THE OSCILLATOR MODEL

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

$$Z_\omega = \int_{-\infty}^{\infty} dx \; e^{-\frac{\beta m\omega^2}{2} x^2}$$

GAUSSIAN INTEGRAL

$\sigma^2 = 1/\beta m\omega^2$

RESULT $\sqrt{2\pi\sigma^2}$

$$Z_\omega = \sqrt{\frac{2\pi}{\beta m\omega^2}}$$

$$\Rightarrow -\beta[F_\omega] = \int_{-\infty}^{\infty} d\omega \; p(\omega) \ln\sqrt{\frac{2\pi}{\beta m\omega^2}}$$

$$p(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\omega^2}{2\sigma^2}}$$

GAUSSIAN DIST OF OSC "FREQ"

$$-\beta[F_\omega] = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi\sigma^2}} e^{-\frac{\omega^2}{2\sigma^2}} \ln\sqrt{\frac{2\pi}{\beta m\omega^2}}$$

$$= -\int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}\,\sigma} e^{-\frac{\omega^2}{2\sigma^2}} \ln \sqrt{\frac{\beta m \omega^2}{2\pi} \cdot \frac{\sigma^2}{\sigma^2}}$$

$$= -2\int_0^{\infty} dy\, e^{-\frac{y^2}{2}} \ln y \frac{1}{\sqrt{2\pi}} \qquad y = \frac{\omega}{\sigma}$$

$$- \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \ln \sqrt{\frac{\beta m \sigma^2}{2\pi}}$$

$$\boxed{\beta [F_\omega] = \ln \sqrt{\frac{\beta m \sigma^2}{2\pi}} + 2\int_0^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \ln y}$$

A STEP FORWARD, USING

$$\int_0^{\infty} dy\, e^{-y^2/2} \ln y =$$

$$-\frac{1}{4}(C + \ln 2)\sqrt{2\pi}$$

$$\beta [F_\omega] = \ln \sqrt{\frac{\beta m \sigma^2}{2\pi}}$$

$$+ \cancel{2}\left(-\frac{1}{\cancel{4}_{2}}\right)\frac{\cancel{\sqrt{2\pi}}}{\cancel{\sqrt{2\pi}}}\left(c+\ln 2\right)$$

$$= \ln\sqrt{\frac{\beta m\sigma^2}{2\pi}} - \frac{c}{2} + \ln\frac{1}{\sqrt{2}}$$

$$\boxed{\beta\left[F_w\right] = \ln\sqrt{\frac{\beta m\sigma^2}{4\pi}} - \frac{c}{2}}$$

WITH REPLICAS $\qquad z^n = 1 + n\ln z$

$$-\beta\left[F_w\right] = \lim_{n\to 0}\frac{\left[z^n\right]-1}{n}$$

$$z^n = \int dx_1 \ldots dx_n \; e^{-\frac{\beta}{2}m\omega^2\sum_{a=1}^{n} x_a^2}$$

$$[Z^n] = \int \frac{dw}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{w^2}{2\sigma^2}}$$

$$\underbrace{\int dx_1 \ldots dx_n}_{\int d\Omega_n \int dr \, r^{n-1}} \; e^{-\frac{\beta}{2} m w^2 \underbrace{\sum_{a=1}^{n} x_a^2}_{r^2}}$$

$$[Z^n] = \int d\Omega_n \int_0^\infty dr \, r^{n-1}$$

$$\int \frac{dw}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{w^2}{2\sigma^2} - \frac{\beta m w^2}{2} r^2}$$

$$= \Omega_n \int_0^\infty dr \, r^{n-1} \int \frac{dw}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{w^2}{2} \underbrace{\left(\frac{1}{\sigma^2} + \beta m r^2\right)}_{\frac{1}{\bar{\sigma}^2}}} \qquad \frac{\sqrt{2\pi\bar{\sigma}^2}}{\sqrt{2\pi\bar{\sigma}^2}}$$

$$= \Omega_n \int_0^\infty dr \; r^{n-1} \frac{1}{\sqrt{\cancel{2\pi}\sigma^2}} \sqrt{\frac{\cancel{2\pi}}{\frac{1}{\sigma^2} + \beta m r^2}}$$

$$= \Omega_n \int_0^\infty \underbrace{dr \; r^{n-1}}_{r\,dr \;\; r^{n-2}} \sqrt{\frac{1}{1 + \beta m \sigma^2 r^2}}$$

CHANGE VARIABLES $\quad u^2 = \beta m \sigma^2 r^2$

$$\cancel{2} u \, du = \beta m \sigma^2 \cancel{2} r \, dr$$

$$= \Omega_n \int_0^\infty \frac{u\,du}{\beta m \sigma^2} \left(r^{n-2} \right) \sqrt{\frac{1}{1+u^2}}$$

$$= \frac{\Omega_n}{\beta m \sigma^2} \int_0^\infty du \; u \left(\frac{u^2}{\beta m \sigma^2} \right)^{\frac{n-2}{2}} \sqrt{\frac{1}{1+u^2}}$$

$$= \frac{\Omega_n}{\cancel{\beta m \sigma^2}} \int_0^\infty \frac{du}{(\beta m \sigma^2)^{n/2 - \cancel{1}}} \frac{u^{\cancel{1} + n - \cancel{2}\ 1}}{\sqrt{1+u^2}}$$

$$= \frac{\Omega_n}{(\beta m \sigma^2)^{n/2}} \underbrace{\int_0^\infty \frac{du\ u^{n-1}}{\sqrt{1+u^2}}}_{\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\sqrt{\pi}}}$$

DIVERGES FOR $n \to 0$ PROBLEM !

USING $\Omega_n = \dfrac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)} = \dfrac{\pi^{n/2}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}$

CHECK $n=2$: $\Omega_2 = \pi / \Gamma(2) = \pi$ $\underline{\text{ok}}$

$n=3$: $\Omega_3 = \dfrac{\pi^{3/2}}{\Gamma(5/2)} = \dfrac{4\pi^{3/2}}{3\pi^{1/2}} = \dfrac{4}{3}\pi$ $\underline{\text{ok}}$

FOR $n \to 0$ KEEPING $\mathcal{O}(n)$:

$$\Omega_n \to \frac{1 + \frac{n}{2}\ln\pi}{1 - c\,\frac{n}{2}} = \left(1 + \frac{n}{2}\ln\pi\right)\left(1 + \frac{n}{2}c\right)$$

$$= 1 + \frac{n}{2}\left(\ln\pi + c\right)$$

$$(\beta m\sigma^2)^{n/2} \longrightarrow 1+\frac{n}{2}\ln(\beta m\sigma^2)$$

RATIO $$1+\frac{n}{2}\left[\ln\pi + c - \ln(\beta m\sigma^2)\right]$$

$$= 1 + n\frac{c}{2} + n\ln\sqrt{\frac{\pi}{\beta m\sigma^2}}$$

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

$$Z_\omega = \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta m\omega^2}{2}x^2}$$

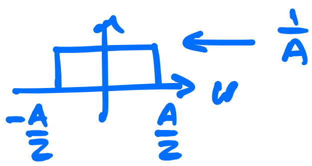
GAUSSIAN INTEGRAL

$\sigma^2 = 1/\beta m\omega^2$

RESULT $\sqrt{2\pi\sigma^2}$

$$Z_\omega = \sqrt{\frac{2\pi}{\beta m\omega^2}}$$

NOW CONSIDER A FLAT DIST $p(\omega) =$



$$[\ln Z_\omega] = \int_{-A/2}^{A/2} d\omega\, \frac{1}{A}\ln\sqrt{\frac{2\pi}{\beta m\omega^2}}$$

$$\left[\qquad \int^{A/2} \qquad \right]$$

$$= \frac{1}{A}\left[\ln\sqrt{2\pi}\, A - 2\int_0 dw\, \frac{1}{2}\ln(\beta m w^2)\right]$$

$$= \ln\frac{\sqrt{2\pi}}{\sqrt{\beta m}} - \frac{1}{\cancel{2}A}\, 2\int_0^{A/2} dw\, \cancel{2}\ln w$$

$$= \ln\sqrt{\frac{2\pi}{\beta m}} - \frac{1}{A}\int_0^{A/2} dw\, \ln w$$

$y = \ln w \qquad dy = \frac{1}{w}dw \qquad w = e^y \qquad dw = e^y dy$

$$= \ln\sqrt{\frac{2\pi}{\beta m}} - \frac{1}{A}\int_{\text{"}\ln 0\text{"}}^{\ln A/2} dy\, e^y\, y$$

DOESN'T REALLY HELP

RANDOM PERTURBATION $w \to w + \delta w$

δw GAUSSIAN DIST

$$[Z^n] = \int \frac{d\delta w}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\delta w)^2}{2\sigma^2}}$$

$$\int dx_1 \dots \int dx_n\, e^{-\beta\frac{1}{2}m(w+\delta w)^2\sum_{a=1}^{n} x_a^2}$$

n

$$= \int \prod_{a=1}^{n} dx_a \underbrace{\int \frac{d\delta\omega}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\delta\omega)^2}{2\sigma^2} - \frac{\beta m}{2}(\omega+\delta\omega)^2 \sum_{a=1} x_a^2}}_{\text{GAUSSIAN INTEGRAL}}$$

$$\int \frac{dz}{\sqrt{2\pi\sigma^2}}\, \underbrace{e^{-\frac{z^2}{2\sigma^2} - A(z+z_0)^2}}$$

$$e^{-\frac{z^2}{2\sigma^2} - Az^2 - 2Az_0 z - Az_0^2}$$

$$e^{-Az_0^2} \cdot e^{-\frac{1}{2\sigma^2}\left[z^2 + A2\sigma^2 z^2 + 2Az_0 2\sigma^2 z\right]}$$

$$= e^{-Az_0^2}\, e^{-\frac{1}{2\sigma^2}\left[z^2(1+2\sigma^2 A) + 4Az_0\sigma^2 z - \frac{(2Az_0\sigma^2)^2}{1+2\sigma^2 A} + \frac{(2Az_0\sigma^2)^2}{1+2\sigma^2 A}\right]}$$

$$= e^{-Az_0^2 - \frac{1}{2\sigma^2}\frac{(2Az_0\sigma^2)^2}{1+2\sigma^2 A}}\ \frac{1}{\sqrt{1+2\sigma^2 A}}$$

$z_0 = \omega$ & $A = \beta m \sum_{a=1}^{n} x_a^2$

THEN

$$[Z^n] = \int \prod_{a=1}^{n} dx_a$$

$$e^{-\beta m \sum_{a=1}^{n} x_a^2 - \frac{1}{2\bar{\sigma}^2} \frac{\left(2\beta m \sum_{a=1}^{n} x_a^2 w \bar{\sigma}^2\right)^2}{1+2\bar{\sigma}^2 \beta m \sum_{a=1}^{n} x_a^2}} \quad \frac{1}{\sqrt{1+2\bar{\sigma}^2 \beta m \sum_{a} x_a^2}}$$

BACK TO THE FREE-ENERGY EXPRESSION

$$-\beta[F_\omega] = \lim_{n\to 0} \frac{[z^n]-1}{n}$$

$$\boxed{-\beta[F_\omega] = -\ln\left(\sqrt{\frac{\beta m\sigma^2}{\pi}}\right) + \frac{C}{2}}$$

<u>CHECK FACTOR 2</u> !

TO COMPARE WITH

$$\beta[F_\omega] = \ln\sqrt{\frac{\beta m\sigma^2}{4\pi}} - \frac{C}{2}$$