\{s_i^a\}} \int \prod_{i \neq j} dJ_{ij} P(J_{ij})\ e^{-\beta \sum_{a=1}^n \sum_{i \neq j} J_{ij} s_i^a s_j^a}$$

$$\boxed{[Z_N^n(\beta, J)] = \mathrm{Tr}_{\{s_i^a\}}\ e^{-\beta H_{\text{eff}}[\{s_i^a\}]}}$$

$H_{\text{eff}}[\{s_i^a\}]$ does not have any randomness but couples the replicas

$$\boxed{\sum_a \sum_{i \neq j} J_{ij} s_i^a s_j^a \ \Rightarrow\ \sum_{i \neq j} \left(\sum_a s_i^a s_j^a\right)^2}$$

Replica method

In more detail

$$\boxed{[Z_N^n(\beta, J)] = \text{Tr}_{\{s_i^a\}}\; e^{-\beta H_{\text{eff}}[\{s_i^a\}]}}$$

$H_{\text{eff}}[\{s_i^a\}]$ does not have any randomness but couples the replicas

$$\sum_{i\neq j} \left(\sum_a s_i^a s_j^a\right)^2 = \sum_{i\neq j} \sum_a \sum_b s_i^a s_j^a s_i^b s_j^b \sim \sum_{ab} \sum_i s_i^a s_i^b \sum_j s_j^a s_j^b$$

One sees Q_{ab} here, introduce their definition via a delta or apply Hubbard-Stratonovich

Once this done, one can exchange the trace (the sum over spin configurations) and the integral over Q_{ab} and end up with

$$[Z_N^n(\beta, J)] \propto \int \prod_{ab} dQ_{ab} e^{-F(Q_{ab})}$$

Replica method

For the SK model

$Q_{ab} = q_{ab}$ and $p = 2$

$$
\begin{aligned}
\beta F(q_{ab}) &= -\frac{N\beta^2 J^2}{2}\left[-\sum_{a\neq b} q_{ab}^p + n\right] - N\ln\zeta(q_{ab}) \ , \\
\zeta(q_{ab}) &= \sum_{s_a} e^{-\beta H(q_{ab},s_a)} \ , \\
H(q_{ab}, s_a) &= -J\sum_{ab} q_{ab}s_a s_b - h\sum_a s_a \ ,
\end{aligned}
$$

Replica method

In more detail

$$\boxed{[Z_N^n(\beta,J)] = \text{Tr}_{\{s_i^a\}}\, e^{-\beta H_{\text{eff}}[\{s_i^a\}]} \propto \int \prod_{ab} dQ_{ab} e^{-F(Q_{ab})}}$$

$H_{\text{eff}}[\{s_i^a\}]$ and Q_{ab} do not have any randomness but couple the replicas

The elements of Q_{ab} can be evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

At the saddle-point level one identifies $Q_{ab}^{sp} = N^{-1}\langle \sum_i s_i^a s_i^b \rangle$

The spin glass transition is from the paramagnetic state with $Q_{a\neq b} = 0$ to a spin glass state with $Q_{a\neq b} \neq 0$ as the temperature is decreased.

Replica method

SK model : replica symmetric Ansatz

Permutation symmetry between replicas $\Rightarrow$

Insert $Q_{a\neq b} = q$ and $Q_{aa} = 1$ in the effective Hamiltonian

Saddle-point with respect to q and $n \to 0$

$$\boxed{q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-z^2/2}\, \tanh^{2}(\beta J \sqrt{q} z)}$$

Note the similarity with the equation for m in the Curie-Weiss model

$q = 0$ for $T \geq T_c = J$

$q \neq 0$ for $T < T_c = J$

Problem I Is this solution stable ? No

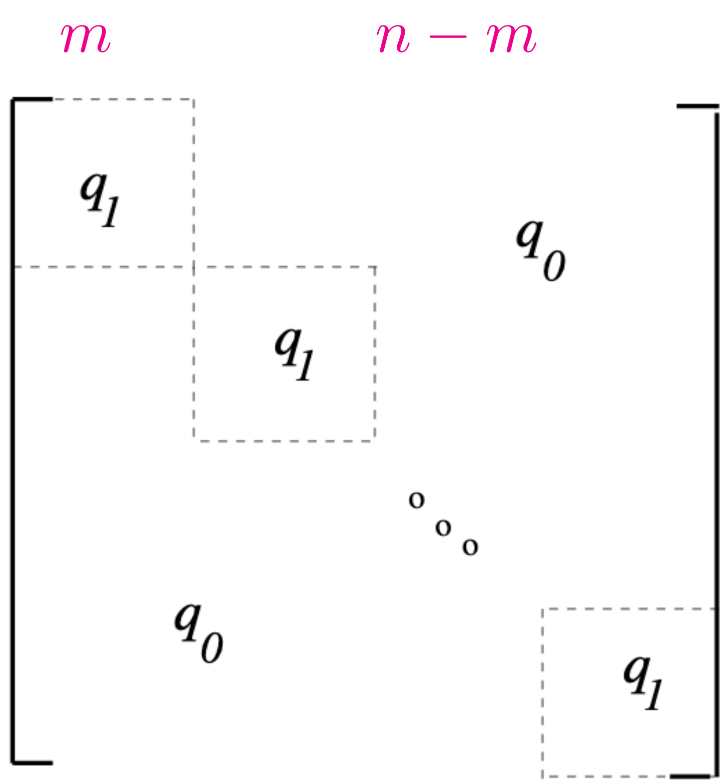
Problem II Does it have a zero-temperature vanishing entropy ? No

Problem III Ground state energy density $e = -0.77 \pm 0.01$ while the replica symmetric value $e = -0.798$, is three standard deviations smaller (in units of J)

Replica method

SK model: one step replica symmetry breaking

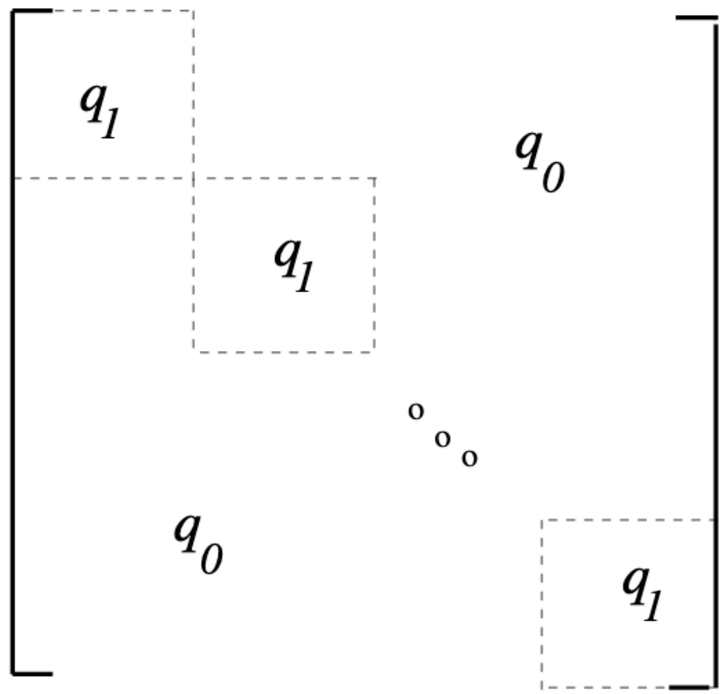
Permutation symmetry broken



$n \times n$ matrix divided in diagonal blocks of size $m \times m$ and the rest

Replica method

SK model : one step replica symmetry breaking



Problem I Stability : improved but not solved

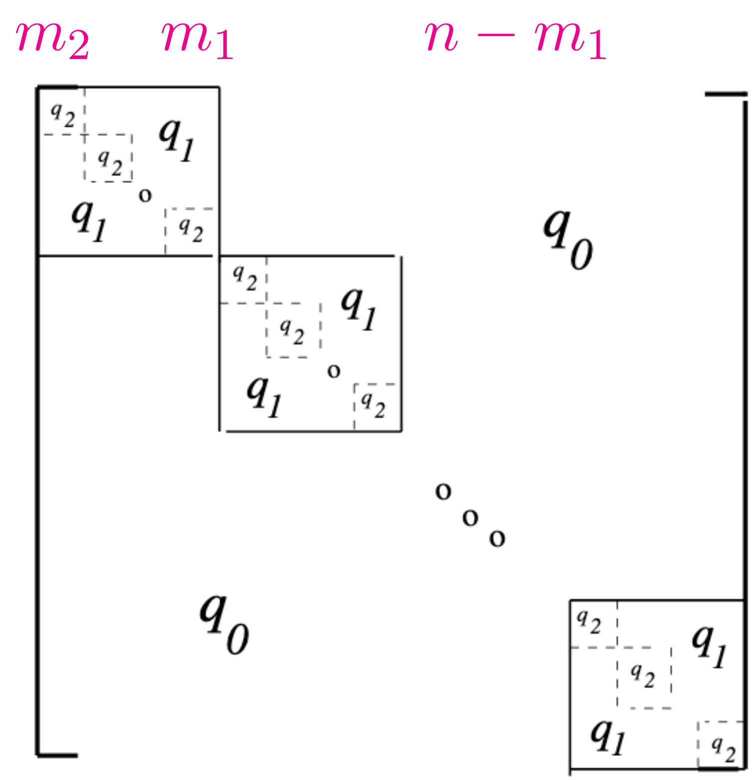
Problem II Zero-temperature entropy : improved but not solved

Problem III e closer to numerical value

Replica method

SK model: two step replica symmetry breaking

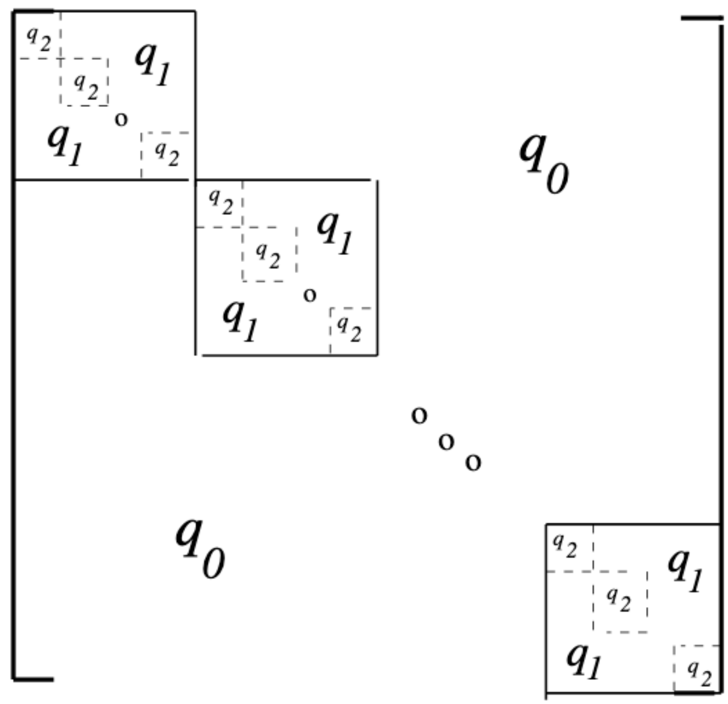
Permutation symmetry broken



$n \times n$ matrix divided in diagonal blocks of size $m_2 \times m_2$, and the rest in blocks of size $m_1 \times m_1$ and the rest

Replica method

SK model : two step replica symmetry breaking



Problem I Stability : improved but not solved

Problem II Zero-temperature entropy : improved but not solved

Problem III e closer to numerical value

Replica method

SK model : full replica symmetry breaking

Blocks of size m_i with parameter q_i

e.g. for replica symmetric case one block a single q.

∞ number of breaking steps, that is, of blocks

$m_i \mapsto x$ and the parameter $q_i \mapsto q(x)$

$$[\langle s_i \rangle^2] = \int_0^1 dx\, q(x) = \int \frac{dx}{dq}\, dq\, q(x) = \int dq\, P(q)\, q$$

with $$\boxed{P(q) = \frac{dx}{dq}}$$

Problem I Stability : solved

Problem II Zero-temperature entropy : solved $S = 0$

Problem III e in agreement with numerical value within numerical accuracy $e = -0.7633$

BACK TO ORDER PARAMETER AND EXPLANATION OF $P(q) = \frac{dx}{dq}$

EDWARDS- ANDERSON

BEFORE KNOWING THAT THERE WERE MANY STATES

$$q_{EA} = \lim_{t \to \infty} \left[\langle s_i(t)\, s_i(0) \rangle \right]$$

DYNAMIC CORRELATION AVERAGED OVER NOISE & QUENCHED RANDOMNESS

EXPECTED

$$q_{EA} = \frac{1}{N} \sum_{i=1}^{N} \left[\langle s_i \rangle^2 \right]$$ TO AVOID signs.

$$q_{EA} = \frac{1}{N} \sum_{i=1}^{N} \left[m_i^2 \right] \qquad m_i = \langle s_i \rangle$$

WHAT HAPPENS WITH THIS WHEN WE KNOW THAT THERE ARE MANY STATES?

"STATE RESOLVED" BEFORE AVERAGING OVER $\{J_{ij}\}$:

$$q^{\alpha}_{EA_J} = \frac{1}{N} \sum_{i=1}^{N} (m_i^{\alpha})^2$$

DEPENDS ON $\{J_{ij}\}$

WE CAN AVERAGE OVER THE STATES LIKE THIS

$$q_{EA_J} = \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha} w_{\alpha}^{J} (m_i^{\alpha})^2$$

w_{α}^{J} WEIGHT OF STATE α FOR $\{J_{ij}\}$

eg. HOMOGENEOUS FM $w_{\alpha} = 1/2$ $m_i^{\alpha} = m$

$$q = \underbrace{\frac{1}{N} \sum_i}_{1} \underbrace{\sum_{\alpha=1}^{2} \frac{1}{2}}_{1} m^2 = m^2 \qquad \underbrace{\boxed{q = m^2}}_{FM}$$

OR WE CAN SAY THAT WE AVERAGE THE m_i

$$m_i = \sum_a w_\alpha^J m_i^\alpha$$

AND THEN BUILD

$$q_J = \frac{1}{N} \sum_i \left(\sum_\alpha w_\alpha^J m_i^\alpha \right)^2$$

$$= \frac{1}{N} \sum_i \underbrace{\sum_{\alpha\beta} w_\alpha^J w_\beta^J m_i^\alpha m_i^\beta}_{\text{IT MIXES DIFF. STATES}}$$

WE SEE THAT IN CASES WITH MANY STATES

$$\boxed{q_{EA_J} \neq q_J}$$

DEFINE THE OVERLAP BETWEEN STATES

$$q_{\alpha\beta}^J = \frac{1}{N} \sum_i \langle s_i \rangle^\alpha \langle s_i \rangle^\beta = \frac{1}{N} \sum_i m_i^\alpha m_i^\beta$$

EDWARDS- ANDERSON = SELF- OVERLAP

$$q_{EA_J} = \frac{1}{N} \sum_i (m_i^\alpha)^2$$ (PARTICULAR CASE)

STATISTICS OF POSSIBLE OVERLAPS

$$\boxed{P_J(q) = \sum_{\alpha\beta} w_\alpha^J w_\beta^J \, \delta(q - q_{\alpha\beta})}$$

AND THIS ONE WE ACCESS WITH THE REAL REPLICA NUMERICAL EXPERIMENT

$\Longrightarrow$ PEAKS IN RSB

RE-WRITING OF q_J USING $P_J(q)$

$$q_J = \frac{1}{N} \sum_i \langle s_i \rangle^2$$

$$= \frac{1}{N} \sum_i \sum_\alpha \sum_\beta w_\alpha^J w_\beta^J m_i^\alpha m_i^\beta$$

$$= \sum_\alpha \sum_\beta w_\alpha^J w_\beta^J \underbrace{\frac{1}{N} \sum_i m_i^\alpha m_i^\beta}_{q_{\alpha\beta}}$$

$$= \sum_\alpha \sum_\beta w_\alpha^J w_\beta^J q_{\alpha\beta}$$

WE DEFINED $\quad P_J(q) = \sum_\alpha \sum_\beta w_\alpha^J w_\beta^J \delta(q - q_{\alpha\beta})$

USE IT TO REWRITE

$$\boxed{q_J = \int dq' \, P_J(q') \, q'} \qquad (1)$$

NEXT, AVERAGE OVER DISORDER

$$\boxed{P(q) = [P_J(q)] = \left[\sum_{\alpha\beta} w_\alpha^J w_\beta^J \delta(q - q_{\alpha\beta})\right]}$$

NOW, WITH A CALCULATION SHOWN IN THE NOTES

$$q = \frac{1}{N} \sum_i [\langle s_i \rangle^2]$$

AVERAGE OVER DISORDER

CAN BE WRITTEN, WITHIN REPLICA CALC. AS

FULL THERMAL AVERAGE

REPLICA EXPRESSIONS

$$\boxed{\begin{aligned} q &= \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} q_{ab} \\ P(q) &= [P_J(q)] = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \delta(q-q_{ab}) \end{aligned}}$$

THUS, KNOWING q_{ab}, ITS STRUCTURE, WE KNOW q OR BETTER STILL $P(q)$

RELATE $P(q)$ TO $q(x)$ THE PARISI STRUCTURE OF THE REPLICA MATRIX

WE CLAIMED ABOVE

$$q = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} q_{ab}$$

$$= \int_0^1 dx \; q(x)$$

BREAKING OF REPLICA BLOCKS NOW CONT PARAM.

$$= \int_0^1 \underbrace{dq \, \frac{dx}{dq}}_{dx} \; q(x)$$

$$= \int_0^1 dq \left(q \cdot \frac{dx}{dq} \right)$$

WE ALSO KNOW FROM (1)

$$q_J = \int_0^1 dq' \, P_J(q') \, q'$$

$$[q_J] = \int_0^1 dq' \, \underbrace{[P_J(q')]}_{P(q')} \, q'$$

$$= \int_0^1 dq' \, P(q') \, q'$$

COMPARING

$$\boxed{P(q) = \frac{dx}{dq}}$$