Advanced Statistical Physics: Quenched random systems

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No material is perfectly homogeneous: *impurities* of different kinds are distributed randomly throughout the samples. In ultra-cold atom systems, so much studied nowadays, disorder can be realised, for example, using speckle laser light. It is quite natural to expect that disorder will change the equilibrium and dynamical properties of the systems on which it acts.

A natural effect of disorder should be to lower the critical temperature of a macroscopic interacting system. Much attention has been payed to the effect of *weak disorder* on phase transitions, that is to say, situations in which the nature of the ordered and disordered phases is not modified by the impurities but the critical phenomenon is. On the one hand, the critical exponents of second order phase transitions might be modified by disorder, on the other hand, disorder may smooth out the discontinuities of first order phase transitions rendering them of second order. *Strong disorder* instead changes the nature of the low-temperature phase and before discussing the critical phenomenon one needs to understand how to characterise the new ordered 'glassy' phase.

In this Section we shall discuss several types of *quenched disorder* and models that account for it. We shall also overview some of the theoretical methods used to deal with the static properties of models with quenched disorder, namely, scaling arguments and the droplet theory, mean-field equations, and the replica method.

1.1 Quenched and annealed disorder

Imagine that one mixes some random impurities in a melt and then very slowly cools it down in such a way that the impurities and the host remain in thermal equilibrium. If the interest sets on the statistical properties of the full system, one has to compute the full partition function in which a sum over all configurations of the host components and the impurities has to be performed. This is called *annealed disorder*.

In the opposite case in which upon cooling the impurities do not equilibrate with the host nor with the environment but remain blocked in random fixed positions, one talks about *quenched disorder*. Basically, the relaxation time associated with the diffusion of the impurities in the sample is so long that these remain trapped:

$$t_{\rm obs} \sim 10^4 \text{ sec} \ll t_{\rm diff} \; . \tag{1.1}$$

As concerns the host variables, they have their own, typically microscopic time scale that is much smaller than the observational time scale and therefore fluctuate. For example, in magnetic system, this time scale is the typical time-scale needed to reverse a spin, $\tau_o \sim 10^{-12} - 10^{-15}$ sec $\ll t_{\rm obs}$.

The annealed case is easier to treat analytically but it brings in less theoretical novelties and, in many cases of interest, is of less physical relevance. The quenched one is the one that leads to new phenomena and ideas that we shall discuss next.

Quenched disorder is static. Instead, in annealed disorder the impurities are in thermal equilibrium in the experimental time-scales, and they can simply be included in the statistical mechanic description of the problem, by summing over their degrees of freedom in the partition function.

1.2 Properties

Let us list a number of properties of systems with frozen-in randomness.

1.2.1 Lack of homogeneity

It is clear that the presence of quenched disorder, in the form of random interactions, fields, dilution, *etc.* breaks spatial homogeneity and renders single samples heterogeneous. Homogeneity is recovered though, if one performs an average of all possible realisations of disorder, each weighted with its own probability.

1.2.2 Frustration

We already discussed *frustration* in the context of magnetic models without disorder. It is quite clear that disorder will also introduce frustration in magnetic (and other) systems.

An example of an Ising model with four sites and four links is shown in Fig. 1.1-left, where we took three positive exchanges and one negative one all, for simplicity, with the same absolute value, J. Four configurations minimise the energy, $E_f = -2J$, but none of them satisfies the lower link. One can easily check that any closed loop such that the product of the interactions takes a negative sign is frustrated. Frustration naturally leads to a *higher energy* and a *larger degeneracy* of the number of ground states. This is again easy to grasp by comparing the number of ground states of the frustrated plaquette in Fig. 1.1-left to its unfrustrated counterpart shown on the central panel. Indeed, the energy and degeneracy of the ground state of the unfrustrated plaquette are $E_u = -4J$ and $n_u = 2$, respectively.

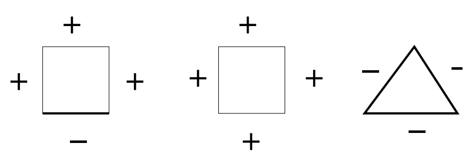


Figure 1.1: A frustrated (left) and an unfrustrated (center) square plaquette for an Ising model with nearest-neighbour interactions. A frustrated triangular plaquette (right).

Frustration may also be due to pure geometrical constraints. The canonical example is an antiferromagnet on a triangular lattice in which each plaquette is frustrated, see Fig. 1.1-right.

In short, frustration arises when the geometry of the lattice and/or the nature of the interactions make impossible the simultaneous minimisation of the energy of all pair couplings between the spins. Any loop of connected spins is said to be frustrated if the product of the signs of connecting bonds is negative. In general, energy and entropy of the ground states increase due to frustration.

Later in this Section, in Eq. (1.19), we will introduce the interaction energy between any pair of spins in a spin-glass sample. Depending on the value of the distance r_{ij} the numerator in this expression can be positive or negative implying that both ferromagnetic and antiferromagnetic interactions exist. This leads to frustration, which means that in any configuration some two-body interactions remain unsatisfied. In other words, there is no spin configuration that minimises all terms in the Hamiltonian.

1.2.3 Random parameters

Each given sample has its own peculiar realisation of the quenched disorder (the interactions between the fluctuating and the frozen-in variables) that is determined by the way in which the sample was prepared. It would be illusory, and quite impossible, to know all details about it. The idea put forward by several theoreticians is to consider that quenched disorder will be typical and hence modelled by random exchanges, fields or potentials, taken from time-independent *probability distribution functions*. What one has to determine first are the characteristics of these distributions (functional form, mean, variance, momenta). Having this modelisation in mind, one then talks about *quenched randomness*.

Exercise 1.1 This exercise provides a useful example of the distinction between *typical* and *average* values of random variables. Consider a random variable z that takes only two values $z_1 = e^{\alpha\sqrt{N}}$ and $z_2 = e^{\beta N}$, with α and β two positive and finite numbers with α unconstrained and $\beta > 1$. The probabilities of the two events are $p_1 = 1 - e^{-N}$ and $p_2 = e^{-N}$. First, confirm that these probabilities are normalised. Second, compute the average $\langle z \rangle$, where the angular brackets indicate average with the probabilities p_1, p_2, and evaluate it in the limit $N \to \infty$. Third, calculate the most probable value taken by z, that we call $z_{\rm typ}$, for typical (indeed, if we were to draw the variable we would typically get this value). Compare and conclude. Now, let us study the behaviour of the quantity $\ln z$ that is also a random variable. Compute its average. By which value of z is it determined? Does $\langle \ln z \rangle = (\ln z)_{\rm typ}$ in the large N limit? Is $\langle \ln z \rangle = \ln \langle z \rangle$? The last result demonstrates the difference between what are called *quenched* and *annealed* averages. Which value is larger? Does the comparison comply with Jensen's inequality? (See App. 1.A.8 for its definition.)

Exercise 1.2 Take two independent identically distributed (*i.i.d.*) random variables x and y that can take values 0 and 2 with probability a half. What are the typical and average values of the product $z = xy$? Are they equal?

Exercise 1.3 Consider a distribution function, $p(x)$, with support on a finite interval centred at $x = 0$. Is the average, $\langle x \rangle$, equal to the typical, $x_{\rm typ}$, value? Under which conditions?

Exercise 1.4 Are the average and typical values of an asymmetric probability distribution function always different? If not, give an example of $p(x)$ with $\langle x \rangle = x_{\rm typ}$.

1.2.4 Self-averageness

Say that the quenched randomness is given by random exchanges (*i.e.*, random J_{ij} in an Ising model). If each sample is characterised by its own realisation of the exchanges, should one expect a totally different behaviour from sample to sample? Fortunately, many generic static and dynamic properties of spin-glasses (and other systems with quenched disorder) do not depend on the specific realisation of the random couplings and are *self-averaging*.

Owing to the fact that each disorder configuration has a probability of occurrence, each physical quantity A depends on it and has a probability distribution $P(A)$ given by $P(A) = \sum_J p(J)\delta(A - A_J)$ where we denoted J a generic disorder realisation. When the size of the system increases one expects (and even proves in some cases) that the distribution $P(A)$ becomes narrower and narrower. Therefore the only quantity which can be observed in the thermodynamic limit is the most probable, or typical value, $A_J^{\rm typ}$, the value around which most of the distribution is concentrated.

For some quantities the typical value is very close to the average over the disorder and

$$A_J^{\rm typ} = [\, A_J \,] \tag{1.2}$$

in the thermodynamic limit. Henceforth, we use square brackets to indicate the average over the random couplings. More precisely, in self-averaging quantities *sample-to-sample fluctuations* with respect to the mean value are expected to be $\mathrm{O}(N^{-a})$ with N the number of variables in the system and $a > 0$. Roughly, observables that involve summing over the entire volume of the system are expected to be self-averaging. In particular, the free-energy density of models with short-ranged interactions is expected to be self-averaging in the infinite size limit. There can be, though, in the same system quantities for which $B_J^{\rm typ} \neq [\, B_J \,]$ even in the thermodynamic limit. We will show examples of both below.

An example: the disordered Ising chain

The meaning of this property can be grasped from the solution of the random bond Ising chain defined by the energy function $H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1}$ with spin variables $s_i = \pm$, for $i = 1, \ldots, N$ and random bonds J_i independently taken from a probability distribution $P(J_i)$. For simplicity, we consider periodic boundary conditions. The disorder-dependent partition function reads

$$Z_J = \sum_{\{s_i = \pm 1\}} e^{\beta \sum_i J_i s_i s_{i+1}} \tag{1.3}$$

and this can be readily computed introducing the change of variables $\sigma_i \equiv s_i s_{i+1}$. (Note that these new variables are not independent, since they are constrained to satisfy $\prod_i \sigma_i = 1$ and one should take it into account to perform the sum. However, this constraint becomes irrelevant in the thermodynamic limit and one can simply ignore it.) One finds

$$Z_J = \prod_i 2\cosh(\beta J_i) \qquad \Rightarrow \qquad -\beta F_J = \sum_i \ln\cosh(\beta J_i) + N \ln 2 \; . \tag{1.4}$$

The partition function is a *product* of *i.i.d.* random variables and it is itself a random variable. If the random exchanges have finite mean and variance the partition function has a log-normal distribution. The free-energy density instead is a *sum* of *i.i.d.* random variables and, using the central limit theorem, in the large N limit becomes a Gaussian random variable narrowly peaked at its maximum. The typical value, given by the maximum of the Gaussian distribution, coincides with the average, $\lim_{N\to\infty}(f_J^{\rm typ} - [\, f_J \,]) = 0$ with $f_J = F_J/N$.

Exercise 1.5 Take a one dimensional Ising model with a Gaussian probability distribution of the interaction strengths J_i, with zero mean and variance J^2. Draw histograms of the partition function Z_J, the total free-energy F_J, and the free-energy density f_J. Study these for increasing value of N. Conclude. Repeat the analysis for different probability distributions of the interaction strengths. In particular, consider distribution functions with fat tails, that is to say, with power law decays. What is the difference?

Exercise 1.6 Take a particle with mass m in a one dimensional harmonic potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 \tag{1.5}$$

with the real frequency ω taken from a probability distribution $p(\omega)$. The position of the particle is given by the real variable x.

1. Compute the free-energy at fixed ω and inverse temperature $\beta = 1/(k_B T)$, with k_B Boltzmann's constant, focusing only on the potential energy part.
2. Find an expression for the probability distribution of F_ω for a generic $p(\omega)$.
3. Are the fluctuations Gaussian?
4. Express the disorder averaged free-energy $[F_\omega]$ for a generic distribution of ω.
5. Calculate the disorder averaged free-energy $[F_\omega]$ for a Gaussian distribution of ω with zero mean and variance σ^2.
6. Explain how the typical value of the free-energy, $F_{\rm typ}$, should be obtained, again for a generic $q(\omega)$.
7. Determine $F_{\rm typ}$ for a Gaussian distribution of ω with zero mean and variance σ^2.
8. Explain the self-averaging property and the conditions under which we proved it in the lectures.
9. Is this model self-averaging? Discuss the result found for the simple harmonic oscillator clearly and justify your answer. Do you expect a phase transition? Why?

A useful integral for this exercise is $\int_0^\infty dy\, e^{-ay^2} \ln y = -\frac{1}{4}\left(C + \ln 4a\right)\sqrt{\frac{\pi}{a}}$ where C is a constant given by $-C = \int_0^1 dx\ \ln\ln 1/x$.

General argument

A simple argument justifies the self-averageness of the free-energy density in generic finite dimensional systems with short-range interactions. Let us divide a, say, cubic system of volume $V = L^d$ in n subsystems, say also cubes, of volume $v = \ell^d$ with $V = nv$. If the interactions are short-ranged, the total free-energy is the sum of two terms, a contribution from the bulk of the subsystems and a contribution from the interfaces between the subsystems: $-\beta F_J = \ln Z_J = \ln \sum_{\rm conf} e^{-\beta H_J({\rm conf})} \approx \ln \sum_{\rm conf} e^{-\beta H_J({\rm bulk}) - \beta H_J({\rm surf})} = \ln \sum_{\rm bulk} e^{-\beta H_J({\rm bulk})} + \ln \sum_{\rm surf} e^{-\beta H_J({\rm surf})} = -\beta F_J^{\rm bulk} - \beta F_J^{\rm surf}$. We have already neglected the contributions from the interaction between surface and bulk. If, moreover, the interaction extends over a short distance σ and the linear size of the boxes is $\ell \gg \sigma$, the surface energy is negligible with respect to the bulk one, same for the entropic contributions to the free-energy, and $-\beta F_J \approx \ln \sum_{\rm bulk} e^{-\beta H_J({\rm bulk})}$. In the thermodynamic limit, the disorder dependent free-energy is then a sum of $n = (L/\ell)^d$ independent random numbers, each one being the disorder dependent free-energy of the bulk of each subsystem: $-\beta F_J \approx \sum_{k=1}^n \ln \sum_{{\rm bulk}_k} e^{-\beta H_J({\rm bulk}_k)}$. In the limit of a very large number of subsystems ($L \gg \ell$ or $n \gg 1$) the central limit theorem (see App. 1.A.9) implies that the free-energy density is Gaussian distributed with the maximum reached at a value $f_J^{\rm typ}$ that coincides with the average over all realisations of the randomness $[\, f_J \,]$. Moreover, the dispersion about the typical value of the total free-energy vanishes in the large n limit, $\sigma_{F_J}/[\, F_J \,] \propto \sqrt{n}/n = n^{-1/2} \to 0$ and the one of the x free-energy density, or intensive free-energy, $f_J = F_J/N$, as well, $\sigma_{f_J}/[f_J] = O(n^{-1/2})$. In a sufficiently large system the typical f_J is then very close to the averaged $[\, f_J \,]$ and one can compute the latter to understand the static properties of typical systems. This is very convenient from a calculational point of view.

Exercise 1.7 Take a two dimensional Ising model with a Gaussian probability distribution of the interaction strengths J_{ij}, with zero mean and variance J^2. Draw histograms of the partition function Z_J, the total free-energy F_J, and the free-energy density f_J. Study these for increasing value of N. Conclude. Repeat the analysis for different probability distributions of the interaction strengths. In particular, consider distribution functions with fat tails, that is to say, with power law decays. What is the difference?

Lack of self-averageness in the correlation functions

Once one has $[f_J]$, one derives all disordered average thermal averages by taking derivatives of the disordered averaged free-energy with respect to sources introduced in the partition function. For example,

$$[\, \langle \, s_i \, \rangle \,] = - \left. \frac{\partial [\, F_J \,]}{\partial h_i} \right|_{h_i=0} , \tag{1.6}$$

$$[\, \langle \, s_i s_j \, \rangle - \langle \, s_i \, \rangle \langle \, s_j \, \rangle \,] = -T \left. \frac{\partial^2 [\, F_J \,]}{\partial h_i h_j} \right|_{h_i=0} , \tag{1.7}$$

with $H_J \to H_J - \sum_i h_i s_i$. Connected correlation functions, though, *are not* self-averaging quantities. This can be seen, again, studying the random bond Ising chain. Take $i < j$. One can easily check that

$$\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = Z_J^{-1} \frac{\partial}{\partial \beta J_{j-1}} \dots \frac{\partial}{\partial \beta J_i} Z_J = \tanh(\beta J_i) \dots \tanh(\beta J_j) \;, \tag{1.8}$$

where we used $\langle s_i \rangle = 0$ (valid for a distribution of random bonds with zero mean) and the dots indicate all sites on the chain between the ending points i and j, *i.e.* $i+1 \leq k \leq j-1$. The last expression is a product of random variables and it is not equal to its average (1.7) – not even in the large separation limit $|\vec{r}_i - \vec{r}_j| \to \infty$.

See the **TD** on correlation functions for more details on them.

1.2.5 Annealed disorder

The thermodynamics of a system with annealed disorder is obtained by averaging the partition function over the impurity degrees of freedom,

$$Z = [\, Z_J \,] \tag{1.9}$$

since one needs to do the partition sum over the disorder degrees of freedom as well. In general, this calculation does not present any particular difficulty. In some cases, the annealed average gives a good description of the high temperature phases of problems with quenched randomness but it fails to predict the phase transition and the low temperature properties correctly.

In general, one can prove

$$f^{\text{quenched}} \geq f^{\text{annealed}} \;. \tag{1.10}$$

Exercise 1.8 Prove this inequality using Jensen's inequality (see App. 1.A.8).

1.3 Random geometries

In this Section we very briefly present some problems in which the quenched randomness is of pure geometric (and not energetic) origin.

1.3.1 Percolation

The understanding of *fluid flow in porous media* needs, as a first step, the understanding of the static geometry of the connected pores. The typical example, that gave the name to the problem, is coffee percolation, where a solvent (water) filter or trickle through the permeable substance that is the coffee grounds and in passing picks up soluble constituents (the chemical compounds that give coffee its color, taste, and aroma).

Another problem that needs the comprehension of a static random structure is the one of *conduction across a disordered sample*. Imagine that one mixes randomly a set of conducting and insulating islands. Whether the mix can conduct an electric current from one end to the other of the container is the question posed, and the answer depends on the structure formed by the conducting islands.

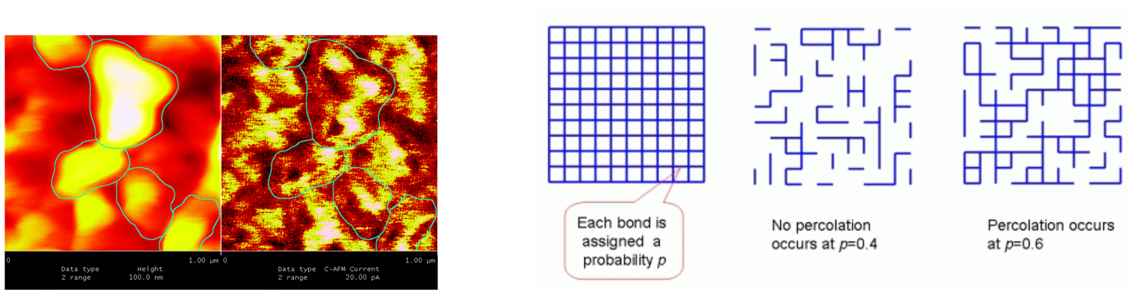


Figure 1.2: Left: a measurement of the topography (left) and local current (right) in an inhomogeneous mixture of good and bad conducting polymers. The brighter the zone the more current passing through it. Several grains are contoured in the left image. Right: an example of bond percolation.

Percolation [3, 4, 5, 6] is a simple *geometric* problem with a *critical threshold*. It is very helpful since it allows one to become familiar with important concepts of *critical phenomena* such as fractals, scaling, and renormalisation group theory in a very intuitive way. Moreover, it is not just a mathematical model, since it is at the basis of the understanding of the two physical problems mentioned above among many others.

Site dilute lattices with missing vertices are intimately related to the site percolation problem. Imagine that one builds a lattice by occupying a site with probability p (and not occupying it with probability $1-p$). For $p = 0$ the lattice will be completely empty while for $p = 1$ is will be totally full. For intermediate values of p, on average, order pL^d sites will be occupied, with L the linear size of the lattice. *Site percolation theory* is about the geometric and statistical properties of the structures thus formed. In particular, it deals with the behaviour of the clusters of nearest neighbour occupied sites.

Similarly, one can construct bond dilute lattices and compare them to the *bond percolation* problem.

The site percolation problem describes, for example, a binary alloy or dilute ferromagnetic crystal, also called a doped ferromagnet. The question in this context is how much dilution is needed to destroy the ferromagnetic order in the sample at a given temperature. The bond percolation problem corresponds to a randomly blocked maze through which the percolation of a fluid can occur. Many other physical problems can be set in terms of percolation: the distribution of grain size in sand and photographic emulsions, the vulcanisation of rubber and the formation of cross-linked gels, the propagation of an infection, etc.

The main interest lies on characterising the statistical and geometric properties of the *clusters* on a lattice of linear size L as a function of the probability p. The clusters are connected ensembles of nearest neighbour sites. Their easiest geometric property is their *size*, defined as the number of sites that compose them. Other geometric properties are also interesting and we will define them below.

The percolation problem is specially interesting since it has a threshold phenomenon, with a critical value p_c at which a first spanning cluster that goes from one end of the lattice to the opposite in at least one of the Cartesian directions appears. For $p < p_c$ there are only finite clusters, for $p > p_c$ there is a *spanning cluster* as well as finite clusters.

The first natural question is whether the value p_c depends on the particular sample studied or not, that is to say, whether it suffers from *sample-to-sample fluctuations*. All samples are different as the sites erased or the links cut are not the same. The threshold value is therefore a random variable and it does not take the same value for different samples. The 'surprise' is that the mean-square deviations of p_c from its mean value vanish as a power law with the system size,

$$\delta^2_{p_c}(N) \equiv \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} (p_c^{(k)} - \overline{p}_c)^2 \simeq C^2 N^{-\nu} \; , \qquad \overline{p}_c \equiv \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} p_c^{(k)} \; , \tag{1.11}$$

with k labelling different measurements and $\mathcal{N}$ counting its total number. N the number of sites in the sample. (C turns out to be 0.54 and $\nu = 1.3$ in $d = 2$.) In the infinite system size limit, p_c does not fluctuate from sample to sample.

One can then count the number of sites belonging to the largest cluster and compare this number to the total number of sites in the sample:

$$r_L(p) \equiv \frac{N_{\max}(p)}{N} \; . \tag{1.12}$$

This is, again, a fluctuating quantity that, in the infinite system size limit does no longer fluctuate and defines

$$r_\infty \equiv \lim_{L\to\infty} r_L(p) \; . \tag{1.13}$$

The precise definition of the *critical threshold* p_c involves the infinite size limit and it can be given by

$$r_\infty(p) = \lim_{L\to\infty} r_L(p) = \begin{cases} 0 & \text{for} \quad p < p_c \\ > 0 & \text{for} \quad p > p_c \end{cases} \tag{1.14}$$

where $r_\infty(p)$ denotes the fraction of sites belonging to the largest cluster in the finite lattice with linear size L. In the magnetic application of percolation, this means that the magnetisation vanishes for $p < p_c$ and it takes the value that the magnetisation takes on the largest cluster for $p > p_c$ (as in both cases the magnetisation on the finite clusters is independent and averages to zero).

An equivalent definition of the critical threshold p_c is given by

$$P_\infty(p) = \lim_{L\to\infty} P_L(p) = \begin{cases} 0 & \text{for} \quad p < p_c \\ 1 & \text{for} \quad p > p_c \end{cases} \tag{1.15}$$

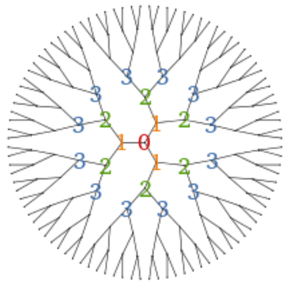


Figure 1.3: A Bethe lattice with coordination number $z = 3$. The root is labeled 0 and the generations 1, 2 and 3 are shown with different colours. Figure taken from Wikipedia.

where $P_L(p)$ denotes the probability of there being a percolating cluster in the finite lattice with linear size L.

The percolation threshold p_c depends on the lattice geometry and its dimensionality. Moreover, it is not the same for bond percolation and site percolation. Exact results are known for special lattices as the Cayley tree. Examples of how these results are found are given in [3]. Numerical data for finite dimensional lattices are complemented by rigorous upper and lower bounds and the outcome of series expansions for the mean cluster value. Harris showed that $p_c \geq 1/2$ for the bond percolation problem on a planar square lattice and the numerics suggests $p_c = 1/2$. Fisher put several bounds on p_c on various $2d$ lattices for the site and bond problem. In particular, $p_c \geq 1/2$ for site percolation on planar regular lattices with no crossings.

1.3.2 Bethe lattices and random graphs

The *Bethe lattice* is a tree, in which each site has z neighbours and each branch gives rise to $z - 1$ new branches, see Fig. 1.3. Two important properties of these lattices are:
- there are no closed loops.
- the number of sites on the border is of the same order of magnitude as the total number of sites on the lattice.
- It is a *rooted tree*, with all other nodes arranged in shells around the root node, also called the origin of the lattice.

Exercise 1.9 Show that the total number of sites on the Bethe lattice with $z = 3$ and g generations (or the distance from the site designed as the central one) is $n_{\text{tot}} = 3\ 2^{g-1}$. Prove that the surface to volume ratio tends to 1/2.

Due to its distinctive topological structure, the statistical mechanics of lattice models on this graph are often exactly solvable.

Exercise 1.10 Take a hypercubic lattice in d dimensions and estimate the surface to volume ratio. Show that this ratio tends to a finite value only if $d \to \infty$.

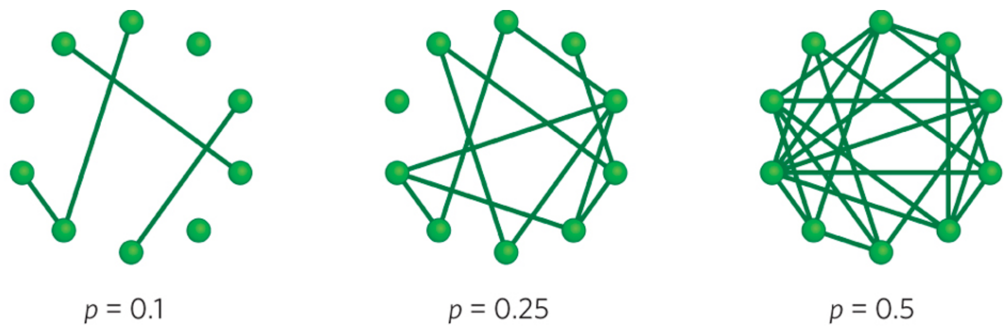


Figure 1.4: Random graphs with $N = 10$ and different probabilities p of joining two nodes.

A *random graph* is obtained by starting with a set of n isolated vertices and adding successive edges between them at random. A popular ensemble is the one denoted $G(n, p)$, in which every possible edge occurs independently with probability $0 < p < 1$. Random graphs with fixed connectivity are also commonly used.

Random graphs are used in social sciences modeling (nodes representing individuals and edges the friendship relationship), technology (interconnections of routers in the Internet, pages of the WWW, or production centers in an electrical network), biology (interactions of genes in a regulatory network) [1, 2]. Disordered systems are usually defined on random graphs, especially the ones motivated by combinatorial optimisation.

1.4 Energetic models

Let us briefly describe here some representative models with quenched randomness.

1.4.1 Dilute spin models

Lattice models with site or link dilution are [7]

$$H_J^{\text{site dil}} = -J \sum_{\langle ij \rangle} s_i s_j \epsilon_i \epsilon_j \; , \qquad H_J^{\text{link dil}} = -J \sum_{\langle ij \rangle} s_i s_j \epsilon_{ij} \; , \tag{1.16}$$

with $P(\epsilon_i = 1, 0) = p, 1-p$ in the first case and $P(\epsilon_{ij} = 1, 0) = p, 1-p$ in the second. These models are intimately related to Percolation theory [3, 4, 5, 6]. Physically, dilution is realised by vacancies or impurity atoms in a crystal.

Exercise 1.11 Take the site dilute Ising model under a uniform magnetic field defined on any graph. Prove that its annealed average is equivalent to the spin one Blume-Capel model under a field. Here are some hints to prove this statement. Write the disorder average partition function of the dilute model as

$$[Z_\epsilon] = \sum_{n=0}^{N} p^n (1-p)^{N-n} Z_n \tag{1.17}$$

with $Z_n = \sum_{\{s_i=\pm 1\}} \sum_{\{\epsilon_i=1,0\}} \exp(-\beta H_J^{\text{site dil}}[\{s_i\}])$ where $n = \sum_i \epsilon_i$ is the occupation number. Rewrite it as

$$[Z_\epsilon] = \sum_{\{\epsilon_i\}} \sum_{\{s_i\}} \exp(-\beta H_J^{\text{site dil}}[\{s_i\}] - \beta\mu \sum_i \epsilon_i) \tag{1.18}$$

with $\mu = \beta^{-1}[\ln p - \ln(1-p)]$. Now, use a transformation of variables between $\{s_i, \epsilon_i\}$ and $\sigma_i = \epsilon_i s_i$, for all i and identify, after this change, the partition function of the Blume-Capel model.

1.4.2 Spin-glass models

Spin-glasses are alloys in which magnetic impurities substitute the original atoms in positions randomly selected during the chemical preparation of the sample [8, 9, 10, 11, 12, 13, 14]. The interactions between the impurities are of RKKY type:

$$V_{\text{rkky}} = -J \, \frac{\cos(2k_F r_{ij})}{r_{ij}^3} \, s_i s_j \tag{1.19}$$

with $r_{ij} = |\vec{r}_i - \vec{r}_j|$ the distance between them and s_i a spin variable that represents their magnetic moment. Clearly, the initial location of the impurities varies from sample to sample. The time-scale for diffusion of the magnetic impurities is much longer than the time-scale for spin flips. Thus, for all practical purposes the positions $\vec{r}_i$ can be associated to quenched random variables distributed according to a uniform probability distribution that in turn implies a probability distribution of the exchanges. This is called *quenched disorder*.

In early 70s *Edwards and Anderson* proposed a rather simple model that should capture the main features of spin-glasses [15]. The interactions (1.19) decay with a cubic power of the distance and hence they are relatively short-ranged. This suggests to put the spins on a regular cubic lattice model and to trade the randomness in the positions into random nearest neighbour exchanges, independently and identically distributed according to a Gaussian probability distribution function (pdf):

$$H_J^{\text{EA}} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad \text{with} \qquad P(J_{ij}) = (2\pi\sigma^2)^{-\frac{1}{2}} \, e^{-\frac{J_{ij}^2}{2\sigma^2}} \; . \tag{1.20}$$

The precise form of the probability distribution of the exchanges is supposed not to be important (though some authors claimed that there might be non-universality with respect to it, see however [16] where this is refuted at least in the random field Ising model case).

Another natural choice is to use bimodal exchanges

$$P(J_{ij}) = p\delta(J_{ij} - J_0) + (1-p)\delta(J_{ij} + J_0) \tag{1.21}$$

with the possibility of a bias towards positive or negative interactions depending on the parameter p. A tendency to non-zero average J_{ij} can also be introduced in the Gaussian pdf.

A natural extension of the EA model in which all spins interact has been proposed by Sherrington and Kirkpatrick

$$H_J^{\mathrm{SK}} = -\sum_{i\neq j} J_{ij} s_i s_j \tag{1.22}$$

and it is called the *SK model* [17]. The interaction strengths J_{ij} are taken from a Gaussian pdf and they scale with N in such a way that the thermodynamic limit is non-trivial:

$$P(J_{ij}) = (2\pi\sigma_N^2)^{-\frac{1}{2}}\, e^{-\frac{J_{ij}^2}{2\sigma_N^2}} \qquad\qquad \sigma_N^2 = J^2 N \ . \tag{1.23}$$

The first two-moments of the exchange distribution are $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2/N$. This is a case for which a mean-field theory is expected to be exact.

1.4.3 Neural networks

In the biological context disordered models have been used to describe *neural networks*, *i.e.* an ensemble of many neurons (typically $N \sim 10^9$ in the human brain) with a very elevated connectivity. Indeed, each neuron is connected to $\sim 10^4$ other neurons and receiving and sending messages *via* their axons. Moreover, there is no clear-cut notion of distance in the sense that axons can be very long and connections between neurons that are far away have been detected. Hebb proposed that the memory lies in the connections and the peculiarity of neural networks is that the connectivity must then change in time to incorporate the process of learning [18].

The simplest neural network models represent neurons with Boolean variables or spins, that either fire or are quiescent. The interactions link pairs of neurons and they are assumed to be symmetric (which is definitely not true). The memory of an object, action, *etc.* is associated to a certain pattern of neuronal activity. It is then represented by an N-component vector in which each component corresponds to the activity of each neuron (configuration of the spins). Finally, sums over products of these patterns constitute the interactions. One can then study, for example, the number of chosen specific patterns that the network can store and later recall, or one can try to answer questions on average, as how many typical patterns can a network of N neurons store. The models then become fully-connected or dilute models of spins in interaction with the exchanges

$$J_{ij} = \frac{1}{p}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \qquad\qquad \xi_i^\mu = \pm 1 \qquad \text{with prob } 1/2 \ . \tag{1.24}$$

ξ_i^μ are the components of an N vector labelled by μ, the μth pattern stored by the network. The quenched disorder is in the J_{ij}s. This is the *Hopfield model* [19], based on *Hebb's rule* [18], and more details can be found in the book [20].

1.4.4 Glass models

A further extension of the EA model is called the *p-spin model* [21]

$$H_J^{\text{p-spin}} = -\sum_{i_1<\dots<i_p} J_{i_1\dots i_p} s_{i_1}\dots s_{i_p} \tag{1.25}$$

with $p \geq 3$. The sum can also be written as $\sum_{i_1<i_2<\dots<i_p} = 1/p! \sum_{i_1\neq i_2 \neq \dots \neq i_p}$. The exchanges are now taken from a Gaussian probability distribution

$$P(J_{ij}) = (2\pi\sigma_N^2)^{-\frac{1}{2}}\, e^{-\frac{J_{ij}^2}{2\sigma_N^2}} \qquad\qquad \sigma_N^2 = J^2 p!/(2N^{p-1})\ . \tag{1.26}$$

with $[J_{i_1\dots i_p}] = 0$ and $[J^2_{i_1\dots i_p}] = \frac{J^2 p!}{N^{p-1}}$. Indeed, an extensive free-energy is achieved by scaling $J_{i_1\dots i_p}$ with $N^{-(p-1)/2}$. This scaling can be justified as follows. Imagine that at low temperatures the spins acquire local equilibrium expectation values that we call m_i. The 'local field' that they induce are $h_i = 1/(p-1)! \sum_{ii_2\neq i_p} J_{ii_2\dots i_p} m_{i_2}\dots m_{i_p}$ and they should be of order one. Contrary to ferromagnetic models, the m_i's take plus and minus signs in the disordered case as there is no tendency to align all moments in the same direction. In particular, we estimate the order of magnitude of this term by working at $T = 0$ and taking $m_i = \pm 1$ with probability $\frac{1}{2}$, since there is no external magnetic field nor a non-vanishing mean of the exchanges that could bias the local order in one or the other direction. In order to keep the discussion simple, let us take $p = 2$. In this case, if the strengths J_{ij}, are of order one, h_i is a sum of N *i.i.d.* random variables, with zero mean and unit variance,[1] and h_i is a random variable with zero mean and variance equal to N. Therefore, one can argue that h_i is of order $\sqrt{N}$. To make it finite we then chose J_{ij} to be of order $1/\sqrt{N}$ or, in other words, we impose $[J_{ij}^2] = J^2/N$. The generalization to $p > 2$ is straightforward.

We classify this model in the "glass" class since it has been shown that its behaviour mimics the one of so-called fragile glasses for $p > 2$ [22, 23, 24].

1.4.5 Vector spins and spherical models

Extensions to *vector spins* with two (XY), three (Heisenberg) or N components also exist. In the former cases can be relevant to describe real samples. One usually keeps the modulus of the spins fixed to be 1 in these cases.

There is another way to extend the spin variables and it is to use a *spherical constraint* [25, 26],

$$-\infty \leq s_i \leq \infty \qquad\qquad \sum_{i=1}^{N} s_i^2 = N\ . \tag{1.27}$$

[1]The calculation goes as follow: $\langle F_i \rangle = \sum_j J_{ij} \langle m_j \rangle = 0$ and $\langle F_i^2 \rangle = \sum_{jk} J_{ij} J_{ik} \langle m_j m_k \rangle = \sum_j J_{ij}^2$

In this case, the spins s_i are the components of an N-dimensional vector, constrained to be an N-dimensional sphere.

1.4.6 Optimization problems

Cases that find an application in *computer science* [27, 28] are defined on random graphs with fixed or fluctuating finite connectivity. In the latter case one places the spins on the vertices of a graph with links between couples or groups of p spins chosen with a probability c. These are *dilute spin-glasses* on graphs (instead of lattices).

Optimisation problems can usually be stated in a form that requires the minimisation of a cost (energy) function over a large set of variables. Typically these *cost functions* have a very large number of local minima – an exponential function of the number of variables – separated by barriers that scale with N and finding the truly absolute minimum is hardly non-trivial. Many interesting optimisation problems have the great advantage of being defined on random graphs and are then mean-field in nature. The mean-field machinery that we will discuss at length is then applicable to these problems with minor (or not so minor) modifications due to the finite connectivity of the networks.

Let us illustrate this kind of problems with two examples. The *graph partitioning* problem consists in, given a graph $G(N, E)$ with N vertices and E edges, to partition it into smaller components with given properties. In its simplest realisation the uniform graph partitioning problem is how to partition, in the optimal way, a graph with N vertices and E links between them in two (or k) groups of equal size $N/2$ (or N/k) and the minimal number of edges between them. Many other variations are possible. This problem is encountered, for example, in computer design where one wishes to partition the circuits of a computer between two chips. More recent applications include the identification of clustering and detection of cliques in social, pathological and biological networks.

Another example, that we will map to a spin model, is *k-satisfiability (k-SAT)*. The problem is to determine whether the variables of a given Boolean formula can be assigned in such a way to make the formula evaluate to 'TRUE'. Equally important is to determine whether no such assignments exist, which would imply that the function expressed by the formula is identically 'FALSE' for all possible variable assignments. In this latter case, we would say that the function is unsatisfiable; otherwise it is satisfiable.

We illustrate this problem with a concrete example. Let us use the convention x for the requirement $x =$ TRUE and $\overline{x}$ for the requirement $x =$ FALSE. For example, the formula $C_1 : x_1$ OR $\overline{x}_2$ made by a single clause C_1 is satisfiable because one can find the values x_1 = TRUE (and x_2 free) or x_2 = FALSE (and x_1 free), which make $C_1 : x_1$ OR $\overline{x}_2$ TRUE. This formula is so simple that 3 out of 4 possible configurations of the two variables solve it. This example belongs to the $k = 2$ class of satisfiability problems since the clause is made by two literals (involving different variables) only. It has $M = 1$ clauses and $N = 2$ variables.

Harder to decide formulæ are made of M clauses involving k literals required to take the true value (x) or the false value ($\overline{x}$) each, these taken from a pool of N variables. An example in 3-SAT is

$$\mathrm{F} = \begin{cases} C_1 \ : \ x_1 \ \text{OR} \ \overline{x}_2 \ \text{OR} \ x_3 \\ C_2 \ : \ \overline{x}_5 \ \text{OR} \ \overline{x}_7 \ \text{OR} \ x_9 \\ C_3 \ : \ x_1 \ \text{OR} \ \overline{x}_4 \ \text{OR} \ x_7 \\ C_4 \ : \ x_2 \ \text{OR} \ \overline{x}_5 \ \text{OR} \ x_8 \end{cases} \tag{1.28}$$

All clauses have to be satisfied simultaneously so the formula has to be read

$$\text{F: } C_1 \ \text{AND} \ C_2 \ \text{AND} \ C_3 \ \text{AND} \ C_4 \quad . \tag{1.29}$$

It is not hard to believe that when $\alpha \equiv M/N \gg 1$ the problems typically become unsolvable while many solutions exist for $\alpha \ll 1$. One could expect to find a sharp threshold between a region of parameters $\alpha < \alpha_c$ where the formula is satisfiable and another region of parameters $\alpha \geq \alpha_c$ where it is not.

In *random k-SAT* an instance of the problem, i.e. a formula, is chosen at random with the following procedure: first one takes k variables out of the N available ones. Second one decides to require x_i or $\overline{x}_i$ for each of them with probability one half. Third one creates a clause taking the OR of these k literals. Forth one returns the variables to the pool and the outlined three steps are repeated M times. The M resulting clauses form the final formula.

The Boolean character of the variables in the k-SAT problem suggests to transform them into Ising spins, i.e. x_i evaluated to TRUE (FALSE) will correspond to $s_i = 1$ (-1) . The requirement that a formula be evaluated TRUE by an assignment of variables (i.e. a configuration of spins) will correspond to the ground state of an adequately chosen energy function. In the simplest setting, each clause will contribute zero (when satisfied) or one (when unsatisfied) to this cost function. There are several equivalent ways to reach this goal. The fact that the variables are linked together through the clauses suggests to define k-uplet interactions between them. We then choose the interaction matrix to be

$$J_{ai} = \begin{cases} 0 & \text{if} \quad \text{neither } x_i \text{ nor } \overline{x}_i \ \in C_a \\ 1 & \text{if} \quad x_i \ \in C_a \\ -1 & \text{if} \quad \overline{x}_i \ \in C_a \end{cases} \tag{1.30}$$

and the energy function as

$$H_J[\{s_i\}] = \sum_{a=1}^{M} \delta(\sum_{i=1}^{N} J_{ai} s_i, -k) \tag{1.31}$$

where $\delta(x, y)$ is a Kronecker-delta that equals one when the arguments are identical and zero otherwise. This cost function is easy to understand. The Kronecker delta contributes one to the sum over a only if the k non-vanishing terms in the sum $\sum_{i=1}^{N} J_{ai} s_i$ are equal to -1. This can happen when $J_{ai} = 1$ and $s_i = -1$ or when $J_{ai} = -1$ and $s_i = 1$. In both cases the condition on the variable x_i is not satisfied. Since this is required from all the variables in the clause, the clause itself and hence the formula are not satisfied.

Another way to represent a clause in an energy function is to consider, for instance for C_1 above, the term $(1-s_1)(1+s_2)(1-s_3)/8$. This term vanishes if $s_1 = 1$ or $s_2 = -1$ or $s_3 = 1$ and does not contribute to the total energy, that is written as a sum of terms of this kind. It is then simple to see that the total energy can be rewritten in a way that resembles strongly physical spin models,

$$H_J[\{s_i\}] = \frac{M}{2^K} + \sum_{R=1}^{K} (-1)^R \sum_{i_1<\cdots<i_R} J_{i_1\ldots i_R} s_{i_1} \ldots s_{i_R} \tag{1.32}$$

and

$$J_{i_1\ldots i_R} = \frac{1}{2^K} \sum_{a=1}^{M} J_{ai_1} \ldots J_{ai_R} \; . \tag{1.33}$$

These problems are "solved" numerically, with algorithms that do not necessarily respect physical rules. Thus, one can use non-local moves in which several variables are updated at once – as in cluster algorithms of the Swendsen-Wang type used to beat critical slowing down close to phase transitions – or one can introduce a temperature to go beyond cost-function barriers and use dynamic local moves that do not, however, satisfy a detail balance. The problem is that with hard instances of the optimization problem none of these strategies is successful. Indeed, one can expect that glassy aspects, such as the proliferation of metastable states separated by barriers that grow very fast with the number of variables, can hinder the resolutions of these problems in polynomial time, that is to say a time that scales with the system size as N^ζ, for *any* algorithm. These are then *hard combinatorial problems*.

1.4.7 Random bond and random field ferromagnets

Let us now discuss some, *a priori* simpler cases. An example is the *Mattis random magnet* with generic energy (1.25) in which the interaction strengths are given by [29]

$$J_{i_1\ldots i_p} = \xi_{i_1} \ldots \xi_{i_p} \qquad \text{with} \qquad \xi_j = \pm 1 \quad \text{with} \quad \text{prob} = 1/2 \tag{1.34}$$

for any p and any kind of graph. In this case a simple *gauge transformation*, $\eta_i \equiv \xi_i s_i$, allows one to transform the disordered model in a ferromagnet, showing that there was no true frustration in the system.

Random bond ferromagnets (RBFMs) are systems in which the strengths of the interactions are not all identical but their sign is always positive. One can imagine such a exchange as the sum of two terms:

$$J_{ij} = J + \delta J_{ij} \; , \qquad \text{with } J > 0 \quad \text{and} \quad \delta J_{ij} \qquad \text{small and random} \; . \tag{1.35}$$

There is no frustration in these systems either. Ising models of this kind have also been used to describe fracture in materials, where the J_{ij} represents the local force needed to break the material and it is assumed the fracture occurs along the surface of minimum total rupture force [30].

As long as all J_{ij} remain positive, this kind of disorder should not change the two bulk phases with a paramagnetic-ferromagnetic second-order phase transition. Moreover the up-down spin symmetry is not broken by the disorder. The disorder just changes the local tendency towards ferromagnetism that can be interpreted as a change in the *local critical temperature*. Consequently, this type of disorder is often called random-T_c disorder, and it admits a Ginzburg-Landau kind of description, with a random distance from criticality, $\delta u(\vec{r})$,

$$F^{\text{random mass}}[m(\vec{r})] = \int d^dr \left\{-h\, m(\vec{r}) + [r + \delta r(\vec{r})]\, m^2(\vec{r}) + (\nabla\, m(\vec{r}))^2 + u\, m^4(\vec{r}) + \dots\right\} \ . \tag{1.36}$$

The disorder couples to the m^2 term in the free-energy functional. In quantum field theory, this term is called the mass term and, therefore, random-T_c disorder is also called *random-mass* disorder. (In addition to random exchange couplings, random-mass disorder can also be realized by random dilution of the spins.)

Link randomness is not the only type of disorder encountered experimentally. Random fields, that couple linearly to the magnetic moments, are also quite common; the classical model is the *ferromagnetic random field Ising model* (RFIM) [16, 31]:

$$H_J^{\text{rfim}} = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i s_i h_i \qquad \text{with} \qquad P(h_i) = (2\pi\sigma^2)^{-\frac{1}{2}}\, e^{-\frac{h_i^2}{2\sigma^2}} \ . \tag{1.37}$$

The *dilute antiferromagnet in a uniform magnetic field* is believed to behave similarly to the ferromagnetic random field Ising model. Experimental realisations of the former are common and measurements have been performed in samples like $Rb_2Co_{0.7}Mg_{0.3}F_4$.

Exercise 1.12 Take a ferromagnetic Ising chain under local random fields h_i that are independent and take the values $h_i = \pm h_0$ with probability a half. Prove that this model can be mapped onto another Ising chain under a uniform field h_0 with nearest neighbour interactions with strength J_i that are now quenched random variables. Find the dependence of J_i on the h_is and their probability distribution function.

Note that the up-down Ising symmetry is not preserved in models in the RFIM and any spin model such that the disorder couples to the local order parameter.

In the Ginzburg-Landau description this model reads

$$F[m(\vec{r})] = \int d^dr \left\{-h(\vec{r})\, m(\vec{r}) + r\, m^2(\vec{r}) + (\nabla\, m(\vec{r}))^2 + u\, m^4(\vec{r}) + \dots\right\} \tag{1.38}$$

where $h(\vec{r})$ is the local random variable that breaks the up-down spin symmetry. Whether or not the symmetry is broken globally depends on the probability distribution of the random fields. A particularly interesting situation arises if the distribution is even in h such that the up-down symmetry is globally preserved in the statistical sense.

Random-field disorder is generally stronger than random-mass disorder.

The random fields give rise to many metastable states that modify the equilibrium and non-equilibrium behaviour of the RFIM. In one dimension the RFIM does not order at all, in $d = 2$ there is strong evidence that the model is disordered even at zero temperature, in $d = 3$ it there is a finite temperature transition towards a ferromagnetic state [33]. Whether there is a glassy phase near zero temperature and close to the critical point is still and open problem.

The RFIM at zero temperature has been proposed to yield a generic description of material cracking through a series of avalanches. In this problem one cracking domain triggers others, of which size, depends on the quenched disorder in the samples. In a random magnetic system this phenomenon corresponds to the variation of the magnetisation in discrete steps as the external field is adiabatically increased (the time scale for an avalanche to take place is much shorter than the time-scale to modify the field) and it is accessed using Barkhausen noise experiments [34]. Disorder is responsible for the jerky motion of the domain walls. The distribution of sizes and duration of the avalanches is found to decay with a power law tail and cut-off at a given size. The value of the cut-off size depends on the strength of the random field and it moves to infinity at the critical point.

1.4.8 Random manifolds

Once again, disorder is not only present in magnetic systems. An example that has received much attention is the so-called *random manifold* [35, 36]. This is a d dimensional *directed elastic manifold* moving in an embedding $N + d$ dimensional space under the effect of a quenched random potential. The simplest case with $d = 0$ corresponds to a particle moving in an embedding space with N dimensions. If, for instance $N = 1$, the particle moves on a line, if $N = 2$ it moves on a plane and so on and so forth. If $d = 1$ one has a line that can represent a domain wall, a polymer, a vortex line, *etc.* The fact that the line is directed means it has a preferred direction, in particular, it does not have overhangs. If the line moves in a plane, the embedding space has $(N = 1) + (d = 1)$ dimensions. One usually describes the system with an N-dimensional coordinate, $\vec{\phi}$, that locates in the transverse space each point on the manifold, represented by the internal d-dimensional coordinate $\vec{r}$,

The elastic energy is $H_{\rm elas} = \gamma \int d^dx \, \sqrt{1 + (\nabla\phi(\vec{r}))^2}$ with γ the deformation cost of a unit surface. Assuming the deformation is small one can linearise this expression and get, upto an additive constant, $H_{\rm elas} = \frac{\gamma}{2} \int d^dr \, (\nabla\phi(\vec{r}))^2$.

Disorder is introduced in the form of a random potential energy $V(\vec{\phi}(\vec{r}), \vec{r})$ characterised by its pdf.

The random manifold model is then

$$H_V(\vec{\phi}) = \int d^dr \left[\frac{\gamma}{2} \, (\nabla\phi(\vec{r}))^2 + V(\vec{\phi}(\vec{r}), \vec{r})\right] \, . \tag{1.39}$$

If the random potential is the result of a large number of impurities, the central limit theorem implies that its probability density is Gaussian. Just by shifting the energy scale one can set its average to zero, $[\,V\,] = 0$. As for its correlations, one typically assumes, for simplicity, that they exist in the transverse direction only:

$$[\,V(\vec{\phi}(\vec{r}), \vec{r})V(\vec{\phi}'(\vec{r}'), \vec{r}')\,] = \delta^d(\vec{r} - \vec{r}')\mathcal{V}(\vec{\phi}, \vec{\phi}') \; . \tag{1.40}$$

If one further assumes that there is a statistical isotropy and translational invariance of the correlations, $\mathcal{V}(\vec{\phi}, \vec{\phi}') = W/\Delta^2\ \mathcal{V}(|\vec{\phi} - \vec{\phi}'|/\Delta)$ with Δ a correlation length and $(W\Delta^{d-2})^{1/2}$ the strength of the disorder. The disorder can now be of two types: short-ranged if $\mathcal{V}$ falls to zero at infinity sufficiently rapidly and long-range if it either grows with distance or has a slow decay to zero. An example involving both cases is given by the power law $\mathcal{V}(z) = (\theta + z)^{-\gamma}$ where θ is a short distance cut-off and γ controls the range of the correlations with $\gamma > 1$ being short-ranged and $\gamma < 1$ being long-ranged.

This model also describes *directed domain walls* in random systems. One can derive it in the long length-scales limit by taking the continuum limit of the pure Ising part (that leads to the elastic term) and the random part (that leads to the second disordered potential). In the pure Ising model the second term is a constant that can be set to zero while the first one implies that the ground state is a perfectly flat wall, as expected. In cases with quenched disorder, the long-ranged and short-ranged random potentials mimic cases in which the interfaces are attracted by pinning centres ('random field' type) or the phases are attracted by disorder ('random bond' type), respectively. For instance, random bond disorder is typically described by a Gaussian pdf with zero mean and delta-correlated $[V(\vec{\phi}(\vec{r}), \vec{r}), V(\vec{\phi}'(\vec{r}'), \vec{r}')] = W\Delta^{d-2}\ \delta^d(\vec{r} - \vec{r}')\delta(\vec{\phi} - \vec{\phi}')$.

1.5 Properties of finite dimensional disordered systems

Once various kinds of quenched disorder introduced, a number of questions on their effect on the equilibrium and dynamic properties arise. Concerning the former:

- Are the equilibrium phases qualitatively changed by the random interactions?
- Is the phase transition still sharp, or is it rendered smoother because different parts of the system undergo the transition independently?
- If there is still a phase transition, does its order (first order *vs.* continuous) change?
- If the phase transition remains continuous, does the critical behavior, *i.e.*, the values of the critical exponents, change?

Now, for the latter:

- Is the dynamic behaviour of the system modified by the quenched randomness?

In the following we explain a series of classical results in this field: the use of random matrix theory to study one dimensional problems, the Harris criterium, the proof of non-analyticity of the free-energy of quenched disordered systems close to their critical temperature given by Griffiths, the analysis of droplets and their domain wall stiffness, and the derivation of some exact results derived by Nishimori using gauge invariance.

We first focus on impurities or defects that lead to spatial variations with respect to the tendency to order but do not induce new types of order, that is to say, no changes are inflicted on the two phases at the two sides of the transition. Only later we consider the spin-glass case.

1.5.1 One dimensional cases

Take the disordered Ising chain

$$H = -\sum_{i=1}^{N} (J_i s_i s_{i+1} + h_i s_i) \tag{1.41}$$

with J_i and h_i random exchanges and random fields taken from probability distribution. Impose periodic boundary conditions such that $s_{N+1} = s_1$. The partition function can be evaluated with the *transfer matrix* method introduced by Kramers and Wannier [80] and Onsager [81], see also [82]. Indeed,

$$Z_N = \sum_{\{s_i=\pm 1\}} \prod_{i=1}^{N} e^{\beta J_i s_i s_{i+1} + \beta h_i s_i} = \sum_{\{s_i=\pm 1\}} T_{1 s_1 s_2} T_{2 s_2 s_3} \dots T_{N s_N s_1} = \mathrm{Tr} \prod_{i=1}^{N} T_i \tag{1.42}$$

where T_i are 2×2 matrices in which one takes the two row and column indices to take the values ± 1. Then

$$T_i = \begin{pmatrix} e^{\beta(J_i+h_i)} & e^{\beta(-J_i+h_i)} \\ e^{\beta(-J_i-h_i)} & e^{\beta(J_i-h_i)} \end{pmatrix} . \tag{1.43}$$

Note that, for random exchanges and/or fields Eq. (1.42) is a product of random matrices.

The free-energy per spin is given by

$$-\beta f_N = \frac{1}{N} \ln Z_N = \frac{1}{N} \ln \mathrm{Tr} \prod_{i=1}^{N} T_i \tag{1.44}$$

The thermodynamic quantities like the energy per spin, magnetic susceptibility and other can be computed from this expression. The local quantities, such as local averaged magnetisation or correlation functions are evaluated with the help of the spin operator

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.45}$$

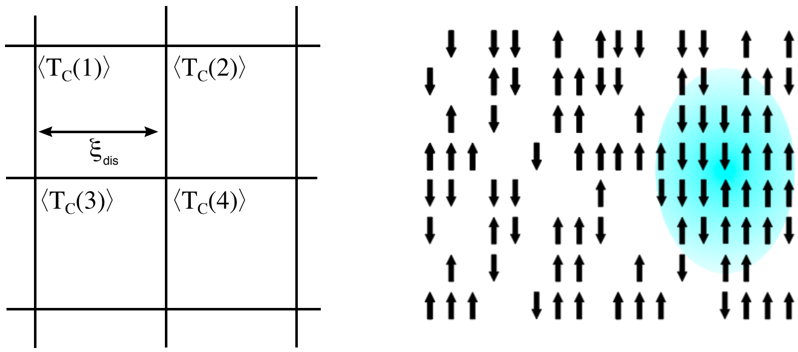


Figure 1.5: Left: scheme of the Harris construction. The disordered system is divided into cells with linear length $\xi_{\rm dis}$, its correlation length. Right: a typical configuration of the dilute Ising ferromagnet. Figures taken from [37].

as

$$
\begin{aligned}
\langle s_i \rangle &= \frac{1}{Z_N} \mathrm{Tr}\; T_1 T_2 \dots T_{i-1} \Sigma T_i \dots T_N && (1.46)\\
\langle s_i s_j \rangle &= \frac{1}{Z_N} \mathrm{Tr}\; T_1 \dots T_{i-1} \Sigma T_i \dots T_{j-1} \Sigma T_j \dots T_N && (1.47)
\end{aligned}
$$

Methods from random matrix theory can then be used to study disordered spin chains [92].

1.5.2 The Harris criterium

The first question to ask is how does the average disorder strength behave under coarse-graining or, equivalently, how is it seen at long distances. This is the question answered by the Harris argument that focuses on the effect of disorder on systems with a conventional order-disorder phase transition, say, a ferromagnetic-paramagnetic one.

The Harris' criterion [40] states that if the specific-heat of a pure system

$$
C^{\rm pure}(T) \simeq |T - T_c^{\rm pure}|^{-\alpha_{\rm pure}} \qquad (1.48)
$$

presents a power-like divergence with

$$
\alpha_{\rm pure} > 0 \;, \qquad (1.49)
$$

the disorder induces a new universality class. Otherwise, if $\alpha_{\rm pure} < 0$, disorder is irrelevant in a renormalisation group sense and the critical behaviour of the model remains unchanged. The criterium does not decide in the marginal case $\alpha_{\rm pure} = 0$ case.

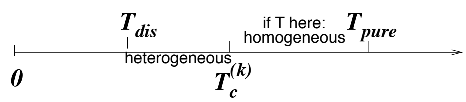
The hyper-scaling relation $2 - d\nu_{\rm pure} = \alpha_{\rm pure}$ allows to rewrite the Harris criterium as

$$
\text{critical behaviour} = \begin{cases} \text{unchanged} & \text{if } \nu_{\rm pure} > 2/d \\ \text{may change} & \text{if } \nu_{\rm pure} < 2/d \end{cases} \qquad (1.50)
$$

where $\nu_{\rm pure}$ is the correlation length exponent

$$\langle \delta s_{\vec{\sigma}}\, \delta s_{\vec{r}} \rangle \simeq e^{-r/\xi_{\rm pure}} \qquad \text{and} \qquad \xi_{\rm pure} \simeq |T - T_c^{\rm pure}|^{-\nu_{\rm pure}} \;, \tag{1.51}$$

of the pure system. In the high temperature phase, $\delta s = s$ since $\langle s \rangle = 0$.



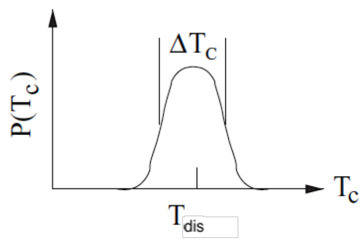


Figure 1.6: Left. The characteristic temperatures. $T_{\rm pure}$ and $T_{\rm dis}$ are the critical temperatures of the pure and disordered systems, respectively. $T_c^{(k)}$ is the critical temperature of the local region with linear size $\xi_{\rm dis}$ labelled k, see the sketch in Fig. 1.5-left. The distance from the disordered critical point is measured by $\Delta T_c^{(k)} = T_c^{(k)} - T_{\rm dis}$ for the critical temperature of block k and $\Delta T = T - T_{\rm dis}$ for the working temperature T. Right: the probability distribution function of the local critical temperatures $T_c^{(k)}$. The width depends on $\xi_{\rm dis}$ and clearly decreases with increasing $\xi_{\rm dis}$ as the local temperatures fluctuate less and less.

The proof of the Harris result is rather simple and illustrates a way of reasoning that is extremely useful [40, 37, 38]. Take the full system with frozen-in disorder at a temperature T slightly above its critical temperature $T_c^{\rm dis}$. Divide it into equal pieces with linear size $\xi_{\rm dis}$, the correlation length of the disordered system at the working temperature, which we will call ξ henceforth. By construction, the spins within each of these blocks behave as a *super-spin* since they are effectively parallel. Because of disorder, each block k has its own local critical temperature $T_c^{(k)}$ determined by the random interactions (or dilution) within the block.

First of all, we notice that if the size of the box is proportional to the system size, and diverges, there is no fluctuation in the critical temperature and $T_c^{(k)} = T_c^{\rm dis}$. Next, we recall that for a ferromagnetic model the critical temperature is proportional to J, the strength of the coupling. Here, we focus on a random bond model in which the couplings are drawn from a probability distribution with positive support, and

$$[J_{ij}] = J \qquad\qquad [J_{ij}^2] - [J_{ij}]^2 = \sigma_J^2 \;, \tag{1.52}$$

with both J and σ_J finite. Call now $\Delta T_c^{(\rm loc)}$ the distance between a generic local critical temperature and the global one. For boxes with linear size $\xi_{\rm dis}$ we can estimate $\Delta T_c^{(\rm loc)}$ using the central limit theorem. Indeed, we can claim that each local $T_c^{(k)}$ is determined by an average of a large number of random variables in the block, the random J_{ij} in the

Hamiltonian. Thus, using (1.52), the dispersion $\Delta T_c^{(\mathrm{loc})}$ should decay as the square root of the block volume,

$$\Delta T_c^{(\mathrm{loc})} \simeq \xi^{-d/2} \; . \tag{1.53}$$

Harris proposes to compare the fluctuations in the local critical temperatures $\Delta T_c^{(k)} \equiv T_c^{(k)} - T_c^{\mathrm{dis}}$ with respect to the global critical one T_c^{dis}, with the distance from the critical point $\Delta T \equiv T - T_c^{\mathrm{dis}} > 0$, taken to be positive:

- If $\Delta T_c^{(k)} < \Delta T$ for all k, $T_c^{(k)} - T_c^{\mathrm{dis}} < T - T_c^{\mathrm{dis}}$ for all k, and all blocks have critical temperature below the working one, $T_c^{(k)} < T$. The system looks then 'homogeneous' with respect to the phase transition and it is at a higher temperature than the critical temperature of each of the blocks.

- If $\Delta T_c^{(k)} > \Delta T$ for some k, such blocks are in the ordered (ferromagnetic) phase while others are in the disordered (paramagnetic) phase, making a uniform transition impossible. The inhomogeneity in the system may then be important.

Require now $\Delta T_c^{(\mathrm{loc})} < \Delta T$, as one approaches the critical region, to have an unmodified critical behaviour, and use also that an unmodified critical behaviour implies $\nu_{\mathrm{dis}} = \nu_{\mathrm{pure}}$. On the one hand, $\Delta T_c^{(\mathrm{loc})} \simeq \xi^{-d/2}$. On the other hand, $\Delta T \simeq \xi^{-1/\nu_{\mathrm{pure}}}$. Therefore,

$$\Delta T_c^{(\mathrm{loc})} < \Delta T \qquad \Rightarrow \qquad d\nu_{\mathrm{pure}} > 2 \; . \tag{1.54}$$

The interpretation of this inequality is the following. If the Harris criterion $d\nu_{\mathrm{pure}} > 2$ is fulfilled, the ratio $\Delta T_c^{(\mathrm{loc})}/\Delta T \simeq \xi^{-d/2+1/\nu_{\mathrm{pure}}}$ goes to zero as the critical point is approached. The system looks less and less disordered on larger length scales, the effective disorder strength vanishes right at criticality, and the disordered system features the same critical behaviour as the clean one. An example of a transition that fullfills the Harris criterion is the ferromagnetic transition in a three-dimensional classical Heisenberg model. Its clean correlation length exponent is $\nu_{\mathrm{pure}} \approx 0.69 > 2/d = 2/3 \approx 0.67$.

In contrast, if $d\nu_{\mathrm{pure}} < 2$, the ratio $\Delta T_c^{(\mathrm{loc})}/\Delta T$ increases upon approaching the phase transition. The blocks differ more and more on larger length scales. Eventually, some blocks are on one side of the transition while other blocks are on the other side. This makes a uniform sharp phase transition impossible. The clean critical behaviour is unstable and the phase transition can be erased or it can remain continuous but with different critical behaviour. More precisely, the disordered system can be in a new universality class featuring a correlation length exponent that fulfills the inequality $d\nu_{\mathrm{dis}} > 2$. Many phase transitions in classical disordered systems follow this scenario, for example the three-dimensional classical Ising model. Its clean correlation length exponent is $\nu_{\mathrm{pure}} \approx 0.63 < 0.67$ and violates the Harris criterion. In the presence of random-mass disorder, the critical behaviour changes and $\nu_{\mathrm{dis}} \approx 0.68$. (Note, however, that the difference between these exponents is tiny!)

In the marginal case $d\nu_{\text{pure}} = 2$, more sophisticated methods are required to decide the stability of the clean critical point.

Chayes et al. [42] turned this argument around to show rigorously that for all the continuous phase transitions in presence of disorder, the correlation-length critical exponent of the disordered system, ν_{dis} verifies $\nu_{\text{dis}} \geq 2/d$, independently of whether or not the critical behaviour is the same as in the uniform system and even when the system does not have a uniform analogue.

Finally, note that the Harris criterion $d\nu_{\text{pure}} > 2$ applies to uncorrelated or short-range correlated disorder. If the disorder displays long-range correlations in space, the inequality needs to be modified because the central-limit theorem estimate of $\Delta T_c^{(\text{loc})}$ changes. In a d dimensional system, with uncorrelated disorder in $d_\perp$ directions and perfectly correlated in $d - d_\perp$ ones, the condition becomes $d_\perp \nu_{\text{pure}} > 2$. If the disorder features isotropic long-range correlations in space that decay as $|\mathbf{r} - \mathbf{r}'|^{-a}$, the criterion is modified to $\min(d, a)\nu > 2$ [**?**].

Exercise 1.13 Consider an Ising model on a square lattice with *striped randomness*. This means that the exchanges J_{ij} are such that when they lie on horizontal links are equal to J_0 while on vertical links they equal $J_0 + \epsilon F(y)$ with y the vertical coordinate of lower site on the vertical bond. $F(y)$ is a random variable taken from a probability distribution with zero mean and finite width σ. This model has long-range correlated disorder. Adapt the Harris criterion to this problem and show that in this case disorder has a dominant effect on the critical behaviour and hence rounds the transition. For details on this problem see Ref. [43].

Long-range correlated disorder is especially important in quantum phase transitions. The reason is the fact that the statistical properties of quantum systems are studied in an imaginary time formulation that makes a d-dimensional quantum problem equivalent to a $d + 1$ dimensional classical one. Along this additional spatial direction, quenched randomness is long-range correlated.

1.5.3 The Griffiths phase

The critical temperature of a spin system is usually estimated from the high temperature expansion and the evaluation of its radius of convergence (see App. 1.B.1). However, Griffiths showed that the temperature at which the free-energy of models with quenched disorder starts being non-analytical falls above the critical temperature where the order parameter detaches from zero [44]. The argument applies to models with second order phase transitions.

Griffiths explained his argument using the dilute ferromagnetic Ising model. First, he argued that the critical temperature of the disordered model should decrease for increasing p, the probability of *empty* sites. This is 'intuitively obvious' since no spontaneous magnetisation can occur at a finite temperature if the probability of occupied sites is less than the critical percolation probability at which an 'infinite cluster' first appears. See

Fig. 1.7 where the phase diagram of the dilute Ising ferromagnet is shown. Besides, below the critical temperature of the corresponding pure system, T_c^0 in the figure, with a finite (exponentially small) probability there exist arbitrary large dense ferromagnetic 'islands' which are critical exactly at the working temperature $T_c(p) < T < T_c^0$. Hence, one can expect the free energy of such random system to be a non-analytic function of the external magnetic field, in the limit $h \to 0$, at any temperature between $T_c(p)$ and T_c^0, *i.e.*, the yellow region in the figure.

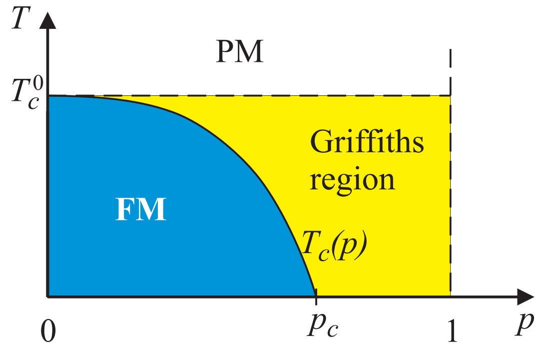


Figure 1.7: The phase diagram of the dilute ferromagnetic Ising model. p is the probability of empty sites in this figure differently from the notation in the main text, figure taken from [37]. With increasing dilution the ordered phase is eventually suppressed.

In the following paragraph we sketch Griffiths' argument and we use his notation in which p is the probability of occupying a site. For any concentration $p < 1$ the magnetisation m is not an analytic function of h at $h = 0$ at any temperature below $T_c^{\rm pure}$, the critical temperature of the regular Ising model $p = 1$. As he explains, this fact is most easily explained for $p < p_c$. The magnetisation m per lattice site in the thermodynamic limit has the form

$$m = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle = \sum_c P(c)m(c) \tag{1.55}$$

where $P(c)$ is the probability that a particular site on the lattice belongs to a cluster c that is necessarily finite for $p < p_c$, and $m(c)$ is the magnetisation density of the cluster c, that is to say $m(c) = N^{-1}(c)\sum_{i\in c}\langle s_i\rangle$ with $N(c)$ the number of sites in the cluster.

Griffiths uses the Yang-Lee theorem, see App. 1.B, to express $m(c)$ as

$$m(c) = 1 + \frac{2z}{N(c)}\sum_{i\in c}\frac{1}{\xi_i - z} \qquad \text{with} \qquad z = e^{-2\beta h} \tag{1.56}$$

and ξ_i, $i = 1, \ldots, N(c)$, complex numbers with $|\xi_i| = 1$. The total magnetisation density is then of the same form

$$m = 1 + zf(z) \qquad f(z) = \sum_i \eta_i(\xi_i - z)^{-1} \tag{1.57}$$

with $\eta_i = 2P(c)/N(c)$. He then argues that this form is analytic for $z < 1$ but non-analytic at $z = 1$ that corresponds to $h = 0$.

A more intuitive understanding of what is going on in the temperature region above the critical temperature of the disordered model, $T_c^{\rm dis}$, and below the critical temperature the pure one, $T_c^{\rm pure}$, can be reached as follows [37]. The effects of quenched disorder show up already in the paramagnetic phase of finite dimensional systems. Below the critical point of the pure case (no disorder) finite regions of the system can order due to fluctuations in the couplings or, in a dilute ferromagnetic model, they can be regions where all sites are occupied, as shown in Fig. 1.5. As such rare regions are finite-size pieces of the clean system, their spins align parallel to each other below the clean critical temperature $T_c^{\rm pure}$. Because they are of finite size, these regions cannot undergo a true phase transition by themselves, but for temperatures between the actual transition temperature $T_c^{\rm dis}$ and $T_c^{\rm pure}$ they act as large superspins.

Note that using the ideas of percolation theory, one can estimate the scaling of $P(c)$ with its size. Recall the one dimensional case. Take a segment of length $L+2$ on the lattice. A cluster of size L will occupy the internal sites with empty borders with probability $(1-p)^L p^2$ since, we recall p is the probability for a site being empty and $(1-p)$ the one of being filled. This is because one needs L contiguous sites to be occupied and its boundary sites be empty. In larger dimensions, this probability will be approximately $(1-p)^{L^d} p^{L^{d-1}}$ with the first factor linked to the filled volume and the second to the empty surface. In the one dimensional case, one has $P(c) = p^2 e^{L \ln(1-p)} \propto e^{-c(p)L}$ where we defined $c(p) = -\ln(1-p) > 0$. In the $d > 1$ case, one can make a harsh approximation and use $P(c) \simeq \exp\{\ln[(1-p)^{L^d} p^{L^{d-1}}]\} = \exp[\ln(1-p)^{L^d} + \ln p^{L^{d-1}}] = \exp[L^d \ln(1-p) + L^{d-1} \ln p] \simeq \exp[-{\rm c}(p) L^d]$ in the large L limit.

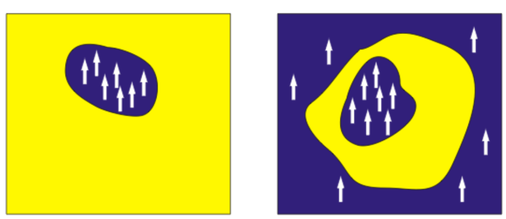


Figure 1.8: Rare regions in a random ferromagnet, figure taken from [37]. On the left, a ferromagnetically ordered region in the paramagnetic bulk ($T > T_c^{\rm dis}$). On the right, a paramagnetic band in a system that is ordered ferromagnetically in a patchwork way ($T < T_c^{\rm dis}$).

The sum in eq. (1.55) is made of two contributions. On the one hand, there are the large clusters that are basically frozen at the working temperature. On the other, there are the free spins that belong to small clusters and are easy to flip at the working temperature. Let us focus on the former. Their magnetic moment is proportional to their volume $m(c) \simeq \mu L^d$. The energy gain due to their alignment with the field is

$\Delta E(c) = -2hm(c) = -2h\mu L^d$ where h is a small uniform field applied to the system, say to measure its susceptibility.

The separation of the clusters in two groups is then controlled by the comparison between $\Delta E(c)$ and the thermal energy: the small clusters with $|\Delta E(c)| < k_B T$ can be flipped by thermal fluctuations, and the large clusters with $|\Delta E(c)| > k_B T$ are frozen in the direction of the field.

The effect of the frozen clusters for which $|\Delta E(c)| > k_B T$ is then

$$m_{\rm frozen}(T,h) \approx \sum_{|\Delta E(c)|>k_B T} P(c) m(c) \approx \int_{L_c}^{\infty} dL\; e^{-\mathrm{c}(p) L^d}\; \mu L^d \tag{1.58}$$

and

$$|\Delta E(c)| = h\mu L^d \approx k_B T \qquad \Rightarrow \qquad L_c^d \approx \frac{k_B T}{\mu h} \tag{1.59}$$

(the numerical factor 2 is irrelevant). This integral can be computed by the saddle-point method and it is dominated by the lower border in the small h limit. The result is

$$m_{\rm frozen}(T,h) \approx e^{-\mathrm{c}(p) L_c^d} = e^{-\mathrm{c}(p) k_B T/(\mu h)} \tag{1.60}$$

where we dropped the contribution from μL^d since it is negligible. The result has an *essential singularity* in the $h \to 0$ limit.

It is important to note that the clusters that contribute to this integral are *rare regions* since they occur with probability $P(c) \simeq e^{-\mathrm{c}(p) L^d}$ that is exponentially small in their volume. Still they are the cause of the non-analytic behaviour of $m(h)$.

The magnetic susceptibility χ can be analysed similarly. Each locally ordered rare region makes a Curie contribution $m^2(c)/k_B T$ to χ. The total rare region susceptibility can therefore be estimated as

$$\chi_{\rm frozen}(T,h) \approx \int_{L_c}^{\infty} dL\; e^{-\mathrm{c}(p) L^d} \mu^2\, L^{2d} \approx e^{-\mathrm{c}(p) L_c^d} \approx e^{-\mathrm{c}(p) k_B T/(\mu h)}\;. \tag{1.61}$$

This equation shows that the susceptibility of an individual rare region does not increase fast enough to overcome the exponential decay of the rare region probability with increasing size L. Consequently, large rare regions only make an exponentially small contribution to the susceptibility.

Rare regions also exist on the ordered side of the transition $T < T_c$. One has to consider locally ordered islands inside holes that can fluctuate between up and down because they are only very weakly coupled to the bulk ferromagnet outside the hole, see Fig. 1.8. This conceptual difference entails a different probability for the rare events as one needs to find a large enough vacancy-rich region around a locally ordered island.

There are therefore slight differences in the resulting Griffiths singularities on the two sides of the transition. In the site-diluted Ising model, the ferromagnetic Griffiths phase comprises all of the ferromagnetic phase for $p > 0$. The phase diagram of the dilute ferromagnetic Ising model is sketched in Fig. 1.7 with p denoting the probability of empty sites in the figure.

1.5.4 Scenario for the phase transitions

The argument put forward by Harris is based on the effect of disorder on average over the local critical temperatures. The intuitive explanation of the Griffiths phase shows the importance of rare regions on the behaviour of global observables such as the magnetisation or the susceptibility. The analysis of the effect of randomness on the phase transitions should then be refined to take into account the effect of *rare regions* (tails in the distributions). Different classes of rare regions can be identified according to their dimension d_{rare} (in the discussion of the dilute ferromagnetic system we used $d_{\text{rare}} = d$. This leaves place for three possibilities for the effect of (still weak in the sense of not having frustration) disorder on the phase transition.

- The rare regions have dimension d_{rare} smaller than the lower critical dimension of the pure problem, $d_{\text{rare}} < d_L$; therefore the critical behaviour is not modified with respect to the one of the clean problem.

- When the rare regions have dimension equal to the lower-critical one, $d_{\text{rare}} = d_L$, the critical point is still of second order with conventional power law scaling but with different exponents that vary in the Griffiths phase. At the disordered critical point the Harris criterium is satisfied $d\nu_{\text{dis}} > 2$.

- Infinite randomness strength, appearing mostly in problems with correlated disorder, lead to a complete change in the critical properties, with unconventional activated scaling. This occurs when $d_{\text{rare}} > d_L$.

In the derivation of this scenario the rare regions are supposed to act independently, with no interactions among them. This picture is therefore limited to systems with short-range interactions.

1.5.5 Domain-wall stiffness and droplets

Let us now just discuss one simple argument that is at the basis of what is needed to derive the *droplet theory* for disordered magnets and spin-glasses, without entering into the complications of the calculations. Let us summarise the kind of fluctuations expected at high, critical and low temperatures.

At very high temperature the configurations are disordered and one does not see large patches of ordered spins.

Close but above the critical temperature T_c finite patches of the system are ordered (in all possible low-temperature equilibrium states) but none of these include a finite fraction of the spins in the sample and the magnetization density vanishes. However, these patches are enough to generate non-trivial thermodynamic properties very close to T_c and the richness of critical phenomena.

At criticality one observes ordered domains of the two equilibrium states at all length scales – with *fractal* properties.

Below the critical temperature thermal fluctuations induce the spin reversal with respect to the order selected by the spontaneous symmetry breaking. It is clear that the structure of *droplets*, meaning patches in which the spins point in the opposite direction to the one of the background ordered state, plays an important role in the thermodynamic behaviour at low temperatures.

M. Fisher and others developed a droplet phenomenological theory for critical phenomena in clean systems. Later D. S. Fisher and D. Huse extended these arguments to describe the effects of quenched disorder in spin-glasses and other random systems; this is the so-called *droplet model*.

Domain-wall stiffness

Ordered phases resist spatial variations of their order parameter. This property is called *stiffness* or *rigidity* and it is absent in high-temperature disordered phases.

More precisely, in an ordered phase the *free-energy cost* for changing one part of the system with respect to another part far away is proportional to k_BT and usually diverges as a power law of the system size. In a disordered phase the information about the reversed part propagates only a finite distance (of the order of the correlation length, see below) and the stiffness vanishes.

Concretely, the free-energy cost of installing a *domain-wall* in a system, gives a measure of the stiffness of a phase. The domain wall can be imposed by special boundary conditions. Compare then the free-energy of an Ising model with linear length L, in its ordered phase, with periodic and anti-periodic boundary conditions on one Cartesian direction and periodic boundary conditions on the $d-1$ other directions of a d-dimensional hypercube. The $\pm$ boundary conditions forces an interface between the regions with positive and negative magnetisations. At $T=0$, the minimum energy interface is a $d-1$ flat hyper-plane and the energy cost is

$$\Delta E(L) \simeq \sigma L^{\theta} \qquad \text{with} \qquad \theta = d-1 \tag{1.62}$$

and $\sigma = 2J$ the *interfacial energy per unit area* or the *surface tension* of the domain wall.

Droplets - generalisation of the Peierls argument

In an ordered system at finite temperature domain walls, surrounding *droplet fluctuations*, or domains with reversed spins with respect to the bulk order, are naturally generated by thermal fluctuations. The study of droplet fluctuations is useful to establish whether an ordered phase can exist at low (but finite) temperatures. One then studies the free-energy cost for creating large droplets with thermal fluctuations that may destabilise the ordered phase, in the way usually done in the simple Ising chain (the Peierls argument).

Indeed, temperature generates fluctuations of different size and the question is whether these are favourable or not. These are the *droplet excitations* made by simply connected regions (domains) with spins reversed with respect to the ordered state. Because of the surface tension, the minimal energy droplets with linear size or radius L will be compact spherical-like objects with volume L^d and surface L^{d-1}. The surface determines their energy and, at finite temperature, an entropic contribution has to be taken into account as well. Simplifying, one argues that the free-energy cost is of the order of L^θ, that is L^{d-1} in the ferromagnetic case but can be different in disordered systems.

Summarising, in system with symmetry breaking the free-energy cost of an excitation of linear size L is expected to scale as

$$\Delta F(L) \simeq \sigma(T) L^\theta \ . \tag{1.63}$$

The sign of θ determines whether thermal fluctuations destroy the ordered phase or not. For $\theta > 0$ large excitations are costly and very unlikely to occur; the order phase is expected to be stable. For $\theta < 0$ instead large scale excitations cost little energy and one can expect that the gain in entropy due to the large choice in the position of these excitations will render the free-energy variation negative. A proliferation of droplets and droplets within droplets is expected and the ordered phase will be destroyed by thermal fluctuations. The case $\theta = 0$ is marginal and its analysis needs the use of other methods.

As the phase transitions is approached from below the surface tension $\sigma(T)$ should vanish. Moreover, one expects that the stiffness should be independent of length close to T_c and therefore, $\theta_c = 0$.

Above the transition the stiffness should decay exponentially

$$\Delta F(L) \simeq e^{-L/\xi} \tag{1.64}$$

with ξ the equilibrium correlation length.

1.5.6 Stability of ordered phases

The classical nucleation theory (CNT) considers the thermally induced generation of stable phase droplets in the metastable surrounding.

The Peierls argument

Let us use a thermodynamic argument to describe the high and low temperature phases of a magnetic system and argue that for short-range interactions a one dimensional system with short-range interactions cannot sustain an order phase at non-zero temperature while one with sufficiently long-range interactions can.

The *free-energy* of a system is given by $F = U - TS$ where U is the internal energy, $U = \langle \mathcal{H} \rangle$, and S is the entropy. The equilibrium state may depend on temperature and it is such that it minimises its free-energy F. A competition between the energetic contribution and the entropic one may then lead to a change in phase at a definite temperature, *i.e.* a different group of micro-configurations, constituting a state, with different macroscopic properties dominate the thermodynamics at one side and another of the transition.

At zero temperature the free-energy is identical to the internal energy U. In a system with nearest-neighbour ferromagnetic couplings between magnetic moments, the magnetic interaction is such that the energy is minimised when neighbouring moments are parallel.

Switching on temperature thermal agitation provokes the reorientation of the moments and, consequently, misalignments. Let us then investigate the opposite, infinite temperature case, in which the entropic term dominates and the chosen configurations are such that entropy is maximised. This is achieved by the magnetic moments pointing in random independent directions.

The competition between these two limits indicates whether a finite temperature transition is possible or not.

Short-range interactions in $d = 1$

At zero temperature the preferred configuration is such that all moments are parallel, the system is fully ordered, and for nearest-neighbour couplings $U = -J\#$ pairs.

For a model with N Ising spins, the entropy at infinite temperature is $S \sim k_B N \ln 2$.

Decreasing temperature magnetic disorder becomes less favourable. The existence or not of a finite temperature phase transitions depends on whether long-range order, as the one observed in the low-temperature phase, can remain stable with respect to *fluctuations*, or the reversal of some moments, induced by temperature. Up to this point, the discussion has been general and independent of the dimension d.

The competition argument made more precise allows one to conclude that there is no finite temperature phase transition in $d = 1$ while it suggests there is one in $d > 1$. Take a one dimensional ferromagnetic Ising model with closed boundary conditions (the case of open boundary conditions can be treated in a similar way),

$$\mathcal{H}([\{s_i\}) = -J \sum_{i=1}^{N} s_i s_{i+1} \;, \tag{1.65}$$

and $s_{N+1} = s_1$. At zero temperature it is ordered and its internal energy is just

$$U_o = -JN \tag{1.66}$$

with N the number of links and spins. Since there are two degenerate ordered configurations (all spins up and all spins down) the entropy is

$$S_o = k_B \ln 2 \tag{1.67}$$

The internal energy is extensive while the entropy is just a finite number. At temperature T the free-energy of the completely ordered state is then

$$F_o = U_o - TS_o = -JN - k_B T \ln 2 \;. \tag{1.68}$$

$$\uparrow\uparrow \cdots \uparrow\uparrow\downarrow\downarrow \cdots \downarrow\downarrow\uparrow\uparrow \cdots \uparrow\uparrow$$

Figure 1.9: A domain wall in a one dimensional Ising system.

This is the *ground state* at finite temperature or global configuration that minimises the free-energy of the system.

Adding a *domain* of the opposite order in the system, *i.e.* reversing n spins, two bonds are unsatisfied and the internal energy becomes

$$U_2 = -J(N-2) + 2J = -J(N-4) \;, \tag{1.69}$$

for any n. Since one can place the misaligned spins anywhere in the lattice, there are N equivalent configurations with this internal energy. The entropy of this state is then

$$S_2 = k_B \ln(2N) \;. \tag{1.70}$$

The factor of 2 inside the logarithm arises due to the fact that we consider a reversed domain in each one of the two ordered states. At temperature T the free-energy of a state with *two domain walls* is

$$F_2 = U_2 - TS_2 = -J(N-4) - k_BT\ln(2N) \;. \tag{1.71}$$

The variation in free-energy between the ordered state and the one with one reversed domain is

$$\Delta F = F_2 - F_o = 4J - k_BT\ln N \;. \tag{1.72}$$

Thus, even if the internal energy increases due to the presence of the domain walls, the increase in entropy is such that the free-energy of the state with a droplet in it is much lower, and therefore the state much more favourable, at any finite temperature T. One can repeat this argument reversing domains within domains and progressively disorder the sample. We conclude that spin flips are favourable and order is destroyed at any non-vanishing temperature. The ferromagnetic Ising chain does not support a non-zero temperature ordered phase and therefore does not have a finite temperature phase transition.

Note that this argument explicitly uses the fact that the interactions are short-ranged (actually, they extend to first neighbours on the lattice only in the example). Systems with sufficiently long-range interactions can have finite temperature phase transitions even in one dimension, as shown below.

Exercise 1.14 Solve the one dimensional Ising chain and confirm that it only orders at zero temperature. Identify the correlation length, $\xi(T)$, from the decay of the connected correlation function, $C(r) \equiv \langle (s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)\rangle|_{|\vec{r}_i-\vec{r}_j|=r} \sim e^{-r/\xi(T)}$, and its temperature dependence.

Power-law decaying interactions in $d = 1$

Take now a one dimensional Ising model

$$\mathcal{H}(\{s_i\}) = -\frac{J}{2}\sum_{i\neq j} J_{ij}s_is_j = -J\sum_{i=0}^{N-1}\sum_{k=1}^{N-i} J_{i\,i+k}s_is_{i+k} \tag{1.73}$$

with open boundary conditions and algebraically decaying ferromagnetic interactions

$$J_{i\,i+k} \sim J\, r_{i\,i+k}^{-\alpha} \equiv J\, r_{i\,i+k}^{-(1+\sigma)} = J\,(ak)^{-(1+\sigma)}\ , \tag{1.74}$$

where $r_{i\,i+k} = |\vec{r}_i - \vec{r}_{i+k}| = ak$, a is the lattice spacing, and we used here the notation in [?] that compared to the one of the Introductory chapter is $\alpha = \sigma + 1$. From the arguments put forward in that chapter, we expect a change in behaviour at $\sigma = 0$ or $\alpha = d = 1$.

In a perfect ferromagnetic configuration the energy is $U_0 = -J\sum_{i=0}^{N-1}\sum_{k=1}^{N-i}(ak)^{-(1+\sigma)}$ that in a continuous limit, $ak \mapsto y$, $a\sum_k \mapsto \int dy$, and $a\sum_i \mapsto \int dx$ reads

$$\begin{aligned} U_0 &\mapsto -\frac{J}{a^2}\int_0^{L-a} dx \int_a^{L-x} dy\, \frac{1}{y^{1+\sigma}} = \frac{J}{a^2}\frac{1}{\sigma}\int_0^{L-a} dx\, [(L-x)^{-\sigma} - a^{-\sigma}] \\ &= \frac{J}{a^2}\frac{1}{\sigma}\frac{1}{1-\sigma}[-a^{1-\sigma} + L^{1-\sigma} - a^{-\sigma}(L-a)]\ . \end{aligned} \tag{1.75}$$

We see that for $\sigma < 0$ the energy is superextensive, $U_0 \propto -L^{1-\sigma}$. One can cure this problem by re-scaling J, $J \mapsto JL^{\sigma-1}$, that is, considering a much weaker interaction strength, scaling with system size. Instead, for $\sigma > 0$ the large system size limit is controlled by the last term and the (still negative) ground state energy is extensive.

We now make an explicit calculation to check whether this system can have long-range order in the cases $\sigma > 0$ (equivalent to $\alpha > d = 1$).

Consider an excitation over the ferromagnetically order state in which n spins on the left point down and $N-n$ spins on the right point up, that is to say, a configuration with a single sharp domain wall (possible because of the open boundary conditions). The excess energy of this excitation with respect to the perfectly ordered ground state in which all spins point up is:

$$\Delta U = 2J\sum_{i=0}^{n}\sum_{j=n-i+1}^{N-i}\frac{1}{(aj)^{1+\sigma}}\ . \tag{1.76}$$

Clearly, if $n = 0$ or $n = N-1$, $\Delta U = 0$. In the continuous space limit, $a \to 0$, the sums can be transformed into integrals

$$\begin{aligned} \Delta U &\mapsto \frac{2J}{a^2}\int_0^z dx \int_{z-x+a}^{L-x} dy\, \frac{1}{y^{1+\sigma}} \\ &= -\frac{2J}{a^2\sigma}\int_0^z dx\, \left[(L-x)^{-\sigma} - (z-x+a)^{-\sigma}\right] \\ &= \frac{2J}{a^2\sigma(1-\sigma)}\left[(L-z)^{1-\sigma} - L^{1-\sigma} - a^{1-\sigma} + (z+a)^{1-\sigma}\right] \end{aligned} \tag{1.77}$$

where we called $L = Na$ the length of the chain and z the placement of the domain wall. We now study this expression in the case $L \gg z \gg a$, that is to say, when the domain wall is placed at a finite distance from the origin compared to the infinite size limit. The contribution of the first two terms in the square brackets is proportional to z/L^σ for $z \ll L$ and negligible for $\sigma > 0$. The third term is just a short-length regularisation depending on the lattice size. The last term is the important one that we approximate as

$$\approx \frac{2J}{a^2\sigma(1-\sigma)}\, z^{1-\sigma} \tag{1.78}$$

using $z \gg a$. Therefore, the excitation energy increases with the length to the reversed domain for $0 < \sigma < 1$ (while in the first-neighbour interaction case it was independent of it). The reversal of large domains is not favourable energetically and this is an indication that long-range order can exist in such a model with $0 < \sigma < 1$. In the case $\sigma < 0$ interactions are strongly long-ranged and order should be even more favorable. (Many mathematical papers from the 60s-80s, by the most celebrated statistical physicists of the time, derived conditions on the decay of the power law interaction to inhibit magnetic order at any non-vanishing temperature.) In contrast, for $\sigma > 1$ the energy of a large droplet is bounded and the entropic term at finite temperature will end up destroying the ferromagnetic order.

A ferromagnet under a magnetic field

Let us study the stability properties of an equilibrium ferromagnetic phase under an applied external field that tends to destabilize it. If we set $T = 0$ the free-energy is just the energy. In the ferromagnetic case the free-energy cost of a spherical droplet of radius R of the equilibrium phase parallel to the applied field embedded in the dominant one (see Fig. 1.10-left) is

$$\Delta F(R) = -2\Omega_d R^d h m_{\text{eq}} + \Omega_{d-1} R^{d-1}\sigma_0 \tag{1.79}$$

where σ_0 is the interfacial free-energy density (the energy cost of the domain wall) and Ω_d is the volume of a d-dimensional unit sphere. We assume here that the droplet has a regular surface and volume such that they are proportional to R^{d-1} and R^d, respectively. The excess free-energy reaches a maximum

$$\Delta F_c = \frac{(d-1)^{d-1}}{d^d}\frac{\Omega_{d-1}^d}{\Omega_d^{d-1}}\left(\frac{\sigma_0}{2hm_{\text{eq}}}\right)^{d-1}\sigma_0 \tag{1.80}$$

at the critical radius

$$R_c = \frac{(d-1)}{d}\frac{\Omega_{d-1}}{\Omega_d}\frac{\sigma_0}{2hm_{\text{eq}}}\,, \tag{1.81}$$

see Fig. 1.10-right ($h > 0$ and $m_{\text{eq}} > 0$ here, the signs have already been taken into account). The free-energy difference vanishes at

$$\Delta F(R_0) = 0 \qquad \Rightarrow \qquad R_0 = \frac{\Omega_{d-1}}{\Omega_d}\frac{\sigma_0}{2hm_{\text{eq}}}\,. \tag{1.82}$$

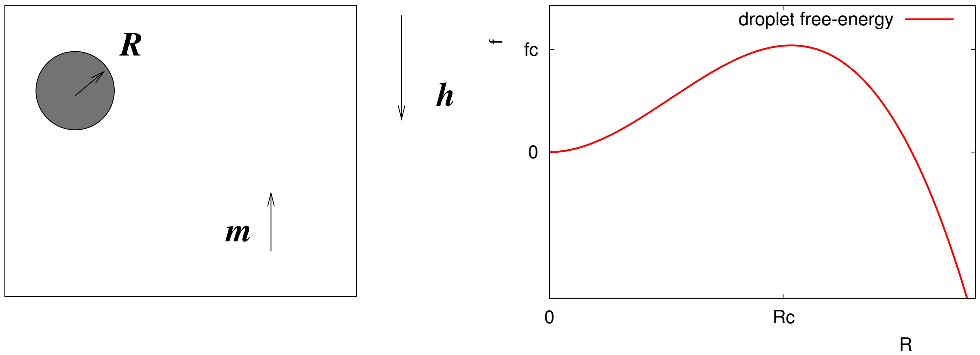


Figure 1.10: Left: the droplet. Right: the free-energy density $f(R)$ of a spherical droplet with radius R.

Several features are to be stressed:

- The barrier vanishes in $d = 1$; indeed, the free-energy is a linear function of R in this case.

- Both R_c and R_0 have the same dependence on $hm_{\rm eq}/\sigma_0$: they monotonically decrease with increasing $hm_{\rm eq}/\sigma_0$ vanishing for $hm_{\rm eq}/\sigma_0 \to \infty$ and diverging for $hm_{\rm eq}/\sigma_0 \to 0$.

- In dynamic terms, the passage above the barrier is done *via thermal activation*; as soon as the system has reached the height of the barrier it rolls on the right side of the 'potential' ΔF and the favourable phase nucleates. It is then postulated that the nucleation time is inversely proportional to the probability of generation of the critical droplet, $t_{\rm nuc}^{-1} \propto P(R_c) \simeq e^{-\beta \Delta F(R_c)}$.

- As long as the critical size R_c is not reached the droplet is not favorable and the system remains positively magnetised.

- In models defined on a lattice, or with anisotropic interactions, the droplets need not be spherical and the particular form they may take has an impact on the results derived above that have to be modified accordingly.

The Imry-Ma argument for the random field Ising model at $T = 0$

Take a ferromagnetic Ising model in a random field, defined in eq. (1.37). In zero applied field and low enough temperature, one may expect a phase transition between a ferromagnetic and a paramagnetic phase at a critical value of the variance of the random fields, $\sigma_h^2 = [h_i^2] \propto h^2$, that sets the scale of the values that these random fields can take. Under the effect of a random field with very strong typical strength, the spins align with the local external fields that point in both directions and the system is paramagnetic. It is, however, non-trivial to determine the effect of a relatively weak random field on the ferromagnetic phase at sufficiently low temperature. The long-range ferromagnetic order could be preserved or else the field could be enough to break up the system into large but finite domains of the two ferromagnetic phases.

A qualitative argument to decide whether the ferromagnetic phase survives or not in presence of the external random field is due to Imry and Ma [45]. Let us fix $T = 0$ and switch on a random field. If a compact domain $\mathcal{D}$ of the opposite order (say down) is created within the bulk of the ordered state (say up) the system pays an energy due to the unsatisfied links lying on the boundary that is

$$\Delta E_{\text{border}} \sim 2JR^{d-1} \tag{1.83}$$

where R is the radius of the domain and $d-1$ is the dimension of the border of a domain embedded in a d dimensional volume, assuming the interface is not fractal. By creating a domain boundary the system can also gain a magnetic energy in the interior of the domain due to the external field:

$$\Delta E_{\text{random field}} \sim -hR^{d/2} \tag{1.84}$$

since there are $N \propto R^d$ spins inside the domain of linear scale R (assuming now that the bulk of the domain is not fractal) and, using the central limit theorem, $-h\sum_{j\in\mathcal{D}} s_i \sim -h\sqrt{N} \propto -hR^{d/2}$. $h \approx \sigma_h$ is the width of the random field distribution. In this discussion we neglected all geometric prefactors that are not important to understand the main parameter dependence in the problem.

One dimensional systems. In $d = 1$ the energy difference is a monotonically decreasing function of R thus suggesting that the creation of droplets is very favourable. There is no barrier to cross to do it. The larger the droplets to form, the better. The system fully disorders.

Dimension lower than two. For any $d < 2$, for small R, the modulus of the random field energy, $\Delta E_{\text{rand field}}$, increases faster with R than the domain wall energy, ΔE_{border}. Therefore, ΔE takes, initially, negative values. For all non-vanishing random fields, there is a critical R *below which* forming domains that align with the local random field becomes favourable. Seen as a function of R, the energy function has a minimum at an R_c that is conveniently written as

$$R_c \sim \left(\frac{J}{h}\right)^{\frac{2}{(2-d)}} \qquad \text{that increases with } \; J/h \;\; \uparrow \tag{1.85}$$

and later crosses zero at an R_0 that scales with J/h in the same way. In particular,

$$R_0 \propto R_c \to \infty \qquad \text{for} \qquad h/J \to 0 \qquad \text{for} \qquad d < 2 \; . \tag{1.86}$$

We note that while this energy function is negative, and this is the case for all R in the limit $h \to 0$ in $d < 2$, it is convenient for the system to reverse droplets with radius $R < R_0$

and thus break the ferromagnetic order in pieces eventually disordering the full system. Consequently, *the uniform ferromagnetic state is unstable* against domain formation for arbitrary random field strength. In other words, in dimensions $d < 2$ random-field disorder prevents spontaneous symmetry breaking. In the resulting Imry-Ma state there is only short-range ordering within randomly oriented domains of average size R_0 which depends on the ratio J/h in the form given above.

Dimension equal to two. This is a marginal case. The function $\Delta E(R)$ is linear and the slope depends on the sign of $2J - h$ (but we neglected many numerical factors) and a more refined study is needed to decide what the system does in this case.

Dimension larger than two. The functional form of the total energy variation $\Delta E = \Delta E_{\text{border}} + \Delta E_{\text{random field}}$ as a function of R is again characterised by $\Delta E \to 0$ for $R \to 0$ and $\Delta E \to \infty$ for $R \to \infty$. The function has a minimum at

$$R_c \sim \left(\frac{h}{J}\right)^{\frac{2}{d-2}} \qquad \text{that increases with } \; h/J \; \uparrow \tag{1.87}$$

and crosses zero at $R_0 \propto R_c$ to approach ∞ at $R \to \infty$. The comparison between these two energy scales yields

$$2JR_0^{d-1} \sim hR_0^{d/2} \qquad \Rightarrow \qquad R_0 \sim \left(\frac{h}{2J}\right)^{\frac{2}{d-2}} \tag{1.88}$$

In particular,

$$R_0 \propto R_c \to 0 \qquad \text{for} \qquad h/J \to 0 \; . \tag{1.89}$$

Therefore, in $d > 2$ the energy difference also decreases from $\Delta E(R = 0) = 0$ to reach a negative minimum at R_c, and then increases back to pass through zero at R_0 and diverge at infinity. The main difference with the $d < 2$ case is the dependence of R_c and R_0 with h/J, the fact that both vanish, for $h/J \to 0$ in $d > 2$. In consequence, in $d > 2$, under an infinitesimal field, it *is not favourable* to reverse domains and long-range ferromagnetic order can be sustained in the sample.

In the arguments above, it has been very important the change in parameter dependence of the R_0 occurring in $d = 2$,

$$\lim_{h/J \to 0} R_0(h/J) = \begin{cases} 0 & \text{if} \quad d > 2 \; , \\ \infty & \text{if} \quad d < 2 \; . \end{cases} \tag{1.90}$$

With this argument one cannot show the existence of a phase transition at h_c nor the nature of it. The argument is such that it suggests that order can be supported by the system at zero temperature and small fields in $d > 2$.

The length scale beyond which domains destroy the uniform state, the so-called breakup length R_B, depends sensitively on the random field. One estimates by comparing

$$|\Delta E_{\text{border}} \sim \Delta R_{\text{random field}} \quad \Rightarrow R_B \sim \frac{h}{J} R_B^{1-d/2} \tag{1.91}$$

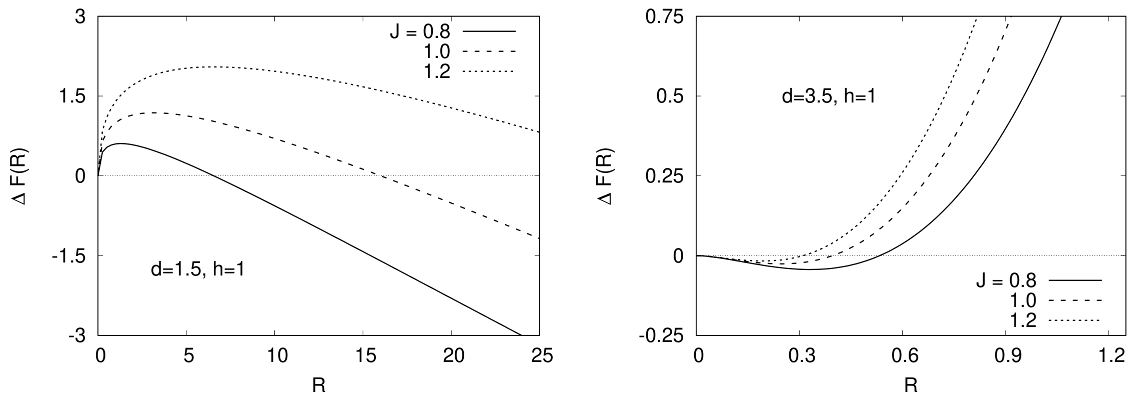


Figure 1.11: The excess free-energy ΔF as a function of the radius of the droplet in the RFIM, that is to say, the sum of Eq. (1.83) and Eq. (1.84), for the parameters given in the two panels. Notice that $d < 2$ in the left panel and $d > 2$ in the right one.

For the marginal dimension $d = 2$, the dependence becomes exponential, $R_B \sim e^{-cJ/h}$, implying that domains become important only at very large scales for weak random fields.

The argument has at least two drawbacks that have been discussed in the literature and shown to be not important for the final conclusions. One is that one should count the number of possible contours with a given length to take into account an entropic contribution to the bubble's free-energy density at non-vanishing temperature. Another one is that one should consider the possibility of there being contours within contours. Both problems have been taken care of, see e.g. [33].

There are rigorous proofs that random fields destroy long-range order (and thus prevent spontaneous symmetry breaking) in all dimensions $d \leq 2$ for discrete (Ising) symmetry and in dimensions $d \leq 4$ for continuous (Heisenberg) symmetry [32]. The existence of a phase transition from a FM to a PM state at zero temperature in the $3d$ RFIM was shown in [33].

The results above hold for short-range correlated disorder. Long-range correlated random fields with correlations that decay as $|\mathbf{r} - \mathbf{r}'|^{-a}$ have stronger effects if $a < d$. In this case, domain formation is favoured for $a < 2$ whereas the uniform ferromagnetic state is stable for $a > 2$.

An elastic line in a random potential

A similar argument has been put forward by Larkin [35] for the random manifold problem.

The interfacial tension, σ, will tend to make an interface, forced into a system as flat as possible. However, this will be resisted by thermal fluctuations and, in a system with random impurities, by quenched disorder.

Let us take an interface model of the type defined in eq. (1.39) with $N = 1$. If one assumes that the interface makes an excursion of longitudinal length L and transverse

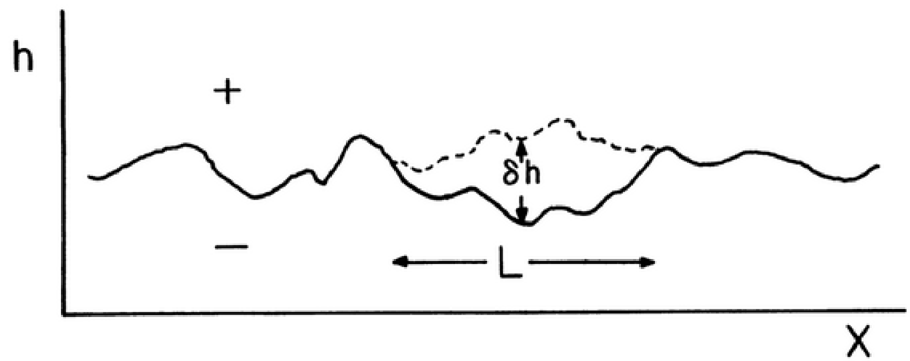


Figure 1.12: Illustration of an interface modeled as a directed manifold. In the example, the domain wall separates a region with positive magnetisation (above) from one with negative magnetisation (below). The line represents a lowest energy configuration that deviates from a flat one due to the quenched randomness. An excitation on a length-scale L is shown with a dashed line. The relative displacement is $\delta h \equiv \delta\phi \simeq L^\alpha$ and the excitation energy $\Delta E(L) \simeq L^\theta$. Figure taken from [46].

length ϕ the elastic energy cost is

$$E_{\text{elast}} = \frac{c}{2} \int d^dx \; (\nabla\phi(\vec{x}))^2 \qquad \Rightarrow \qquad \Delta E_{\text{elast}} \sim cL^d(L^{-1}\phi)^2 = cL^{d-2}\phi^2 \tag{1.92}$$

Ignore for the moment the random potential. Thermal fluctuations cause fluctuations of the kind shown in Fig. 1.12. The interfaces *roughens*, that is to say, it deviates from being flat. Its mean-square displacement between two point $\vec{x}$ and $\vec{y}$, or its *width* on a scale L satisfies

$$\langle[\phi(\vec{x}) - \phi(\vec{y})]^2\rangle \simeq T \; |\vec{x} - \vec{y}|^{2\zeta_T} \tag{1.93}$$

with ζ_T the *roughness exponent*.

The elastic energy cost of an excitation of length L is then

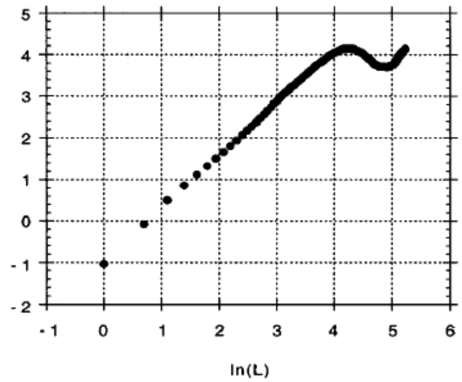
$$\Delta E_{\text{elast}}(L) \simeq cL^{d-2}\phi^2(L) \simeq cTL^{d-2}L^{2\zeta_T} \tag{1.94}$$

and this is of order one if

$$\zeta_T = \frac{2-d}{2} \; . \tag{1.95}$$

In the presence of quenched randomness, the deformation energy cost competes with gains in energy obtained from finding more optimal regions of the random potential. Naively, the energy gain due to the randomness is

$$\int d^dx \; V \simeq [W^2 L^d]^{1/2} \simeq W L^{d/2} \tag{1.96}$$



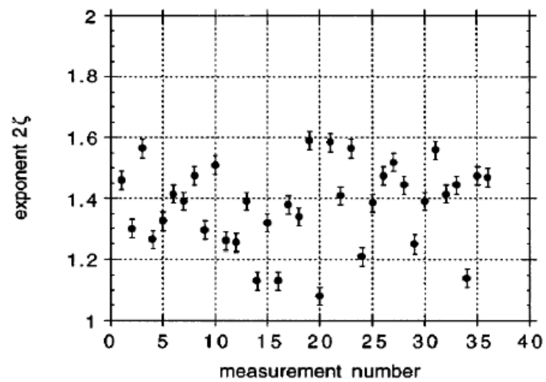


Figure 1.13: The interface width and the roughness exponent in a magnetic domain wall in a thin film. The value measured $\zeta_D \simeq 0.6$ is compatible with the Flory value 2/3 expected for a one dimensional domain wall in a two dimensional space ($N = 1$ and $d = 1$ in the calculations discussed in the text.) [36].

and the balance with the elastic cost, assumed to be the same as with no disorder, yields

$$cTL^{d-2}L^{2\zeta_D} \simeq WL^{d/2} \qquad \Rightarrow \qquad \zeta_D = \frac{4-d}{2} \tag{1.97}$$

This result turns out to be an upper bound of the exponent value [46]. It is called the *Flory exponent* for the roughness of the surface. One then concludes that for $d > 4$ disorder is irrelevant and the interface is flat ($\phi \to 0$ when $L \to \infty$). Since the linearization of the elastic energy [see the discussion leading to eq. (1.39)] holds only if $\phi/L \ll 1$, the result (1.97) may hold only for $d > 1$ where $\alpha < 1$.

Destruction of first order phase transitions under randomness

A first order phase transition is characterized by macroscopic phase coexistence at the transition point. For example, at the liquid-gas phase transition of a fluid, a macroscopic liquid phase coexists with a macroscopic vapour phase. Random-mass disorder locally favours one phase over the other. The question is whether the macroscopic phases survive in the presence of disorder or the system forms domains (droplets) that follow the local value of the random-mass.

Consider a single domain or droplet (of linear size L) of one phase embedded in the other phase. The free energy cost due to forming the surface is

$$\Delta F_{\text{surf}} \sim \sigma L^{d-1} \tag{1.98}$$

where σ is the surface energy between the two phases. The energy gain from the random-mass disorder can be estimated via the central limit theorem, resulting in a typical magnitude of

$$|\Delta F_{\text{dis}}| \sim W^{1/2} L^{d/2} \tag{1.99}$$

where W is the variance of the random-mass disorder. (We are here loosely using free-energy language either because we work at zero temperature or because we assume that the scalings of free-energies and energies are the same.) Note the similarity of these scalings with the ones derived for the random field Ising model.

The macroscopic phases are stable if $|\Delta F_{\text{dis}}| < \Delta F_{\text{surf}}$, but this is impossible in dimensions $d \leq 2$ no matter how weak the disorder is. In dimensions $d > 2$, phase coexistence is possible for weak disorder but will be destabilized for sufficiently strong disorder.

We thus conclude that random-mass disorder destroys first-order phase transitions in dimensions $d \leq 2$. In many examples, the first-order transition is replaced by ('rounded to') a continuous one, but more complicated scenarios cannot be excluded.

The 3d Edwards-Anderson model in a uniform magnetic field

A very similar reasoning is used to argue that there cannot be spin-glass order in an Edwards-Anderson model in an external field [47, 48]. The only difference is that the domain wall energy is here assumed to be proportional to L^y with an *a priori* unknown d-dependent exponent y that is related to the geometry of the domains.

Comments

These arguments are easy to implement when one knows the equilibrium states (or one assumes what they are). They cannot be used in models in which the energy is not a slowly varying function of the domain wall characteristics.

1.5.7 Consequences of the gauge invariace

H. Nishimori used a *local gauge transformation* to derive a series of exact results for averaged observables of finite dimensional disordered systems [12].

The idea follows the steps by which one easily proves, for example, that the averaged local magnetisation of a ferromagnetic Ising model vanishes, that is to say, one applies a transformation of variables within the partition sum and evaluates the consequences over the averaged observables. For example,

$$\langle s_i \rangle = \frac{1}{Z} \sum_{\{s_j = \pm 1\}} s_i \; e^{\beta J \sum_{ij} s_i s_j} = \frac{1}{Z} \sum_{\{s_j = \pm 1\}} (-s_i) \; e^{\beta J \sum_{ij} s_i s_j} = -\langle s_i \rangle \; . \tag{1.100}$$

This immediately implies $\langle s_i \rangle = 0$ and, more generally, the fact that the average of any odd function under $\{s_i\} \to \{-s_i\}$ vanishes exactly.

In the case of disordered Ising systems, one is interested in observables that are averaged over the random variables weighted with their probability distribution. The local gauge transformation that leaves the Hamiltonian unchanged involves a change of spins accompanied by a transformation of the exchanges:

$$\overline{s}_i = \eta_i s_i \qquad \overline{J}_{ij} = \eta_i \eta_j J_{ij} \tag{1.101}$$

with $\eta_i = \pm 1$. The latter affects the couplings probability distribution as this one, in general, is not gauge invariant. For instance, the bimodal pdf $P(J_{ij}) = p\delta(J_{ij} - J) + (1 - p)\delta(J_{ij} + J)$ can be rewritten as

$$P(J_{ij}) = \frac{e^{K_p J_{ij}/J}}{2\cosh K_p} \qquad \text{with} \qquad e^{2K_p} = \frac{p}{1-p} \;, \tag{1.102}$$

as one can simply check. $\tau_{ij} \equiv J_{ij}/J$ are just the signs of the J_{ij}. Under the gauge transformation $P(J_{ij})$ transforms as

$$\overline{P}(\overline{J}_{ij})d\overline{J}_{ij} = P(J_{ij})dJ_{ij} \qquad \Rightarrow \qquad \overline{P}(\overline{J}_{ij}) = P(J_{ij}(\overline{J}_{ij}))\,\frac{dJ_{ij}}{d\overline{J}_{ij}} \tag{1.103}$$

that implies

$$\overline{P}(\overline{J}_{ij}) = \frac{e^{K_p \overline{J}_{ij}/(\eta_i\eta_j J)}}{2\cosh K_p}\frac{1}{\eta_i\eta_j} \qquad \Rightarrow \qquad \overline{P}(\overline{J}_{ij}) = \eta_i\eta_j\,\frac{e^{K_p \overline{J}_{ij}\eta_i\eta_j/J}}{2\cosh K_p} \tag{1.104}$$

For instance, applying the gauge transformation to the internal energy of an Ising spin-glass model with bimodal disorder, after a series of straightforward transformations one finds

$$[\langle H_J \rangle]_J = -N_B J \tanh K_p \tag{1.105}$$

with N_B the number of bonds in the lattice, under the condition $\beta J = K_p$. This relation holds for any lattice. The constraint $\beta J = K_p$ relates the inverse temperature $J/(k_B T)$ and the probability $p = (\tanh K_p + 1)/2$. The curve $\beta J = K_p$ connects the points $(p = 1, T = 0)$ and $(p = 1/2, T \to \infty)$ in the (p, T) phase diagram and it is called the *Nishimori line*.

The proof of the relation above goes as follows. The full pdf of the interactions is

$$P(\{J_{ij}\}) = \prod_{\langle ij \rangle} P(J_{ij}) \tag{1.106}$$

and the average of any disorder dependent quantity is expressed as

$$[A_J] = \sum_{\{J_{ij}=\pm J\}} \prod_{\langle ij \rangle} P(J_{ij}) A_J \tag{1.107}$$

The disorder average Hamiltonian reads

$$[\langle H_J \rangle]_J = \sum_{\{J_{ij}\}} \frac{e^{K_p \sum_{\langle ij \rangle} J_{ij}/J}}{(2\cosh K_p)^{N_B}}\;\frac{\sum_{\{s_i\}}(-\sum_{\langle ij \rangle} J_{ij}s_i s_j)\; e^{\beta \sum_{\langle ij \rangle} J_{ij}s_i s_j}}{\sum_{\{s_i\}} e^{\beta \sum_{\langle ij \rangle} J_{ij}s_i s_j}} \tag{1.108}$$

with N_B the number of bonds in the graph or lattice. Performing the gauge transformation, that is, changing s_i and J_{ij} into the barred ones,

$$[\langle H_J \rangle]_J = \sum_{\{J_{ij}\}} \frac{e^{K_p \sum_{\langle ij \rangle} J_{ij}\eta_i\eta_j/J}}{(2\cosh K_p)^{N_B}}\;\frac{\sum_{\{s_i\}}(-\sum_{ij} J_{ij}s_i s_j)\; e^{\beta \sum_{\langle ij \rangle} J_{ij}s_i s_j}}{\sum_{\{s_i\}} e^{\beta \sum_{\langle ij \rangle} J_{ij}s_i s_j}} \tag{1.109}$$

where the gauge invariance of the Hamiltonian has been used and the spins and interactions have been renamed J_{ij} and s_i. As this is independent of the choice of the parameters $\{\eta_i\}$ used in the transformation, one can sum over all possible 2^N choices and divide by this number keeping the result unchanged:

$$[\langle H_J \rangle]_J = \frac{1}{2^N} \sum_{\{J_{ij}\}} \frac{\sum_{\{\eta_i\}} e^{K_p \sum_{\langle ij \rangle} J_{ij}\eta_i\eta_j/J}}{(2\cosh K_p)^{N_B}} \frac{\sum_{\{s_i\}} (-\sum_{\langle ij \rangle} J_{ij}s_is_j)\, e^{\beta \sum_{\langle ij \rangle} J_{ij}s_is_j}}{\sum_{\{s_i\}} e^{\beta \sum_{\langle ij \rangle} J_{ij}s_is_j}} \tag{1.110}$$

If β is chosen to be $\beta = K_p/J$ the sum over the spins in the denominator (the partition sum in the normalisation) cancels out the sum over the parameters $\eta_i = \pm 1$ introduced via the gauge transformation. The sum over J_{ij} and the remaining sum over the spin configurations can be rewritten

$$[\langle H_J \rangle]_J = \frac{1}{2^N} \frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial \beta} \right) \sum_{\{s_i\}} \prod_{\langle ij \rangle} \sum_{\{J_{ij}=\pm J\}} e^{\beta J_{ij} s_i s_j} . \tag{1.111}$$

Changing now variables in the sum over $J_{ij} = \pm J$ to $\tau_{ij} = J_{ij}s_is_j = \pm J$,

$$\begin{aligned}
[\langle H_J \rangle]_J &= \frac{1}{2^N} \frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial \beta} \right) \sum_{\{s_i\}} \prod_{\langle ij \rangle} \sum_{\tau_{ij}=\pm J} e^{\beta \tau_{ij}} \\
&= \frac{1}{2^N} \frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial \beta} \right) 2^N (2\cosh K_p)^{N_B} ,
\end{aligned} \tag{1.112}$$

where the sum over the spin configurations yields the 2^N factor and the sum over the independent τ_{ij} configurations yields the last factor. Finally, taking the derivative with respect to β:

$$[\langle H_J \rangle]_J = -N_B J \tanh K_p \tag{1.113}$$

with $K_p = \beta J$, defining the Nishimori line in the phase diagram. In the particular case $p = 1/2$, $K_{1/2} = \frac{1}{2} \ln 1 = 0$, which means $\beta = 0$ and therefore $T \to \infty$, not such an interesting case. But for other values of p this line is not trivial.

Exercice 1.15 For Gaussian distributed quenched randomness there also exists a Nishimori line and the averaged internal energy can also be computed exactly on this line. Work out this case.

Many other relations of this kind exist and are explained in [12]. A timely application appeared recently [64] where the gauge transformation was used to put bounds on the Jarzynski relation [65] for the work done in a non-equilibrium transformation of a spin-glass on the Nishikori line.

1.6 Spin-glasses

Let us now discuss a problem in which disorder is so strong as to modify the nature of the low temperature phase [8, 9, 10, 11, 13, 14]. If this is so, one needs to define a new order parameter, capable of identifying order in this phase. We focus on equilibrium properties hereafter.

1.6.1 The ferromagnetic order parameter

The paramagnetic-ferromagnetic transition in a model with no quench randomness is characterised by the local magnetisation, $m_i = \langle s_i \rangle$, or the global magnetisation density, $m = N^{-1} \sum_{i=1}^{N} m_i$, that detach from zero at T_c if the thermodynamic average $\langle \ldots \rangle$ is computed on 'half' phase space to counteract the global spin reversal symmetry of the Hamiltonian. Otherwise, both quantities are identical to zero at all temperatures.

In finite size systems m is distributed around the (two) equilibrium infinite-size limit values, with peaks that get narrower and narrower for larger and larger system sizes. The local magnetisations m_i are also distributed around the (two) equilibrium infinite-size limit m values.

1.6.2 The spin-glass order parameter

The spin-glass equilibrium phase is one in which spins 'freeze' in randomly-looking configurations. In finite dimensions these configurations are spatially irregular. A snapshot looks statistical identical to a high temperature paramagnetic configuration in which spins point in any direction (two if the spins are Ising like). However, while at high temperatures the spins flip rapidly and another snapshot taken immediately after would look completely different from the previous one, at low temperatures two snapshots taken at close times are highly correlated. Similarly, two snapshots taken at the same very long time but on different realisations of the same experiment, that is to say, after quenching the same sample in the same way, are also very similar.

Let us use the language of Ising models in the following.

In a spin-glass state the *local magnetisation* is expected to take a non-zero value, $m_i = \langle s_i \rangle \neq 0$, where the average is interpreted in the restricted sense introduced in the discussion of ferromagnets, that we shall call here within a *pure state*.[2] Instead, the *total magnetisation density*, $m = N^{-1} \sum_{i=1}^{N} m_i$, vanishes since one expects to have as many averaged local magnetisation pointing up ($m_i > 0$) as pointing down ($m_i < 0$) with each possible value of $|m_i|$. Therefore,

$$m_i \neq 0 \qquad \text{but} \qquad m = 0 \, . \tag{1.114}$$

[2] the notion of a pure state will be made more precise below. A mathematical definition can be given by it lies beyond the scope of these lectures.

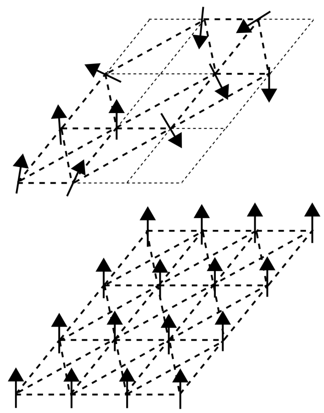


Figure 1.14: A spin configuration in a Heisenberg spin-glass and in an Ising ferromagnet.

Thus, the total magnetisation density, m, of a spin-glass vanishes at all temperatures and it is not a good order parameter.

The spin-glass transition is characterised by a finite peak in the *linear magnetic susceptibility* and a diverging *non-linear magnetic susceptibility*. Let us discuss the former first and show how it yields evidence for the freezing of the local magnetic moments. For a generic magnetic model such that the magnetic field couples linearly to the Ising spin, $H_J[\{s_i\}] \to H_J[\{s_i\}] - \sum_i h_i s_i$, the linear susceptibility is related, via the static *fluctuation-dissipation theorem* to the correlations of the fluctuations of the magnetisation:

$$\chi_{ij} \equiv \left. \frac{\partial \langle\, s_i \,\rangle_h}{\partial h_j} \right|_{h=0} = \beta \,\langle\, (s_i - \langle\, s_i \,\rangle)(s_j - \langle\, s_j \,\rangle) \,\rangle \; . \tag{1.115}$$

The averages in the rhs are taken without perturbing field. This relation is proven by using the definition of $\langle\, s_i \,\rangle_h$ and simply computing the derivative with respect to h_j.

Exercise 1.16 Prove Eq. (1.115).

In particular,

$$\chi_{ii} = \beta \,\langle\, (s_i - \langle\, s_i \,\rangle)^2 \,\rangle = \beta \left(1 - m_i^2\right) \geq 0 \; , \tag{1.116}$$

with $m_i = \langle\, s_i \,\rangle$. The total susceptibility measured experimentally is $\chi \equiv N^{-1} \sum_{ij} \chi_{ij}$. On the experimental side we do not expect to see $O(1)$ sample-to-sample fluctuations in this global quantity. On the analytical side one can use a similar argument to the one presented in Sect. 1.2.4 to argue that χ should be *self-averaging* (it is a sum over the entire volume of site-dependent terms). Thus, the experimentally observed susceptibility of sufficiently large samples should be given by

$$\chi = [\,\chi\,] = N^{-1} \sum_{ij} [\,\chi_{ij}\,] \approx N^{-1} \sum_i [\,\chi_{ii}\,] = N^{-1} \sum_i \beta \left(1 - [\,m_i^2\,]\right) \; , \tag{1.117}$$

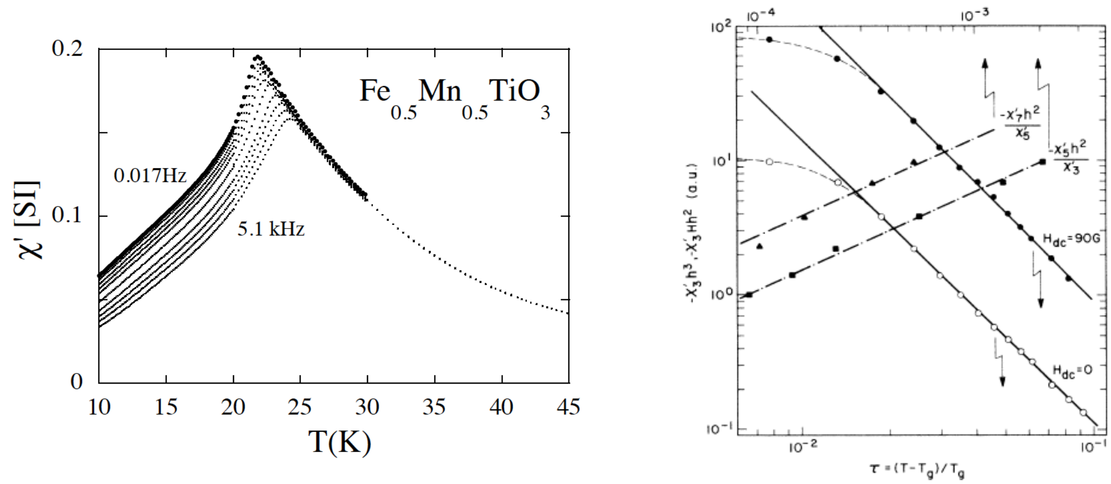


Figure 1.15: Left: The ac-susceptibility of $Fe_{0.5}Mn_{0.5}TiO_3$ at logarithmically evenly spaced frequencies from 0.017 Hz to 1.7 kHz (top to bottom) [13]. Right: Temperature dependence of $-\chi_3$ (vertical axis) above T_c measured at 10 Hz in static fields of 0 (open circles) and 90 G (solid circles) as a function of reduced temperature τ (lower axis). The slope is $-\gamma$. Plot of the susceptibility ratios $-\chi_5' h^2/\chi_3'$, $-\chi_7' h^2/\chi_5'$ (top axis) as a function of $-\chi_3$ (vertical axis) in zero field. The slope is $1+\beta/\gamma$ [90, 91].

since we can expect that cross-terms cancel under the disorder average.[3] The fall of χ at low temperatures with respect to its value at T_c, *i.e.* the *cusp* observed experimentally, signals the freezing of the *local magnetizations*, m_i, in the non-zero values that are more favourable thermodynamically. Note that this argument is based on the assumption that the measurement is done in equilibrium. The linear ac susceptibility of a spin-glass sample is shown in the left panel in Fig. 1.15.

Thus, the natural *global order parameter* that characterises the spin-glass transition is

$$q \equiv N^{-1} \sum_i [\, m_i^2 \,] \tag{1.118}$$

as proposed in the seminal 1975 Edwards-Anderson paper [15]. q vanishes in the high temperature phase since all m_i are zero but it does not in the low temperature phase since the square power takes care of the different signs. Averaging over disorder eliminates the site dependence. Thus, q is also given by

$$q = [\, m_i^2 \,] \; . \tag{1.119}$$

These definitions, reasonable as they seem at a first glance, hide a subtle distinction that we discuss below.

[3]Note that χ_{ij} can take negative values. Moreover, the sum over $i \neq j$ has $O(N^2)$ terms of different sign and then central limit theorem implies that, if they are uncorrelated, the result is $O(N)$ that once normalised by N yields a value $O(1)$. The further average over the J_{ij} yields the vanishing result.

1.6.3 Two or many pure states

Let us keep *disorder fixed* and imagine that once the global spin inversion symmetry has been taken into account there still remain more than one pure or equilibrium states in the selected sample. Consider the disorder-dependent quantity

$$q_J = N^{-1} \sum_i m_i^2 \tag{1.120}$$

where the m_i depend on the realisation of the exchanges but we do not write the subindex J explicitly to lighten the notation. Then, two possibilities for the statistical average in $m_i = \langle\, s_i \,\rangle$ have to be distinguished:

- If we interpret it in the same restricted sense as the one discussed in the paramagnetic - ferromagnetic transition of the usual Ising model, *i.e.* under a pinning field that selects *one* chosen pure state, in (1.120) we define a disorder dependent *Edwards-Anderson parameter*,

$$q_{J\,\mathrm{EA}}^{\alpha} = N^{-1} \sum_i^N (m_i^\alpha)^2 \;, \tag{1.121}$$

  where we label α the selected pure state. Although $q_{J\,\mathrm{EA}}^{\alpha}$ could depend on α it turns out that in all known cases it does not and the α label in $q_{J\,\mathrm{EA}}^{\alpha}$ is superfluous. In addition, $q_{J\,\mathrm{EA}}$ could fluctuate from sample to sample since the individual m_i's do. It turns out that in the thermodynamic limit $q_{J\,\mathrm{EA}}$ does not fluctuate. Therefore, later we will use

$$q_{\mathrm{EA}} = q_{J\,\mathrm{EA}} \;. \tag{1.122}$$

- If, instead, the statistical average in m_i^α runs over *all* possible equilibrium states (on half the phase space, that is to say, eliminating spin-reversal) the quantity (1.120) has non-trivial contributions from overlaps between different states. Imagine each state has a probability weight w_α^J (in the ferromagnetic phase of the Ising model one has only one (two) pure states with $w_1 = w_2 = 1/2$) then

$$q_J = N^{-1} \sum_{i=1}^N \left(\sum_\alpha w_\alpha^J m_i^\alpha \right)^2 \;. \tag{1.123}$$

In the ferromagnetic transition $q = q_{\mathrm{EA}} = m^2$, and q_{EA} and q are identical order parameters.

In the disorder case, $q_{J\,\mathrm{EA}}^{\alpha}$ takes the same value on all equilibrium states independently of there being only two (as in the usual ferromagnetic phase) or more (as we shall see appear in fully-connected spin-glass models). Therefore it does not allow us to distinguish between the two-state and the many-state scenarii. Instead, q_J does.

It is important to note that which are the pure states in the model depends on the quenched disorder realization.

The parameter q in Eq. (1.118), that involves a further average over quenched disorder, is then

$$q = [\, q_J \,] \; . \tag{1.124}$$

Having defined a disorder-dependent order parameter, q_J, and its disorder average, q, that explains the decay of the susceptibility below T_c, we still have to study whether this order parameter characterises the low temperature phase completely. It will turn out that the knowledge of the disorder-averaged q is not enough, at least in fully-connected and dilute spin-glass models. Indeed, one needs to consider the disorder-dependent *probability distribution* of the fluctuating q_J, $P_J(q_J)$, see Fig. 1.16. The more pertinent definition of an order parameter as being given by such a probability distribution allows one to distinguish between the simple, two-state, and the complex, many-state, scenarii.

In practice, a way to compute the probability distribution of the order parameter is by using an *overlap* – or correlation – between two spin configurations, say $\{s_i\}$ and $\{\sigma_i\}$, defined as

$$q^J_{s\sigma} = N^{-1} \sum_i \langle\, s_i \sigma_i \,\rangle \tag{1.125}$$

where $\langle \dots \rangle$ is an unrestricted thermal average. $q^J_{s\sigma}$ takes values between -1 and 1. It equals one if $\{s_i\}$ and $\{\sigma_i\}$ differ in a number of spins that is smaller than $O(N)$, it equals -1 when the two configurations are totally anti-correlated – with the same proviso concerning a number of spins that is not $O(N)$ – and it equals zero when $\{s_i\}$ and $\{\sigma_i\}$ are completely uncorrelated. Other values are also possible. Note that the *self-overlap* of a configuration with itself is identically one for Ising spins.

The overlap can be computed by running a Monte Carlo simulation, equilibrating a sample and recording many equilibrium configurations. With them one computes the overlap and should find a histogram with two peaks at $q_{\rm EA}$ and $-q_{\rm EA}$ (the values of the overlap when the two configurations fall in the same pure state or in the sign reversed ones) and, in cases with many different pure states, other peaks at other values of $q^J_{s\sigma}$. This is observed in the $3d$ EA model as exemplified in Fig. 1.16. Note that $q^J_{s\sigma}$ is related to the q definition above. A related definition is the one of the *Hamming distance*:

$$d^J_{s\sigma} = N^{-1} \sum_{i=1}^{N} \langle\, (s_i - \sigma_i)^2 \,\rangle = 2(1 - q^J_{s\sigma}) \; . \tag{1.126}$$

Figure 1.16 shows the probability distribution $P_J(q)$ obtained from a MC simulation of the $3d$ EA model. The external peaks are at $q_{\rm EA}$, cases in which the two copies are taken in the same equilibrium state. In the first panel there are only two states, one and its reversed. In the other figures other peaks appear associated with the existence of more than one state and the overlap between them. They are sampled differently in the various panels since the temperature of the simulation is changed.

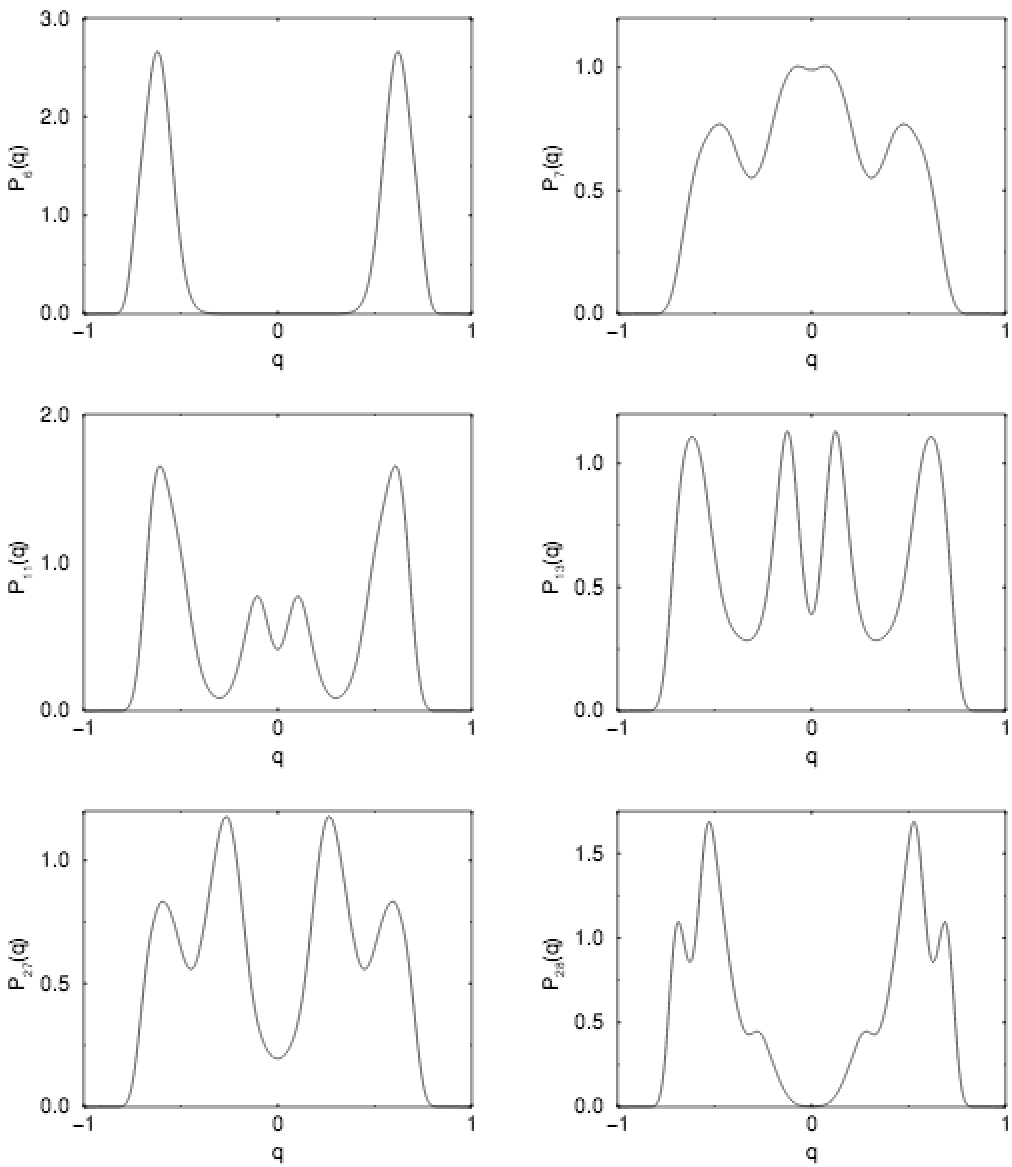


Figure 1.16: Monte Carlo simulations of the $3d$ Edwards-Anderson model. The disorder-dependent overlap probability distribution function, $P_J(q)$, for different choices of the random couplings. Figure taken from [93].

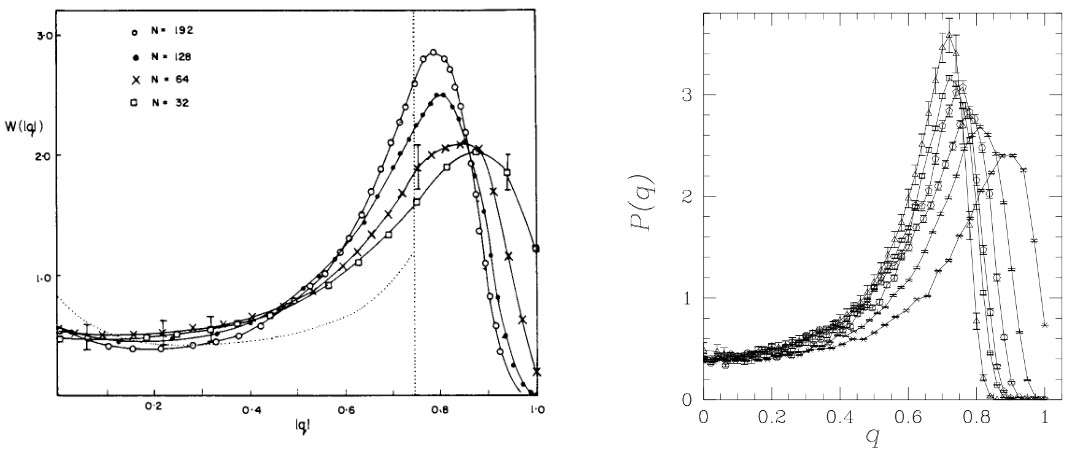


Figure 1.17: Monte Carlo simulations of the Sherrington-Kirkpatrick (left) and $3d$ Edwards-Anderson (right) models. The disorder averaged overlap distribution function, $[P_J(q)]$, for different system sizes. Left figure taken from [94] with maximal system size $N = 192$ and Right figure taken from [95], with linear system sizes $L = 4, 6, 8, 10, 12, 16$. The dotted line in the left plot is an approximate solution to Parisi's equation for the SK model.

Instead, Fig. 1.17 displays the disorder averaged $P(q)$ for a $3d$ Edwards-Anderson model at low temperatures. The dotted line is the theoretical prediction for the Sherrington-Kirkpatrick model that we will discuss below. It has a sharp peak at $q_{\rm EA}$ and a non-vanishing continuous weight at all values of $q < q_{\rm EA}$. The various lines represent numerical data for different system sizes. The questions is whether the intermediate part will remain non-vanishing in the infinite size limit or whether it will eventually vanish.

1.6.4 Pinning fields

In the discussion of the ferromagnetic phase transition we established that one of the two equilibrium states, related by spin reversal symmetry, is chosen by a small pinning field that is taken to zero after the thermodynamic limit, $\lim_{h\to 0}\lim_{N\to\infty}$.

In a problem with quenched disorder it is no longer feasible to choose and apply a magnetic field that is correlated to the statistical averaged local magnetization in a single pure state since this configuration is not known! Moreover, the remanent magnetic field that might be left in any experience will not be correlated with any special pure state of the system at hand.

Which is then the statistical average relevant to describe experiments? We shall come back to this point below.

1.6.5 Divergent susceptibility

In a pure magnetic system with a second-order phase transition the susceptibility of the order parameter to a field that couples linearly to it diverges when approaching the transition from both sides. In a paramagnet, one induces a local magnetisation with a local field

$$m_i = \langle\, s_i \,\rangle = \sum_{j=1}^{N} \chi_{ij} h_j \tag{1.127}$$

with χ_{ij} the linear susceptibilities, the magnetic energy given by $E = E_0 - \sum_i s_i h_i$, and the field is set to zero at the end of the calculation. Using this expression, the order parameter in the high temperature phase becomes

$$q = q_{\rm EA} = \frac{1}{N}\sum_{i=1}^{N} [\, m_i^2 \,] = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} [\, \chi_{ij}\chi_{ik} h_j h_k \,] \tag{1.128}$$

If the applied fields are random and taken from a probability distribution such that $\overline{h_j h_k} = \sigma^2 \delta_{jk}$ one can replace $h_j h_k$ by $\sigma^2 \delta_{jk}$ and obtain

$$q = \frac{1}{N}\sum_{i=1}^{N} [\, m_i^2 \,] = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} [\, \chi_{ij}^2 \,]\, \sigma^2 \equiv \chi_{SG}\, \sigma^2 \; . \tag{1.129}$$

σ^2 acts as a field conjugated to the order parameter $q_{\rm EA}$. (One can also argue that a uniform field looks random to a spin-glass sample and therefore the same result holds. It is more natural though to use a trully random field since a uniform one induces a net magnetization in the sample.) The *spin-glass susceptibility* is then defined as

$$\chi_{SG} \equiv \frac{1}{N}\sum_{ij} [\, \chi_{ij}^2 \,] = \frac{\beta^2}{N}\sum_{ij} [\, (\langle\, s_i s_j \,\rangle - \langle\, s_i \,\rangle\langle\, s_j \,\rangle)^2 \,] = \frac{\beta^2}{N}\sum_{ij} [\, \langle\, s_i s_j \,\rangle^2 \,] \tag{1.130}$$

and one finds that it diverges as $T \to T_c^+$ as expected in a second-order phase transition. (Note that there is no cancelation of crossed terms because of the square.) Indeed, the divergence of χ_{SG} is related to the divergence of the non-linear magnetic susceptibility that is measurable experimentally and numerically. An expansion of the total mangnetization in powers of a uniform field h acting as $E \to E - h\sum_i s_i$ is

$$M_h = \chi h - \frac{\chi^{(3)}}{6} h^3 + \dots \;, \tag{1.131}$$

and the first non-linear susceptibility is then given by

$$-\chi^{(3)} \equiv \left.\frac{\partial^3 M_h}{\partial h^3}\right|_{h=0} = -\beta^{-1} \left.\frac{\partial^4 \ln Z_h}{\partial h^4}\right|_{h=0} = -\frac{\beta^3 N}{3} \left\langle \left(\sum_i s_i \right)^4 \right\rangle_c \tag{1.132}$$

with the subindex c indicating that the quartic correlation function is connected. Above T_c, $m_i = 0$ at zero field,

$$\chi^{(3)} = \beta^3 \sum_{ijkl} (\langle\, s_i s_j s_k s_l \,\rangle - 3\langle\, s_i s_j \,\rangle\langle\, s_k s_l \,\rangle) = \frac{\beta^3}{N} 3 \left(4N - 6 \sum_{ij} \langle\, s_i s_j \,\rangle^2 \right) , \qquad (1.133)$$

and one can identify χ_{SG} when $i = k$ and $j = l$ plus many other terms that we assume are finite. Then,

$$\chi^{(3)} = \beta(\chi_{SG} - \frac{2}{3}\beta^2) . \qquad (1.134)$$

This quantity can be accessed experimentally. A careful experimental measurement of $\chi^{(3)}$, $\chi^{(5)}$ and $\chi^{(7)}$ demonstrated that all these susceptibilities diverge at T_c [90, 91], see the right panel in Fig. 1.15.

1.6.6 Phase transition and scaling

Having identified an order parameter, the linear and the non-linear susceptibility one can now check whether there is a static phase transition and, if it is of second order, whether the usual scaling laws apply. Many experiments have been devoted to this task. It is by now quite accepted that Ising spin-glasses in $3d$ have a conventional second order phase transition. Still, the exponents are difficult to obtain and there is no real consensus about their values. There are two main reasons for this: one is that as T_c is approached the dynamics becomes so slow that equilibrium measurements cannot really be done. Critical data are thus restricted to $T > T_c$. The other reason is that the actual value of T_c is difficult to determine and the value used has an important influence on the critical exponents. Possibly, the most used technique to determine the exponents is *via* the scaling relation for the non-linear susceptibility:

$$\chi_{nl} = t^{\beta} f\left(\frac{h^2}{t^{\gamma+\beta}}\right) \qquad (1.135)$$

with $t = |T - T_c|/T_c$ and one finds, approximately, the values given in Table 1 to be compared with the values for the ferromagnetic transitions summarized in Table 1.

No cusp in the specific heat of spin-glasses is seen experimentally. Since one expects a second order phase transition this means that the divergence of this quantity must be very weak.

The critical exponents satisfy the usual relations

$$\begin{aligned} &\gamma = \nu(2-\eta) && \alpha = 2 - \nu d \\ &\beta = (2+\alpha-\gamma)/2 && 2\beta\delta = 2+\alpha+\gamma \end{aligned} \qquad (1.136)$$

1.6.7 The droplet theory

	d	β	γ	δ	α	ν	η
FM	∞	1	1	2	-1	1/2	0
FM	3	0.326	1.237	4.790	0.110	0.630	0.036
SK	∞	1	1	2	-1	1/2	0
Exp	3	1	2.2	3.1	x		

Table 1: Critical exponents in the Ising ferromagnetic and spin-glass transitions. $d \to \infty$ corresponds to the mean-field results and the Sherrington-Kirkpatrick model. Experiments measuring the critical exponents of an Ising spin-glass were reported in [97], for example, and are given in the last row.

Results for $\pm J$ model at $p = 0.5$ taken from H. Katzgraber, M. Körner, and A. P. Young, Phys. Rev. B **73**, 224432 (2006).

authors	year	T_c	ν	η
Ogielski, Morgenstern	1985	1.20(5)	1.2(1)	
Ogielski	1985	1.175(25)	1.3(1)	-0.22(5)
Singh, Chakravarty	1986	1.2(1)	1.3(2)	
Bhatt, Young	1985	1.2(2)	1.3(3)	-0.3(2)
Kawashima, Young	1996	1.11(4)	1.7(3)	-0.35(5)
Bernardi et al	1996	1.165(10)		-0.245(20)
Berg, Janke	1998	1.12(1)		-0.37(4)
Palassini, Caracciolo	1999	1.156(15)	1.8(2)	-0.26(4)
Mari, Campbell	1999	1.20(1)		-0.21(2)
Ballesteros et al.	2000	1.138(10)	2.15(15)	-0.337(15)
Mari, Campbell	2001	1.190(15)		-0.20(2)
Mari, Campbell	2002	1.195(15)	1.35(10)	-0.225(25)
Nakamura et al	2003	1.17(4)	1.5(3)	-0.4(1)
Pleimling, Campbell	2005	1.19(1)		-0.22(2)
Katzgraber et al.	2006	1.120(4)	2.39(5)	-0.395(17)
our result	2007	1.101(5)	2.53(8)	-0.384(9)

Figure 1.18: List of critical exponents of the $3d$ EA model. The last results for ν and η were determined with numerical simulations using system sizes $L = 3 - 28$ [96].

The droplet theory is a phenomenological model that assumes that the low temperature phase of a spin-glass model has only two equilibrium states related by an overall spin flip. It is then rather similar to a ferromagnet, only that the nature of the order in the two equilibrium states is not easy to see, it is not just most spins pointing up or most spins pointing down with some thermal fluctuations within. At a glance, one sees a disordered paramagnetic like configuration and a more elaborate order parameter has to be measured to observe the order. The spin-glass phase is then called a *disguised ferromagnet* and a usual spontaneous symmetry breaking (between the two equilibrium states related spin reversal symmetry) leading to usual ergodicity breaking is supposed to take place at T_c.

Once this assumption has been done, renormalisation group arguments are used to describe the scaling behaviour of several thermodynamic quantities. The results found are then quantitatively different from the ones for a ferromagnet but no *novelties* appear.

1.7 Mean-field treatment at fixed disorder: TAP

In the previous section we analysed finite dimensional models, with approximation methods and a few exact ones. Here, we focus on what are called mean-field models, or models defined on the complete graph in such a way that the mean-field treatment becomes exact. We present two techniques: the Thouless-Anderson-Palmer (TAP) approach, we give a hint on how the cavity method works, and we discuss the replica trick.

1.7.1 The TAP equations

Disordered models have quenched random interactions. Due to the fluctuating values of the exchanges, one expects that the equilibrium configurations be such that *in each equilibrium state* the spins freeze in different directions. The local averaged magnetizations need not be identical, on the contrary one expects $\langle s_i \rangle = m_i$ and, if many states exist, each of them can be identified by the vector $(m_1, \dots, m_N)$.

Let us focus on the Sherrington-Kirkpatrick model, defined by

$$H_J^{\mathrm{SK}} = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j - \sum_i h_i^{\mathrm{ext}} s_i \tag{1.137}$$

with interaction strengths J_{ij} taken from a Gaussian pdf and scaled with N in such a way that the thermodynamic limit is non-trivial:

$$P(J_{ij}) = (2\pi\sigma_N^2)^{-\frac{1}{2}} \, e^{-\frac{J_{ij}^2}{2\sigma_N^2}} \tag{1.138}$$

and external applied field h_i^{ext}. The first two-moments of the exchange distribution are $[J_{ij}] = 0$ and $[J_{ij}^2] = \sigma_N^2 = J^2/N$, implying that $J_{ij} \simeq 1/\sqrt{N}$. This scaling is justified by

$$h_i^{\mathrm{loc}} \equiv \sum_{j(\neq i)} J_{ij} \langle s_j \rangle \simeq \sum_{j(\neq i)} \frac{1}{\sqrt{N}} m_j \simeq \frac{1}{\sqrt{N}} \sum_{j(\neq i)} m_j \simeq 1 \tag{1.139}$$

because of the central limit theorem. (Note that one can express the energy in terms of the local magnetizations, $H = -\sum_i m_i h_i^{\mathrm{loc}}$, and that this is $O(N)$ as it should, since one expects that the local magnetization will be aligned with the local field and hence all terms in the sum be positive.) We will use this scaling at length below.

One may try to use the naive mean-field equations, see the derivation in App. 1.C, generalised to local variational parameters m_i, to characterise the low temperature properties of these models at *fixed quenched disorder*:

$$m_i = \tanh\left(\beta h_i^{\mathrm{loc}}\right) = \tanh\left(\sum_{j(\neq i)} \beta J_{ij} m_j + \beta h_i^{\mathrm{ext}}\right) \tag{1.140}$$

and determine then the different $\{m_i^\alpha\} = (m_1^\alpha, \dots, m_N^\alpha)$ values from them, with the label α indicating the possibility of there being many solutions to these equations. It is important to reckon that, in this discussion, the $m_i = \langle s_i \rangle$ are assumed to be averaged in each thermodynamic state (with no mixture between them).

It has been shown by Thouless-Anderson-Palmer (TAP) [49] that these equations are not completely correct even in the fully-connected disordered case: a term which is called the *Onsager reaction term* is missing. This term represents the reaction of the spin i: the magnetisation of the spin i produces a field $h'_{j(i)} = J_{ji} m_i = J_{ij} m_i$ on spin j; this field induces a magnetisation $m'_{j(i)} = \chi_{jj} h'_{j(i)} = \chi_{jj} J_{ij} m_i$ on the spin j. This magnetisation, in turn, produces a field $h'_{i(j)} = J_{ij} m'_{j(i)} = J_{ij} \chi_{jj} J_{ij} m_i = \chi_{jj} J_{ij}^2 m_i$ on the site i. The equilibrium fluctuation-dissipation relation between susceptibilities and connected correlations implies $\chi_{jj} = \beta \langle (s_j - \langle s_j \rangle)^2 \rangle = \beta(1-m_j^2)$ and one then has $h'_{i(j)} = \beta(1-m_j^2) J_{ij}^2 m_i$. The idea of Onsager – or *cavity method* – is that one has to study the ordering of the spin i in the absence of its own effect on the rest of the system. Thus, the total field produced by the sum of $h'_{i(j)} = \beta(1-m_j^2) J_{ij}^2 m_i$ over all the spins j with which it can connect, has to be subtracted from the mean-field created by the other spins in the sample, *i.e.*

$$h_i^{\mathrm{loc}} = \sum_{j(\neq i)} J_{ij} m_j + h_i^{\mathrm{ext}} - \beta m_i \sum_{j(\neq i)} J_{ij}^2 (1-m_j^2) \tag{1.141}$$

where h_i^{ext} is the external field. The equations then read

$$m_i = \tanh \left[\sum_{j(\neq i)} (\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1-m_j^2)) + \beta h_i^{\mathrm{ext}} \right] .$$

The reason why the reaction term does not appear in the mean-field equations for ferromagnets, the well-known Curie-Weiss equation $m = \tanh(\beta J m + \beta h)$ is that it is sub-leading with respect to the first one. We now discuss why it is not so in the disordered case. Let us study the orders of magnitude, as powers of N, of each term in the r.h.s. In the first term

$$\sum_{j(\neq i)} J_{ij} m_j \simeq \sum_{j(\neq i)} \frac{1}{\sqrt{N}} m_j \simeq \frac{1}{\sqrt{N}} \sum_{j(\neq i)} m_j \simeq 1 \tag{1.142}$$

because of the central limit theorem. In the second term

$$\sum_{j(\neq i)} J_{ij}^2 (1-m_j^2) \simeq \sum_{j(\neq i)} \frac{1}{N} (1-m_j^2) \simeq 1 \tag{1.143}$$

because J_{ij} appears squared, $1-m_j^2$ is positive (or zero), and all terms in the sum are (semi)positive definite. Thus, in disordered systems the reaction term is of the same order of the usual mean-field; a correct mean-field description has to include it. In the ferromagnetic case this term can be neglected since it is sub-leading in N, since $J_{ij}^2 = J^2/N^2$ in this case, obtaining $\mathcal{O}(1/N)$ for the second term while the first one is $\sum_{j(\neq i)} (1/N)\, m_j = \mathcal{O}(1)$ (since $m_j = m$ in this case).

The argument leading to the Onsager reaction term can be generalised to include the combined effect of the magnetisation of spin i on a sequence of spins in the sample, *i.e.* the effect on i on j and then on k that comes back to i. For a loop joining i with $i_1, i_2, \dots, i_n$ and back to i, the field created at site i due to its own magnetization m_i is

$$h'_i = \beta^n \sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} J_{ii_1} J_{i_1 i_2} \dots J_{i_n i} (1 - m_{i_1}^2)(1 - m_{i_2}^2) \dots (1 - m_{i_n}^2)\, m_i \; . \tag{1.144}$$

Using orders of magnitud (unless $n = 1$ which is special because the two J's are the same and the sign is lost $J_{ii_1}^2$),

$$h'_i \sim k_B T \, \frac{(\beta J)^{n+1}}{N^{(n+1)/2}} \, N^{n/2} \, (1 - m_{i_1}^2)(1 - m_{i_2}^2) \dots (1 - m_{i_n}^2)\, m_i \sim N^{-1/2} \; . \tag{1.145}$$

Using the fact that there is a sum over a very large number of elements, J_{ij}^2 can be replaced by its site-independent variance $[J_{ij}^2] = J^2/N$ in the last term in (1.142). Introducing the Edwards-Anderson parameter

$$q_{\rm EA} = \frac{1}{N} \sum_{i=1} m_i^2 \tag{1.146}$$

a simplified expression of the TAP equations follows:

$$m_i = \tanh \left[\beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 J^2 (1 - q_{\rm EA})\, m_i + \beta h_i^{\rm ext} \right] \; . \tag{1.147}$$

The generalisation of the argument leading to the reaction term to p spin interactions

$$H_J[\{s_i\}] = - \sum_{i_1 < \cdots < i_p} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} \qquad\qquad [\, J_{i_1 \dots i_p}^2 \,] = \frac{J^2 p!}{2 N^{p-1}} \tag{1.148}$$

is not so straightforward. An alternative derivation has been given by Biroli [50], using the method in [51]. The TAP equations for p-spin fully connected models read [52]

$$m_i = \tanh \left[\frac{\beta}{(p-1)!} \sum_{i_2 \neq \cdots \neq i_p} \left(J_{ii_2 \dots i_p} m_{i_2} \dots m_{i_p} - \beta J_{ii_2 \dots i_p}^2 (1 - m_{i_2}^2) \dots (1 - m_{i_p}^2)\, m_i \right) \right]$$

where we set the external field to zero, $h_i^{\rm ext} = 0$. The first contribution to the internal field is proportional to $J_{i_1 i_2 \dots i_p} \sim N^{-(p-1)/2}$ and once the $p-1$ sums performed it is of order one. The reaction term instead is proportional to $J_{ii_2 \dots i_p}^2$ and, again, a simple power counting shows that it is $O(1)$. Using the fact that there is a sum over a very large number of elements, $J_{i_1 \dots i_p}^2$ can be replaced by its site-independent variance $[J_{i_1 \dots i_p}^2] = p! J^2/(2N^{p-1})$ in the last term in (1.149). Introducing the Edwards-Anderson parameter $q_{\rm EA} = \frac{1}{N} \sum_{i=1} m_i^2$ as done above,

$$m_i = \tanh \left[\frac{\beta}{(p-1)!} \sum_{i_2 \neq \cdots \neq i_p} J_{ii_2 \dots i_p} m_{i_2} \dots m_{i_p} - \frac{\beta^2 J^2 p}{2} m_i (1 - q_{\rm EA})^{p-1} + \beta h_i^{\rm ext} \right] \; .$$

Finally, we give the TAP equations for the spherical $p \geq 2$ model. The difference with the Ising case lies in the entropic contribution to the free energy which reads

$$-\frac{T}{2} \ln \left(1 - \frac{1}{N} \sum_i m_i^2 \right) = -\frac{T}{2} \ln(1 - q_{\rm EA}) \; . \tag{1.149}$$

The Onsager correction is the same as for Ising spins.

Exercise 1.17 Consider the Hopfield model $\mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j$ in which $J_{ij} = \frac{1}{N} \sum_{\mu=1}^{N_p} \xi_i^\mu \xi_j^\mu$ with $N_p = \alpha N$ and α finite, number of patters. The components of the ξ_i^μ are i.i.d. random variables with zero mean and finite variance, as realized, for example, by a bimodal $p(\xi_i^\mu = \pm 1) = 1/2$.

- Calculate the average $[J_{ij}]$ and show that its variance $[(J_{ij} - [J_{ij}])^2]$ equals α/N.
- What is the order of magnitud of the typical J_{ij} that you can extract from this analysis? Is it the same as for the SK model?
- Given the structure of the J_{ij}'s, what is the average of three nested exchanges $[J_{ii_1} J_{i_1 i_2} J_{i_2 i}]$? And its sign?
- Extend the analysis to the generic case $[J_{ii_1} J_{i_1 i_2} \dots J_{i_n i_1}]$.
- Now, replace the product of exchanges by their average over disorder. Which is the series obtained? Resum it and show that the Onsager correction term is

$$-\frac{\alpha \beta^2 q_{\rm EA}}{1 - \beta(1 - q_{\rm EA})} m_i \tag{1.150}$$

  with $q_{\rm EA} = \frac{1}{N} \sum_{j=1}^N m_j^2$.

The phase transition

The importance of the reaction term becomes clear from the analysis of the linearised equations, expected to describe the second order critical behaviour of the SK model ($p = 2$) in the absence of an applied field. Assuming that all the m_i are small, as expected close to the phase transition, the TAP eqs. (1.147) become

$$m_i \sim \beta \sum_{j (\neq i)} J_{ij} m_j - \beta^2 J^2 m_i + \beta h_i^{\rm ext} \; . \tag{1.151}$$

A change of basis to the one in which the J_{ij} matrix is diagonal leads to $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$. The notation we use is such that J_λ is an eigenvalue of the J_{ij} matrix associated to the eigenvector $\vec{v}_\lambda$. m_λ represents the projection of $\vec{m}$ on the eigenvector $\vec{v}_\lambda$, $m_\lambda = \vec{v}_\lambda \cdot \vec{m}$, with $\vec{m}$ the N-vector with components m_i. The *staggered susceptibility* then reads

$$\chi_\lambda \equiv \left. \frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}} \right|_{\vec{h}^{\rm ext} = \vec{0}} = \beta \, \left(1 - \beta J_\lambda + (\beta J)^2 \right)^{-1} \; . \tag{1.152}$$

Random matrix theory tells us that the eigenvalues of the symmetric matrix, with i.i.d. entries, J_{ij}, taken from a Gaussian or bimodal pdf with zero mean (GOE ensemble), are distributed with the Wigner semi-circle law [55]. For the normalisation of the J_{ij}'s that we used, the largest eigenvalue is $J_\lambda^{\max} = 2J$ [55]. The staggered susceptibility for the largest eigenvalue diverges at $\beta_c J = 1$. Note that without the reaction term the divergence appears at the inexact value $T^* = 2T_c$ (see Sect. 1.8 for the replica solution of the SK model which yields an alternative derivation of the critical temperature).

How to solve the TAP equations

The TAP equations are N non-linear coupled equations at fixed realization of the couplings, taken from their pdf. At sufficiently high temperature, and zero external field, they should admit a single stable solution,

$$m_i = 0 \qquad \text{for all } \; i \; . \tag{1.153}$$

Lowering the temperature, solutions $\{m_i \neq 0\}$ should start appearing but, how should one find them?

The usual graphical solution of the Curie–Weiss mean-field equation for the single order parameter m of the paramagnetic - ferromagnetic transition is not feasible with $N \gg 1$ order parameters to be fixed simultaneously. This same equation can be solved iteratively and, at low temperature, the two solutions $m \neq 0$ are the attractors of any seed $m(t_0) \neq 0$ that one can propose, with t_0 the initial "time" of the iteration.

It was known for long the a naive iterative solution of the TAP equations for the SK model was not converging at low temperatures. E. Bolthausen proposed in 2012 a way to iterate that does converge and yields correct results [56]. The idea is that the Onsager reaction term has to be delayed in the iteration time, in a similar way in which a Langevin equation with multiplicative noise,

$$\dot{x}(t) = f(x(t)) + g(x(t))\xi(t) \; , \qquad \langle \xi(t) \rangle = 0 \; , \quad \langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t-t') \tag{1.154}$$

once discretized

$$x(t_k) - x(t_{k-1}) = f(\overline{x}(t_{k-1}))\Delta t + g(\overline{x}(t_{k-1}))\xi(t_{k-1})\Delta t \tag{1.155}$$

with $\Delta = t_k - t_{k-1}$ and $\overline{x}(t_k) = \alpha x(t_k) + (1-\alpha)\alpha x(t_{k-1})$, is interpreted in the Ito $\alpha = 0$ convention, and becomes

$$x(t_k) = x(t_{k-1}) + f(x(t_{k-1}))\Delta t + g(x(t_{k-1}))\xi(t_{k-1})\Delta t \; . \tag{1.156}$$

The convenience of using the Ito convention in solving Langevin equations lies in the fact that the new value $x(t_k)$ appears only in the left-hand-side and one can simply construct the solution step by step in time.

Going back to the TAP equations, the m_j in the Onsager reaction term has to be computed with the Gibbs measure of the system without the spin s_i. One then delays it with respect to the contribution of the usual local field generated by the neighbours of i.

All in all, the proposal is to use

$$m_i(t_k) = \tanh\left[\beta \sum_{j(\neq i)} J_{ij} m_j(t_{k-1}) - \beta^2 \sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2(t_{k-2})) m_i(t_{k-2})\right] . \qquad (1.157)$$

In this way, the iteration converges to non-trivial solutions at low temperatures. Having said so, more recent studies by Aspelmeier et al. suggest to use other iteration methods which, they claim, are more efficient [58].

The TAP free-energy density

The TAP equations are the *extremization conditions* on the *TAP free-energy density*

$$\begin{aligned} f_J^{\text{tap SK}}(\{m_i\}) = & -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \sum_i h_i^{\text{ext}} m_i - \frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2) \\ & + T\sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right] \end{aligned} \qquad (1.158)$$

for the SK model, and

$$\begin{aligned} f_J^{\text{tap p-spin}}(\{m_i\}) = & -\frac{1}{p!}\sum_{i_1\neq\dots\neq i_p} J_{i_1\dots i_p} m_{i_1}\dots m_{i_p} - \sum_i h_i^{\text{ext}} m_i \\ & -\frac{\beta J^2}{4}\left[(p-1)q_{\text{EA}}^p - p q_{EA}^{p-1} + 1\right] \\ & + T\sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right] \end{aligned} \qquad (1.159)$$

for the Ising p-spin model and, to simplify the notation we replaced $J_{i_1\dots i_p}$ by $J[p!/(2N^{p-1})]^{\frac{1}{2}}$ in the Onsager term and we introduced q_{EA} [52]. In the spherical model the entropic contribution has to be modified [53, 54].

The free-energy density as a function of the local magnetisations m_i defines what is usually called the *free-energy landscape*. Note that this function depends on $N \gg 1$ variables, m_i, and these are not necessarily identical in the disordered case in which the interactions between different groups of spins are different.

The function $f_J^{\text{tap}}(\{m_i\})$ defines a, typically complex, landscape on a space of N dimensions. It has many minima, saddles of all kinds and maxima, as illustrated in Fig. 1.19. This figure has to be interpreted with care, though, since the underlying space is one dimensional (just an axis) while in reality it is large dimensional (N axes).

The TAP free-energy landscape depends explicitly on the choice of the coupling strengths. So, it can be very different from one choice of the interactions to another. However, unless one picks a very uncommon realization of the couplings, the statistical properties of all these cases turn out to be the same, in the large N limit.

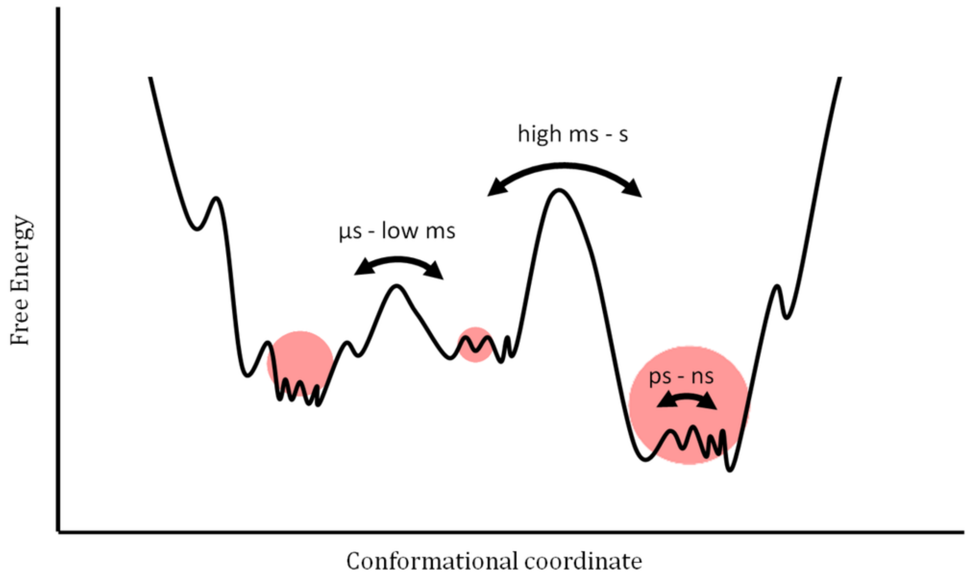


Figure 1.19: Schematic representation of a rugged free-energy landscape. Application to protein folding [63].

The stability properties of each extreme $\{m_l^\alpha\}$ are given by the eigenvalues of the Hessian matrix

$$\mathcal{H}_{ij}^J \equiv \left. \frac{\partial f_J^{\rm tap}(\{m_k\})}{\partial m_i \partial m_j} \right|_{\{m_l^\alpha\}} . \tag{1.160}$$

The number of positive, negative and vanishing eigenvalues determine then the number of directions in which the extreme is a minimum, a maximum or marginal. The sets $\{m_l^\alpha\}$ for which $f_J^{\rm tap}(\{m_l^\alpha\})$ is the absolute minima yield a first definition of equilibrium or pure states.

The TAP equations apply to $\{m_i\}$ and not to the configurations $\{s_i\}$. The values of the $\{m_i^\alpha\}$ are determined as extrema of the TAP free-energy density, $f_J^{\rm tap}(\{m_i\})$, and they not need to be the same as those of the energy, $H_J(\{s_i\})$, a confusion sometimes encountered in the glassy literature. The coincidence of the two can only occur at $T \to 0$.

The complexity or configurational entropy

There are a number of interesting questions about the extrema of the TAP free-energy landscape, or even its simpler version in which the Onsager term is neglected, that help us understanding the static behaviour of disordered systems:

- For a given temperature, T, how many solutions to the mean-field equations exist? The *total number of solutions* can be calculated using

$$\mathcal{N}_J(T) = \sum_\alpha \prod_i \int_{-1}^{1} dm_i \, \delta(m_i - m_i^\alpha) = \prod_i \int_{-1}^{1} dm_i \, \delta(\mathrm{eq}_i^J) \left| \det \frac{\partial \mathrm{eq}_i^J}{\partial m_j} \right| . \tag{1.161}$$

$\{m_i^\alpha\}$ are the solutions to the TAP equations that we write as $\{\text{eq}_i^J = 0\}$. The last factor is the normalization of the delta function after the change of variables, it ensures that we count one each time the integration variables touch a solution to the TAP equations independently of their stability properties. We made explicit the fact the this quantity depends on temperature. The determinant is the one of the Hessian of the free-energy density, since it is the one of the second derivative of it with respect to the local magnetizations. The first one yields the equations and the second one is the one explicitly written above.

This is the generalization to many variables of the relation $\mathcal{N} = \sum_\alpha \int dx\ \delta(x - x_0^\alpha)$ with x_0^α such that $f(x_0^\alpha) = 0$ (and no degenerate zeroes). This equation can have $\mathcal{N}$ solutions if $f(x)$ is a non linear function. For example, $f(x) = x^2 - 1$ has two zeros, $x_0 = \pm 1$. Take $\int_a^b dx\ \delta(f(x))$, such that in the interval $[a, b]$ the function f is monotonic, and change variables to $y = f(x)$. The integral becomes $\int_{f(a)}^{f(b)} dy\ f(y)/|f'(f^{-1}(y))| = 1/|f'(f^{-1}(y = 0))| = 1/|f'(x_0)|$. Since we want to count 1 for each zero and not this factor, we can start from $\int_a^b dx\ \delta(f(x))\ |f'(x)|$. This is called Kac formula in the mathematical literature. Perform now the change of variables $y = f(x)$. Then, $\int_{f(a)}^{f(b)} dy\ \delta(y)\ |f'(f^{-1}(y))|/|f'(f^{-1}(y))| = 1$ if there is a zero in $[a, b]$. Repeat now for each interval over which the function is monotonic.

We define the *complexity* or *configurational entropy* as the logarithm of the number of solutions at temperature T divided by N:

$$\Sigma_J(T) \equiv N^{-1} \ln \mathcal{N}_J(T) \ . \tag{1.162}$$

The normalization with N suggests that the number of solutions can be an exponential of N. We shall come back to this very important point below.

A point about the origin of this kind of calculation. In the context of disordered system they were pioneered by Bray & Moore [59]. However, in the mathematic literature similar calculations were carried out well before by Kac [60] – who was interested in the averaged number of real zeroes of polynomials with random coefficients – and Rice [61] – who studied the averaged number of zeroes of trigonometric polynomials and the averaged number of times a random walk crosses the origin within a given time interval. The equation (1.161) is nowadays called the *Kac-Rice formula*.

It is instructive to see the Kac-Rice formula at work in a simple example. Take a random function

$$F(t) = \sum_{k=0}^{N} a_k f_k(t) \tag{1.163}$$

where the coefficients a_k are *i.i.d.* random variables taken from a Gaussian probability with

$$[a_k] = 0 \qquad\qquad [a_k a_j] = \sigma_k^2 \delta_{ij} \ . \tag{1.164}$$

The t-dependent functions $F(t)$ and $F'(t)$ are then Gaussian themselves with zero mean and correlations

$$\begin{cases} [F(t)F(t')] = \sum_k \sum_j [a_k a_j]\, f_k(t) f_k(t') = \sum_k \sigma_k^2\, f_k(t) f_k(t')\ , \\ [F(t)F'(t')] = \sum_k \sum_j [a_k a_j]\, f_k(t) f'_k(t') = \sum_k \sigma_k^2\, f_k(t) f'_k(t')\ , \\ [F'(t)F(t')] = \sum_k \sum_j [a_k a_j]\, f'_k(t) f_k(t') = \sum_k \sigma_k^2\, f'_k(t) f_k(t')\ , \\ [F'(t)F'(t')] = \sum_k \sum_j [a_k a_j]\, f'_k(t) f'_k(t') = \sum_k \sigma_k^2\, f'_k(t) f'_k(t')\ . \end{cases} \tag{1.165}$$

Their joint probability at a given t is

$$p(F, F') = \frac{1}{2\pi\Delta}\, e^{-\frac{1}{2\Delta^2}\left(F^2[(F')^2] - 2FF'[FF'] + (F')^2[F^2]\right)} \tag{1.166}$$

where $\Delta^2 = [F^2][(F')^2] - [FF']^2$.

Exercise 1.18 Consider the particular case in which the functions $f_k(t)$ are simple powers t^k and the function $F(t)$ is then a polynomial $P_N(t)$, called *Kac polynomial*.

– Calculate the correlation of $P_N(t)$ and $P_N(t')$ in the case in which $\sigma_k^2 = \sigma^2$ for all k.

Take now $N = 2$, $P_2(t) = a_0 + a_1 t + a_2 t^2$ with coefficients a_0, a_1 and a_2 *i.i.d.* random variables taking real values with probability $p(a_k)$ (not necessarily Gaussian). Note that $p(a_2 = 0) = 0$ since a_2 is a continuous random variable, so the polynomial P_2 has almost surely degree 2. We will call $\mathcal{N}$ the number of real zeroes of P_2, and we will be interested in calculating its average.

– What is the condition on the coefficients to have only real zeroes?
– How can one express the probability of not having real zeroes in terms of a the probability of the condition just derived? In the case in which the probability of the coefficients is $p(a) = 1/(2N)$ for $a \in [-N, N]$ and zero otherwise the explicit calculation yields $P(\mathcal{N} = 0) \sim 0.37$.
– How can one express the probability of $\mathcal{N} = 1$? Give its value.
– What is then the probability of $\mathcal{N} = 2$?
– Calculate the averaged number of real zeroes.

The generalization of this kind of calculation to higher (and large) order polynomials becomes rapidly unfeasible.

We go back to the generic function $F(t)$ defined above. The Kac-Rice formula yields the averaged number of zeroes at time t as

$$
\begin{aligned}
[\mathcal{N}(F,t)] &= \int dF \int dF'\, \delta(F)\, |F'|\, \frac{1}{2\pi\Delta}\; e^{-\frac{1}{2\Delta^2}\left(F^2[(F')^2]-2FF'[FF']+(F')^2[F^2]\right)} \\
&= 2\int_0^\infty dF'\; F'\; \frac{1}{2\pi\Delta}\; e^{-\frac{[F^2]}{2\Delta^2}\,(F')^2} \\
&= \int_0^\infty \frac{dF'[F^2]^{1/2}}{\pi\Delta}\; \frac{F'[F^2]^{1/2}}{\Delta}\; \frac{\Delta}{[F^2]}\; e^{-\frac{1}{2}\left(\frac{F'[F^2]^{1/2}}{\Delta}\right)^2} \\
&= \frac{\Delta}{[F^2]}\int_0^\infty \frac{du}{\pi}\, u\, e^{-\frac{1}{2}\,u^2} \\
&= \frac{\Delta}{[F^2]}\frac{1}{\pi}
\end{aligned}
\tag{1.167}
$$

and we finally integrate this over time, $t \in I$, to get the final result.

- Does $\mathcal{N}_J(T)$ depend on T and does it change abruptly at particular values of T that may or may not coincide with static and dynamic phase transitions?

  If we use the Curie–Weiss theory as guideline, we know that at T_c the free-energy density changes from having a single minimum at $m = 0$ to having two symmetric minima at non-zero values of m and a maximum at $m = 0$. So, it goes from $\mathcal{N} = 1$ to $\mathcal{N} = 3$. With quenched randomness the change can be more abrupt and go from one minimum to an exponentially large number of saddle points.

- One can define a free-energy level dependent complexity

  $$\Sigma_J(f,T) \equiv N^{-1} \ln \mathcal{N}_J(f,T) \tag{1.168}$$

  where $\mathcal{N}_J(f,T)$ is the number solutions in the interval $[f, f + df]$ at temperature T.

  Again, the Curie–Weiss example tells us that below T_c there is one maximum at a higher free-energy density than the two minima.

- From these solutions, one can identify the minima as well as all saddles of different type, *i.e.* with different indices K. These are different kinds of metastable states. Geometry constrains the number of metastable states to satisfy Morse theorem that states $-\sum_{\alpha=1}^{\mathcal{N}_J}(-1)^{\kappa_\alpha} = 1$, where κ_α is the number of negative eigenvalues of the Hessian evaluated at the solution α, for any continuous and well-behaved function diverging at infinity in all directions.

  For example, in the one-dimensional double-well function of the Curie–Weiss $f(m)$ one has $(-1)^0 + (-1)^1 + (-1)^0 = -1$. Basically, there will be one more maximum than minima in a one dimensional function diverging (or at least growing) at the boundaries.

One can then count the number of solutions to the TAP equations of each index, $\mathcal{N}_J(K,T)$, and define the corresponding complexity

$$\Sigma_J(K,T) \equiv N^{-1} \ln \mathcal{N}_J(K,T) \;, \tag{1.169}$$

or even work at fixed free-energy density

$$\Sigma_J(K,f,T) \equiv N^{-1} \ln \mathcal{N}_J(K,f,T) \;. \tag{1.170}$$

Even more interestingly, one can analyse how are the free-energy densities of different saddles organized. For instance, one can check whether all maxima are much higher in free-energy density than saddles of a given type, *etc.*

- What is the barrier, $\Delta f = f_1 - f_0$, between ground states and first excited states? How does this barrier scale with the actual free-energy difference, Δf between these states?

  This is the kind of question asked within the analysis of the reversal of a magnetized system when a magnetic field of the opposite direction is applied. Then one estimates the surface cost of building a bubble of reverse magnetization and compares it to the volume gain of reversing the magnetization within the bubble. This, of course, is a real space argument, not applicable to the mean-field models we are dealing with here. Anyway, this argument gives a free-energy function as a function of the radius of the bubble with a maximum that represents a barrier estimate.

The definitions of complexity given above are disorder-dependent. One might then expect that the complexity will show sample-to-sample fluctuations and be characterized by a probability distribution.

The total *quenched complexity*, $\Sigma^{\text{quenched}}(T)$, which under self-averaging conditions coincides with the averaged one, is the most likely value of the total complixity $\Sigma_J(T)$. From the requirement

$$\max_{\{J_{ij}\}} P(\Sigma_J) \tag{1.171}$$

one gets $J_{ij}^{\max}$ and then

$$\Sigma^{\text{quenched}}(T) = \Sigma_{J_{ij}^{\max}}(T) = [\Sigma_J(T)] = N^{-1}[\ln \mathcal{N}(T)] \;. \tag{1.172}$$

In practice, this is very difficult to compute. Most analytic results concern the *annealed complexity*

$$\Sigma^{\text{annealed}}(T) \equiv N^{-1} \ln [\, \mathcal{N}_J \,] = N^{-1} \ln [\, e^{N\Sigma_J(T)} \,] \;. \tag{1.173}$$

One can show that the annealed complexity is larger or equal than the quenched one applying the Jensen inequality, to the concave logarithm function, as we did with the free-energy:

$$\Sigma^{\text{annealed}}(T) \geq \Sigma^{\text{quenched}}(T) \;. \tag{1.174}$$

Weighted averages

Having identified many solutions to the TAP equations one needs to determine now how to compute statistical averages. A natural proposal is to give a probability weight to each solution, w_α, and to use it to average the value of the observable of interest:

$$\langle O \rangle = \sum_\alpha w_\alpha^J \, O_\alpha \qquad \text{with} \qquad O_\alpha = O(\{m_i^\alpha\}) \tag{1.175}$$

where α labels the TAP solutions, O_α is the value that the observable O takes in the TAP solution α, and w_α^J are their statistical weights, satisfying the normalization condition $\sum_\alpha w_\alpha^J = 1$. Two examples can illustrate the meaning of this average. In a spin-glass problem, if $O = s_i$, then $O_\alpha = m_i^\alpha$. In an Ising model in its ferromagnetic phase, if $O = s_i$, then $O_\alpha = m_i^\alpha = \pm m$ and $w_\alpha = 1/2$. Within the TAP approach one proposes

$$w_\alpha^J = \frac{e^{-\beta F_\alpha^J}}{\sum_\gamma e^{-\beta F_\gamma^J}} \tag{1.176}$$

with F_α^J the total free-energy of the α-solution to the TAP equations. The discrete sum can be transformed into an integral over free-energy densities, introducing the degeneracy of solutions quantified by the free-energy density dependent complexity:

$$\langle O \rangle = \frac{1}{Z_J} \int df \; e^{-N\beta f} \, \mathcal{N}_J(f,T) \, O(f) = \frac{1}{Z_J} \int df \; e^{-N(\beta f - \Sigma_J(f,T))} \, O(f) \; . \tag{1.177}$$

The normalization is the 'partition function'

$$Z_J = \int df \; e^{-N\beta f} \, \mathcal{N}_J(f,T) = \int df \; e^{-N(\beta f - \Sigma_J(f,T))} \; . \tag{1.178}$$

We assumed that the labelling by α can be traded by a labelling by f that implies that at the same free-energy density level f the observable O takes the same value. In the $N \to \infty$ limit these integrals can be evaluated by saddle-point, provided the parenthesis is positive. In order to simplify the calculations, the disorder-dependent complexity is generally approximated with the annealed value introduced in eq. (1.173).

The equilibrium free-energy

The total equilibrium free-energy density, using the saddle-point method to evaluate the partition function Z_J in eq. (1.178), reads

$$-\beta f_{\rm eq}^J = N^{-1} \ln Z_J = \min_f \left[f - k_B T \Sigma_J(f,T) \right] \equiv \min_f \Phi_J(f,T) \; . \tag{1.179}$$

It is clear that $\Phi_J(f,T)$ is the *Landau free-energy density* of the problem with f playing the rôle of the energy and Σ_J of the entropy. If we use $f = (E - k_B T S)/N = e - Ts$ with E the actual energy and S the microscopic entropy one has

$$\Phi_J(f,T) = e - k_B T \left(s + \Sigma_J(f,T) \right) \; . \tag{1.180}$$

Thus, Σ_J is an extra contribution to the total entropy that is due to the exponentially large number of metastable states. Note that we do not distinguish here their stability.

Note that Σ_J is subtracted from TAP free-energy level f. Thus, it is possible that in some cases states lying at a *higher free-energy density* f but being *very numerous* have a *lower total Landau free-energy density* Φ than lower lying states that are less numerous. Collectively, higher states dominate the equilibrium measure in these cases. This phenomenon actually occurs in p-spin models, as explained below.

The order parameter

Now that we know that there can be a large number of states (defined as extrema of the TAP free-energy) we have to be careful about the definition of the spin-glass order parameter.

The *Edwards-Anderson parameter* is understood as a *property of a single state*. Within the TAP formalism one then has

$$q_{J\,\mathrm{EA}}^{\alpha} = N^{-1} \sum_i (m_i^\alpha)^2 \qquad \text{with} \qquad m_i^\alpha = \langle s_i \rangle_\alpha \tag{1.181}$$

being restricted to spin configurations in state α. An average of this quantity over all extrema of the free-energy density yields $\sum_\alpha w_\alpha^J \, q_{J\,\mathrm{EA}}^{\alpha} = \sum_\alpha w_\alpha^J \, N^{-1} \sum_i (m_i^\alpha)^2$.

Instead, the statistical *equilibrium magnetisation*, $m_i = \langle s_i \rangle = \sum_\alpha w_\alpha^J m_i^\alpha$, squared is

$$q_J \equiv \langle \, s_i \, \rangle^2 = m_i^2 = \left(\sum_\alpha w_\alpha^J m_i^\alpha \right)^2 = \sum_{\alpha\beta} w_\alpha^J w_\beta^J \, m_i^\alpha m_i^\beta \; . \tag{1.182}$$

If there are multiple phases, the latter sum has crossed contributions from terms with $\alpha \neq \beta$. These sums, as in a usual paramagnetic-ferromagnetic transition have to be taken over half space-space, otherwise global up-down reversal would imply the cancellation of all cross-terms.

Clearly

$$q_{J\,\mathrm{EA}}^{\alpha} \neq q_J \qquad \text{and} \qquad \sum_\alpha w_\alpha^J \, q_{J\,\mathrm{EA}}^{\alpha} \neq q_J \; . \tag{1.183}$$

1.7.2 Metastable states in two families of models

Let us now give a (very) short summary of what is known on the free-energy landscapes of such disordered systems. The derivation and understanding of the full structure of the TAP free-energy landscape is quite subtle and goes beyond the scope of these Lectures. Still, we shall briefly present their structure for the SK and p-spin models to give a flavor of their complexity.

High temperatures

For all models, at high temperatures $f(\{m_i\})$ is characterized by a single stable absolute minimum in which all local magnetizations vanish, as expected; this is the paramagnetic state. The $m_i = 0$ for all i minimum continues to exist at all temperatures. However, even if it is still the global absolute minimum of the TAP free-energy density, $f_J^{\rm TAP}$, at low temperatures it becomes unstable thermodynamically, and it is substituted as the equilibrium state, by other non-trivial configurations with $m_i \neq 0$ that are the absolute minima of Φ. Note the difference with the ferromagnetic problem for which the paramagnetic solution becomes a minimum below T_c and is hence unstable.

Low temperatures

At low temperature many equilibrium states appear (and not just two as in an Ising ferromagnetic model) and they are not related by symmetry (as spin reversal in the Ising ferromagnet or a rotational symmetry in the Heisenberg ferromagnet). These are characterized by non-zero values of the local magnetizations m_i that are different in different states.

The SK model

The first calculation of the complexity in the SK model appeared in 1980 [66, 59]. After more than 40 years of research the structure of the free-energy landscape in this system is still a matter of discussion. At present, the picture that emerges is the following. The temperature-dependent annealed complexity is a decreasing function of temperature that vanishes only at T_c but takes a very small value already at $\sim 0.6\,T_c$. Surprisingly enough, at finite but large N the TAP solutions come in pairs of minima and saddles of type one [57], that is to say, extrema with only one unstable direction. These states are connected by a mode that is softer the larger the number of spins: they coalesce and become marginally stable in the limit $N \to \infty$. Numerical simulations show that starting from the saddle-point and following the 'left' direction along the soft mode one falls into the minimum; instead, following the 'right' direction along the same mode one falls into the paramagnetic solution. See Fig. 1.20 for a sketch of these results. The free-energy difference between the minimum and saddle decreases for increasing N and one finds, numerically, an averaged $\Delta f \sim N^{-1.4}$. The extensive complexity of minima and type-one saddles is identical in the large N limit, $\Sigma_J(0,T) = \Sigma_J(1,T) + O(N^{-1})$ [58] in such a way that the Morse theorem is respected. The free-energy dependent annealed complexity is a smooth function of f with support on a finite interval $[f_0, f_1]$ and maximum at $f_{\max}$. The Bray and Moore annealed calculation (with supersymmetry breaking) yields $f_{\max} = -0.654$, $\Sigma_J^{\max} = 0.052$, $\Sigma''(f_{\max}) = 8.9$. The probability of finding a solution with free-energy density f can be expressed as

$$p_J(f,T) = \frac{\mathcal{N}_J(f,T)}{\mathcal{N}_J(T)} = \frac{e^{N\Sigma_J(f,T)}}{\mathcal{N}_J(T)} \sim \sqrt{\frac{N\Sigma_J''(f_{\max})}{2\pi}}\; e^{-\frac{N}{2}|\Sigma_J''(f_{\max})|(f-f_{\max})^2}\;, \tag{1.184}$$

where we evaluated the total number of solutions, $\mathcal{N}_J(T) = \int df\; e^{N\Sigma_J(f,T)}$, by steepest descent. The complexity, approximated by its annealed value, vanishes linearly close to

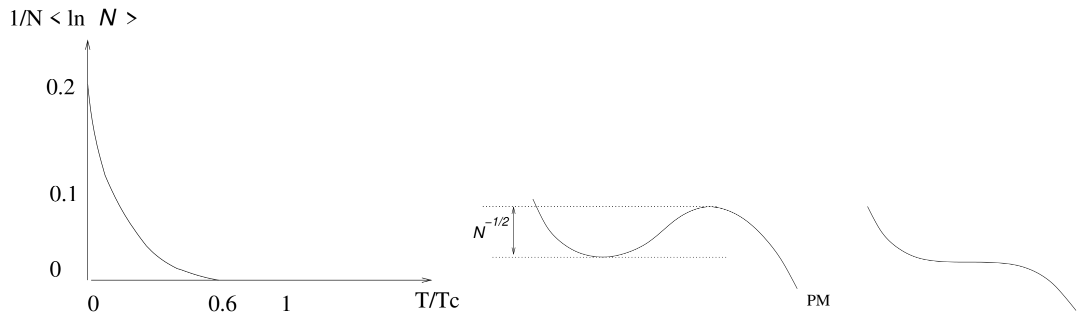


Figure 1.20: Left: sketch of the temperature dependent complexity, $\Sigma_J(T)$, of the SK. It actually vanishes only at T_c but it takes a very small value already at $\sim 0.6\ T_c$. Right: pairs of extrema in the SK model with N large and $N \to \infty$ limit.

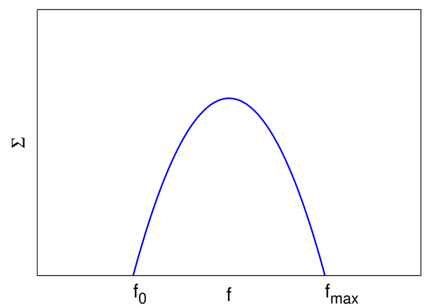


Figure 1.21: The complexity as a function of f for the SK model.

f_0: $\Sigma_J(f,T) \sim \lambda(f - f_0)$ with $\lambda < \beta$.

Only the lowest lying TAP solutions contribute to the statistical weight. The complexity does not contribute to Φ in the large N limit:

$$
\begin{aligned}
\Phi &= \beta f - \Sigma_{\rm ann}(f,T) \simeq \beta f - (f - f_0)\lambda \\
\frac{\partial \Phi}{\partial f} &\simeq \beta - \lambda > 0 \qquad \text{iff} \qquad \beta > \lambda
\end{aligned}
\tag{1.185}
$$

and $\Phi_{\rm min} \simeq \beta f_{\rm min} = \beta f_0$. See Fig. 1.21. The 'total' free-energy density in the exponential is just the free-energy density of each low-lying solution.

The (spherical) p-spin model

The number and structure of saddle-points is particularly interesting in the $p \geq 3$ cases and it is indeed the reason why these models, with a *random first order transition*, have been proposed to mimic the structural glass arrest. The $p \geq 3$ model has been studied in greater detail in the spherical case, that is to say, when spins are not Ising variables but satisfy the global constraint, $\sum_{i=1}^N s_i^2 = N$.

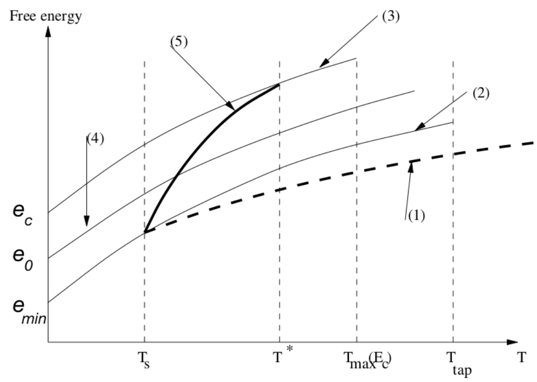


Figure 1.22: The TAP free-energy as a function of T in the spherical p-spin model. Three representative levels with $\{m_i \neq 0\}$ are drawn. (1) Free energy of the paramagnetic solution for $T > T^*$, $f_{\rm tot}$ for $T < T^*$; (2) free energy of the lowest TAP states, with zero temperature energy $e_{\rm min}$; (3) free energy of the highest TAP states, corresponding to $e_{\rm th}$; (4) an intermediate value of e_0 leads to an intermediate value of f at any temperature; (5) $f_{\rm eq}(T)$ the difference between curves (5) and (1) gives the complexity $T\Sigma(f_{\rm eq}(T), T)$ [78]. (Correction: the curvature of these curves is the opposite, they actually look upward.)

Although in general the minima of the mean-field free energy do not coincide with the minima of the Hamiltonian, they do in the spherical p-spin model. Their positions in the phase space does not depend on temperature, while their self-overlap does. At $T = 0$ a state (stable or metastable) is just a minimum (absolute or local) of the energy. For increasing T energy minima get dressed up by thermal fluctuations, and become states but they do nor cross nor merge. Thus, the states can be labeled by their zero-temperature energy density e_0 [53]. (We set $J = 1$ in the equations below.)

The annealed total complexity is given by [54]

$$\Sigma^{\rm ann}(e_0) = \frac{1}{2}\left[-\ln\frac{pz^2}{2} + \frac{p-1}{2}z^2 - \frac{2}{p^2z^2} + \frac{2-p}{p}\right] , \tag{1.186}$$

where z is

$$z = \frac{1}{p-1}\left[-e_0 - \sqrt{{e_0}^2 - e_{\rm th}^2}\right] . \tag{1.187}$$

and it is plotted in Fig. 1.23.

It vanishes at

$$e_0 = e_{\rm min} = f(p) , \tag{1.188}$$

the ground state of the system. This means that below this energy there can still be solutions to the TAP equations but they are not exponential in N in number. The complexity is real for zero-temperature energies $e < e_{\rm th}$ with

$$e_{\rm th} = -\sqrt{\frac{2(p-1)}{p}} . \tag{1.189}$$

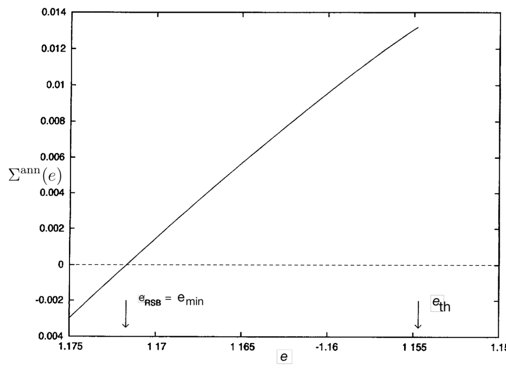


Figure 1.23: The annealed complexity of the spherical p-spin model.

This is the threshold energy density, being the attractor of the long-time limit of the dynamics after a quench [70]. e_{min} is the zero-T energy density that one finds with the replica calculation using a 1-step RSB *Ansatz*, as we shall see below. It can also be shown that below e_{th} minima dominate on average while above it there are still non trivial solutions but they are unstable.

Each zero-temperature state is characterized by a unit N-vector s_i^α and it gives rise to a finite-T state characterized by $m_i^\alpha = \sqrt{q(e,T)} s_i^\alpha$ with $q(e,T)$ given by

$$q^{p-2}(1-q)^2 = T^2 \, \frac{(e + \sqrt{e^2 - e_{\mathrm{th}}^2})^2}{(p-1)^2} \; . \tag{1.190}$$

($q(e, T = 0) = 1$ and at finite T the solution with q closest to 1 has to be chosen.) The self-overlap at the threshold energy, $e = e_{\mathrm{th}}$, is then

$$q_{\mathrm{th}}^{p-2}(1-q_{\mathrm{th}})^2 = T^2 \, \frac{2}{p(p-1)} \; . \tag{1.191}$$

Another way for the q equation to stop having solution, is by increasing the temperature, $T > T_{\mathrm{max}}(e_0)$, at fixed bare energy e_0. This means that, even though minima of the energy do not depend on the temperature, states, i.e. minima of the free energy do. When the temperature gets too high, the paramagnetic state becomes the only pure ergodic one, even though the energy landscape is broken up in many basins of the energy minima. This is just one particularly evident demonstration of the fundamental difference between pure states and energy minima. $T_{\mathrm{max}}(e_0)$ is obtained as the limiting temperature for which eq. (1.190) admits a solution. It is given by

$$T_{\mathrm{max}}(e_0) = \left(\frac{2}{p}\right) \left(\frac{p-1}{-e_0 - \sqrt{{e_0}^2 - e_{\mathrm{th}}^2}}\right) \left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} \; . \tag{1.192}$$

T_{max} is a decreasing function of e_0, and correspond to the ending points of the three curves (2), (3) and (4) in Fig. 1.22. The last state to disappear is the one with minimum energy e_{min}, ceasing to exist at $T_{\mathrm{TAP}} \equiv T_{\mathrm{max}}(e_{\mathrm{min}})$.

Below a temperature T_d, an exponential (in N) number of metastable states contribute to the thermodynamics in such a non-trivial way that their combined contribution to the observables makes it the one of a paramagnet. Even if each of these states is non-trivial (the m_i's are different from zero) the statistical average over all of them yields results that are identical to those of a paramagnet. For example, the free-energy density is $-1/(4T)$ as in the $m_i = 0$ paramagnetic solution. One finds

$$T_d = \sqrt{\frac{p(p-2)^{p-2}}{2(p-1)^{p-1}}} \,. \tag{1.193}$$

In the p-spin models there is a range of temperatures in which high lying states dominate this sum since they are sufficiently numerous so as to have a complexity that renders the combined term $\beta f - \Sigma_J(f,T)$ smaller (in actual calculations the disorder dependent complexity is approximated by its annealed value).

At a lower temperature T_s ($T_s < T_d$) there is an *entropy crisis*, less than an exponential number of metastable states survive, and there is a *static phase transition* to a glassy state.

In short:

- Above T_d the (unique) paramagnetic solution dominates, $q = 0$ and $\Phi = f = -1/(4T)$.
- In the interval $T \in [T_s, T_d]$ an exponentially large number of states (with $q \neq 0$ given by the solution to $pq^{p-2}(1-q) = 2T^2$) dominate the partition sum. $\Phi = -1/(4T)$ appearing as the continuation of the paramagnetic solution.
- At $T < T_s$ the lowest TAP states with $e_0 = e_{\min}$ control the partition sum. Their total free-energy Φ is different from $-1/(4T)$.

This picture is confirmed with other analytical studies that include the use of pinning fields adapted to the disordered situation [68], the effective potential for two coupled real replicas [69], the dynamic approach [70], and the numerical exhaustive determination of all stationary points of the Hamiltonian in systems with size $N \leq 20$ [71].

Low temperatures, entropy crisis

The interval of definition of $\Phi_J(e,T)$ is the same as $\Sigma_J(e)$, that is $e \in [e_{\min} : e_{\text{th}}]$. Assuming that at a given temperature T the energy $e_{\text{eq}}(T)$ minimizing Φ_J lies in this interval, what happens if we lower the temperature? Remember that the complexity is an increasing function of e, as of course is $f(e,T)$. When T decreases we favor states with lower free energy and lower complexity, and therefore e_{eq} decreases. As a result, it must exist a temperature T_s, such that, $e_{\text{eq}}(T_s) = e_{\min}$ and thus, $\Sigma_J(e_{\text{eq}}(T)) = \Sigma_J(e_{\min}) = 0$. Below T_s the bare energy e_{eq} cannot decrease any further: there are no other states below the ground states $e_{\min}$. Thus, $e_{\text{eq}}(T) = e_{\min}$ for each temperature $T \leq T_s$. As a result, if we plot the complexity of equilibrium states $\Sigma_J(e_{\text{eq}}(T))$ as a function of the temperature, we find a discontinuity of the first derivative at T_s, where the complexity vanishes. A thermodynamic transition takes place at T_s: below this temperature equilibrium is no longer dominated by metastable states, but by the lowest lying states, which have zero complexity and lowest free energy density.

We shall show that T_s is the transition temperature found with a replica calculation. The temperature where equilibrium is given for the first time by the lowest energy states, is equal to the static transition temperature. The interpretation of these results in terms of the replica calculation goes as follows. Above T_s the partition function is dominated by an exponentially large number of states, each with high free energy and thus low statistical weight, such that they are not captured by the overlap distribution $P(q)$ of the replica calculation. At T_s the number of these states becomes sub-exponential and their weight nonzero, such that the $P(q)$ develops a secondary peak at $q_s \neq 0$.

The threshold

The analysis of the Hessian at the threshold level, that is, setting $e = e_{\rm th}$, reveals that these states are saddles with an extensive number of flat directions. The threshold level is then like a large flat plateau in a mountain landscape.

One finds that the typical spectrum of the free-energy Hessian in a extreme of the TAP free-energy corresponds to a "shifted" semicircle law, with the lowest eigenvalue $\lambda_{\rm min}$ given (in terms of the parameters of the minimum) by

$$\lambda_{\rm min} = pq^{(p-2)/2}(e_{\rm th} - e_0) \tag{1.194}$$

Hence, for sub-threshold free-energies we have well-defined minima with no "zero-modes" separated by O(N) barriers while right at the threshold $\lambda_{\rm min}$ vanishes.

Finite dimensions

In finite-dimensional systems, only equilibrium states can break the ergodicity, i.e. states with the lowest free energy density. In other words, the system cannot remain trapped for an infinite time in a metastable state, because in finite dimension free energy barriers surrounding metastable states are always finite.

The extra free energy of a droplet of size r of equilibrium phase in a background metastable phase has a positive interface contribution which grows as r^{d-1}, and a negative volume contribution which grows as r^d,

$$\Delta f = \sigma r^{d-1} - \delta f \; r^d \tag{1.195}$$

where here σ is the surface tension and δf is the bulk free energy difference between the two phases. This function has always a maximum, whose finite height gives the free energy barrier to nucleation of the equilibrium phase (note that at coexistence $\delta f = 0$ and the barrier is infinite). Therefore, if initially in a metastable states the system will, sooner or later, collapse in the stable state with lower free energy density. For this reason, in finite dimension we cannot decompose the Gibbs measure in metastable components. When this is done, it is always understood that the decomposition is only valid for finite times, i.e times much smaller than the time needed for the stable equilibrium state to take over. On the other hand, in mean-field systems (infinite dimension), barriers between metastable states may be infinite in the thermodynamic limit, and it is therefore possible to call pure states also metastable states, and to assign them a Gibbs weight w^J_α. We will analyse a mean-field spin-glass model, so that we will be allowed to perform the decomposition above even for metastable states.

Comments

There is a close relationship between the topological properties of the model and its dynamical behavior. In particular, the slowing down of the dynamics above but close to T_d is connected to the presence of saddles, whose instability decreases with decreasing energy. In fact, we have seen that the threshold energy level e_{th} separating saddles from minima, can be associated to the temperature $T_{\rm th} = T_d$, marking the passage from ergodicity to ergodicity breaking. In this context the dynamical transition can be seen as a topological transition. The plateau of the dynamical correlation function, which has an interpretation in terms of cage effect in liquids, may be reinterpreted as a pseudo-thermalization inside a saddle with a very small number of unstable modes.

1.8 Average over disorder: the replica method

A picture that is consistent with the one arising from the naive mean-field approximation but contradicts the initial assumption of the droplet model arises from the *exact* solution of fully-connected spin-glass models. These results are obtained using a method called the *replica trick* that we will briefly present below.

(The first reference to the use of this trick is attributed to Brout [72]. In the context of spin-glasses Edwards-Anderson proposed to use it in their celebrated paper [15]).

In Sect. 1.2.4 we argued that the typical properties of a disordered system can be computed from the disorder averaged free-energy

$$[\, F_J \,] \equiv \int dJ \, P(J) F_J \; , \tag{1.196}$$

where J collects in a compact way all possible sources of quenched disorder. One then needs to average the logarithm of the partition function. In the *annealed approximation* one exchanges the ln with the average over disorder and, basically, considers the interactions equilibrated at the same temperature T as the spins:

$$[\, \ln Z_J \,] \sim \ln [\, Z_J \,] \; . \tag{1.197}$$

This approximation turns out to be correct at high temperatures but incorrect at low ones.

The evaluation of the disordered average free-energy density is difficult for at least two reasons: firstly, virtually all configurations are not translationally invariant; secondly, no factorisation helps one easily reduce the partition sum over one acting on independent variables.

The replica method allows one to compute $[\,F_J\,]$ for fully-connected models. It is based on the smart use of the identity

$$\ln Z_J = \lim_{n\to 0} \frac{Z_J^n - 1}{n} \; . \tag{1.198}$$

The idea is to compute the right-hand-side for finite and integer $n = 1, 2, \ldots$ and then perform the analytic continuation to $n \to 0$. Care should be taken in this step: for some models the analytic continuation may not be unique. (Recall the calculation done using the Potts model with $q \to 1$ that allows one to recover results for the percolation problem.) It turns out that this is indeed the case for the emblematic Sherrington-Kirkpatrick model, as discussed by van Hemmen and Palmer [73] though it has also been recently shown that the free-energy $f(T)$ obtained by Parisi [74] with the replica trick is exact! [75, 76]

Exercise 1.19 Take a particle with mass m in a one dimensional harmonic potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 \tag{1.199}$$

with real frequency ω taken from a probability distribution $p(\omega)$. The position of the particle is given by the real variable x. We will be interested in the canonical equilibrium properties of the particle at inverse temperature $\beta = 1/(k_B T)$, with k_B Boltzmann's constant, as characterized by $Z_\omega = \int dx \, \exp(-\beta V(x))$, focusing only on the potential contribution to the energy.

1. Dimensional analysis. Since Z_ω should be adimensional, show that $\beta m \omega^2$ is adimensional.

2. Direct calculation.

   (a) Compute the free-energy at fixed ω.

   (b) Compute the average over disorder of F_ω. Pose the calculation for generic $p(\omega)$ and then take the particular case $p(\omega) = (2\pi\sigma^2)^{-1/2} \; e^{-\omega^2/(2\sigma^2)}$. Firstly establish the parameter dependence of the result and then go on to show that it reads $-\beta[F_\omega] = [\ln Z_\omega] = C/2 + \ln[4\pi/(\beta m \sigma^2)]^{1/2}$, that is, it depends on the parameters *via* the adimensional variable $\beta m \sigma^2$ only.

3. Replica method calculation.

   (a) Find an expression for the disorder averaged free-energy following the usual steps of the replica method.

   (b) Take now a Gaussian probability with zero mean and variance σ^2 for the frequency ω. What is the averaged free-energy? Does one find the same parameter dependence as with the direct calculation?

A useful integral will be $\int_0^\infty dy \; e^{-ay^2} \ln y = -\frac{1}{4}\,(C + \ln 4a)\,\sqrt{\frac{\pi}{a}}$ where C is a constant given by $-C = \int_0^1 dx \ln\ln 1/x$.

Note that the volume of a sphere of unit radius in an n-dimensional space is $\Omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$. This is the form to exploit in the calculations above.
We also know $\int_0^\infty dr \frac{r^{n-1}}{\sqrt{1+r^2}} = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)$ and we note that this expression diverges for $n \to 0$ because of the divergence of $\Gamma(n/2 \to 0)$ or the logarithmic divergence of the integral for $r \to 0$ if $n < 1$.
The Gamma function admits a series expansion $\Gamma(1+z) = \sum_{k=0}^\infty c_k z^k$ with $c_0 = 1$, $c_1 = -C$, and C the same constant as in the integral above. Therefore, $\Gamma\left(1-\frac{n}{2}\right) = 1 + C\,\frac{n}{2} + \mathcal{O}(n^2)$.

We treat first an example in which the replica structure appears in an exact way and then we deal with the Sherrington-Kirkpatrick model where an *Ansatz* is necessary.

1.8.1 A particle in a harmonic potential under a random force

Consider now a particle confined to move in one dimension, feeling a harmonic potential, and under a random force f taken from a Gaussian pdf with zero mean,

$$V_f(x) = \frac{1}{2}m\omega^2 x^2 - fx \qquad\qquad p(f) = \frac{1}{\sqrt{2\pi f_0^2}}\, e^{-\frac{f^2}{2f_0^2}} \,. \tag{1.200}$$

The potential energy can be rewritten as

$$V_f(x) = \frac{1}{2}m\omega^2\left(x - \frac{f}{m\omega^2}\right)^2 - \frac{f^2}{2m\omega^2} \tag{1.201}$$

where one sees clearly the minimum at $x_{\min} = f/(m\omega^2)$.

Direct calculation

This problem can be solved exactly. The disorder dependent partition function is

$$\begin{aligned} Z_f &= \int_{-\infty}^{\infty} dx\, e^{-\beta(\frac{1}{2}m\omega^2x^2 - fx)} = e^{\frac{\beta f^2}{2m\omega^2}} \sqrt{\frac{2\pi k_BT}{m\omega^2}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\frac{2\pi k_BT}{m\omega^2}}}\, e^{-\frac{\beta m\omega^2}{2}\left(x - \frac{f}{m\omega^2}\right)^2} \\ &= \sqrt{\frac{2\pi k_BT}{m\omega^2}}\, e^{\frac{\beta f^2}{2m\omega^2}} \,, \end{aligned} \tag{1.202}$$

and the free-energy

$$-\beta F_f = \ln Z_f = \frac{1}{2}\,\ln\frac{2\pi k_BT}{m\omega^2} + \frac{\beta f^2}{2m\omega^2} \tag{1.203}$$

which can be readily averaged over f:

$$-\beta[F_f] = \frac{1}{2}\,\ln\left(\frac{2\pi k_BT}{m\omega^2}\right) + \frac{\beta[f^2]}{2m\omega^2} \quad\Rightarrow\quad \boxed{-\beta[F_f] = \frac{1}{2}\,\ln\left(\frac{2\pi k_BT}{m\omega^2}\right) + \frac{\beta f_0^2}{2m\omega^2}} \tag{1.204}$$

Working at fixed f the averaged position of the particle is

$$\langle x \rangle = \frac{1}{\beta} \frac{\partial \ln Z_f}{\partial f} = \frac{1}{\cancel{\beta}} \frac{\cancel{2}\cancel{\beta} f}{\cancel{2} m\omega^2} = \frac{f}{m\omega^2} \; . \tag{1.205}$$

Let us check the dimensions of our result. x cannot have dimensions since Z_f does not have. Hence, $\beta m\omega^2$ and βf do not have either. Therefore, $f/(m\omega^2)$ is adimentional as x. If we now average $\langle x \rangle$ over the random force, we obtain a vanishing result $[\langle x \rangle] = 0$, which is reasonable due to the symmetry of $p(f)$. Similarly,

$$\begin{aligned}
\langle x^2 \rangle &= \frac{1}{\beta^2} \frac{1}{Z_f} \frac{\partial^2 Z_f}{\partial f^2} \\
&= \frac{1}{\beta^2} \frac{1}{\cancel{\sqrt{\frac{2\pi k_B T}{m\omega^2}}}} \frac{1}{\cancel{e^{\frac{\beta f^2}{2m\omega^2}}}} \left[\frac{\beta}{m\omega^2} + \left(\frac{\beta f}{m\omega^2} \right)^2 \right] \cancel{\sqrt{\frac{2\pi k_B T}{m\omega^2}}} \cancel{e^{\frac{\beta f^2}{2m\omega^2}}} \\
&= \frac{1}{\beta^2} \left[\frac{\beta}{m\omega^2} + \left(\frac{\beta f}{m\omega^2} \right)^2 \right] \\
&= \frac{1}{\beta m\omega^2} + \left(\frac{f}{m\omega^2} \right)^2 \\
&= \frac{1}{\beta m\omega^2} + \langle x \rangle^2
\end{aligned} \tag{1.206}$$

and its average over disorder

$$[\langle x^2 \rangle] = \frac{k_B T}{m\omega^2} + \frac{f_0^2}{(m\omega^2)^2} \tag{1.207}$$

with a thermal contribution which would also be present in the absence of the random force and the second term originating in the random force. The first term characterises the position fluctuations

$$[\langle (x - \langle x \rangle)^2 \rangle] = \frac{k_B T}{m\omega^2} \tag{1.208}$$

Replica calculation

We can now try recover this result with the replica method:

$$-\beta [F_f] = [\ln Z_f] = \lim_{n \to 0} \frac{[Z_f^n] - 1}{n} \; . \tag{1.209}$$

The disorder averaged replicated partition function reads

$$\begin{aligned}
[Z_f^n] &= \int \frac{df}{\sqrt{2\pi f_0^2}} e^{-\frac{f^2}{2f_0^2}} \int \prod_{a=1}^n dx_a \; e^{-\beta \sum_{a=1}^n \frac{1}{2} m\omega^2 x_a^2 + \beta f \sum_{a=1}^n x_a} \\
&= \int \prod_{a=1}^n dx_a \; e^{-\beta \sum_{a=1}^n \frac{1}{2} m\omega^2 x_a^2} \int \frac{df}{\sqrt{2\pi f_0^2}} e^{-\frac{f^2}{2f_0^2}} \; e^{\beta f \sum_{a=1}^n x_a} \\
&= \int \prod_{a=1}^n dx_a \; e^{-\beta \sum_{a=1}^n \frac{1}{2} m\omega^2 x_a^2} \quad e^{\frac{\beta^2 f_0^2}{2} (\sum_{a=1}^n x_a)^2} \\
&= \int \prod_{a=1}^n dx_a \; e^{-\beta \sum_{a=1}^n \frac{1}{2} m\omega^2 x_a^2} \quad e^{\frac{\beta^2 f_0^2}{2} \sum_{a,b}^n x_a x_b} \\
&= \int \prod_{a=1}^n dx_a \; e^{-\beta H_{\text{eff}}(\{x_a\})} \qquad (1.210)
\end{aligned}$$

where we identified an effective quadratic Hamiltonian H_{eff} which yields the energy of the interacting n replica system. We can now go on with the calculation

$$\begin{aligned}
[Z_f^n] &= (2\pi k_B T)^{n/2} \int \prod_{a=1}^n \left(\frac{dx_a}{\sqrt{2\pi k_B T}} \right) e^{-\frac{1}{2}\beta \sum_{a,b} x_a \left(m\omega^2 \delta_{ab} - \beta f_0^2 \right) x_b} \\
&= (2\pi k_B T)^{n/2} \int \prod_{a=1}^n \left(\frac{dx_a}{\sqrt{2\pi k_B T}} \right) e^{-\frac{1}{2}\beta \sum_{a,b} x_a G^{-1}_{ab} x_b} \\
&= (2\pi k_B T)^{n/2} \, (\det \mathbb{G})^{1/2} \; . \qquad (1.211)
\end{aligned}$$

We now use the matricial identity

$$\ln(\det \mathbb{G})^{1/2} = \frac{1}{2} \mathrm{Tr} \ln \mathbb{G} \qquad \Rightarrow \qquad \det \mathbb{G} = e^{\frac{1}{2}\mathrm{Tr}\ln\mathbb{G}} \sim 1 + \frac{1}{2}\mathrm{Tr}\ln\mathbb{G} + \mathcal{O}(n^2) \qquad (1.212)$$

if $\mathrm{Tr}\ln\mathbb{G}$ is proportional to n as it will turn out to be. Then,

$$[Z_f^n] \sim [1 + n\ln(2\pi k_B T)^{1/2}]\,[1 + \frac{1}{2}\mathrm{Tr}\ln\mathbb{G}] \sim [1 + n\ln(2\pi k_B T)^{1/2} + \frac{1}{2}\mathrm{Tr}\ln\mathbb{G}]$$

where we also expanded the prefactor for $n \to 0$. We therefore need to calculate $\mathrm{Tr}\ln\mathbb{G}$ since

$$[\ln Z_f] = \frac{1}{2}\ln(2\pi k_B T) + \frac{1}{2} \lim_{n\to 0} \frac{1}{n} \mathrm{Tr}\ln\mathbb{G} \; . \qquad (1.213)$$

The algebra of replica symmetric matrices

The $n \times n$ matrix $\mathbb{G}$ is the inverse of the matrix $\mathbb{G}^{-1}$ with diagonal and off-diagonal elements

$$\begin{aligned}
G^{-1}_{\text{diag}} &= m\omega^2 - \beta f_0^2 & a &= b \, , \\
G^{-1}_{\text{off}} &= -\beta f_0^2 & a &\neq b \, . \qquad (1.214)
\end{aligned}$$

That is, they are all identical on the diagonal, and take another value in the off-diagonal elements. These are the so-called *replica symmetric* matrices.

To be able to calculate $\mathrm{Tr}\ln\mathbb{G}$ we need to identify the elements of $\mathbb{G}$. Let us study the properties of products of two replica symmetry matrices $\mathbb{A}$ and $\mathbb{B}$ yielding another matrix $\mathbb{C}$. Let us call

$$a_{\rm diag} \qquad \text{diagonal elements} \tag{1.215}$$

$$a_{\rm off} \qquad \text{off} - \text{diagonal elements} \tag{1.216}$$

and similarly for the matrix $\mathbb{B}$. Then, a simple calculation yields

$$c_{\rm diag} = a_{\rm diag}b_{\rm diag} + (n-1)a_{\rm off}b_{\rm off} \tag{1.217}$$

$$c_{\rm off} = a_{\rm diag}b_{\rm off} + a_{\rm off}b_{\rm diag} + (n-2)a_{\rm off}b_{\rm off} \tag{1.218}$$

and we find that $\mathbb{C}$ is also a replica symmetric matrix. We can apply this generic result to derive the elements of the square matrix:

$$\mathbb{A}^2{}_{\rm diag} = a_{\rm diag}a_{\rm diag} + (n-1)a_{\rm off}a_{\rm off} \to a^2_{\rm diag} - a^2_{\rm off} \tag{1.219}$$

$$\mathbb{A}^2{}_{\rm off} = a_{\rm diag}a_{\rm off} + a_{\rm off}a_{\rm diag} + (n-2)a_{\rm off}a_{\rm off} \to 2(a_{\rm diag}a_{\rm off} - a^2_{\rm off}) \tag{1.220}$$

where in the last step we took the $n \to 0$ limit. The relation between the three matrices also allows us to identify the elements of the inverse matrix $\mathbb{A}^{-1}$ of a generic replica symmetric matrix $\mathbb{A}$. We require $c_{\rm diag} = 1$ and $c_{\rm off} = 0$, and then

$$1 = a_{\rm diag}b_{\rm diag} + (n-1)a_{\rm off}b_{\rm off} \tag{1.221}$$

$$0 = a_{\rm diag}b_{\rm off} + a_{\rm off}b_{\rm diag} + (n-2)a_{\rm off}b_{\rm off} \tag{1.222}$$

which we solve for $b_{\rm diag}$ and $b_{\rm off}$:

$$b_{\rm off} = \frac{a_{\rm off}}{(n-1)a^2_{\rm off} - a^2_{\rm diag} - (n-2)a_{\rm off}a_{\rm diag}} \tag{1.223}$$

$$b_{\rm diag} = -\frac{b_{\rm off}}{a_{\rm off}}\left[a_{\rm diag} + (n-2)a_{\rm off}\right] \tag{1.224}$$

and in the limit $n \to 0$ become

$$b_{\rm off} = \mathbb{A}^{-1}_{\ \ \rm off} \to \frac{a_{\rm off}}{-a^2_{\rm off} - a^2_{\rm diag} + 2a_{\rm off}a_{\rm diag}} = -\frac{a_{\rm off}}{(a_{\rm diag} - a_{\rm off})^2} \tag{1.225}$$

$$b_{\rm diag} = \mathbb{A}^{-1}_{\ \ \rm diag} \to -\frac{b_{\rm off}}{a_{\rm off}}\left[a_{\rm diag} - 2a_{\rm off}\right] = \frac{a_{\rm diag} - 2a_{\rm off}}{(a_{\rm diag} - a_{\rm off})^2} \tag{1.226}$$

We can now apply this general results to the matrix $\mathbb{A} = \mathbb{G}^{-1}$ to derive;

$$g_{\rm diag} = \frac{(m\omega^2 - \beta f_0^2) - 2(-\beta f_0^2)}{[(m\omega^2 - \beta f_0^2) - (-\beta f_0^2)]^2} = \frac{m\omega^2 + \beta f_0^2}{(m\omega^2)^2} = \frac{1}{m\omega^2}\left(1 + \frac{\beta f_0^2}{m\omega^2}\right) \tag{1.227}$$

$$g_{\rm off} = -\frac{-\beta f_0^2}{[(m\omega^2 - \beta f_0^2) - (-\beta f_0^2)]^2} = \frac{\beta f_0^2}{(m\omega^2)^2} = \frac{1}{m\omega^2}\frac{\beta f_0^2}{m\omega^2} \tag{1.228}$$

which is also a replica symmetric matrix, with the structure

$$\mathbb{G} = \frac{1}{m\omega^2}\left(\mathbb{I} + \frac{\beta f_0^2}{m\omega^2}\mathbb{F}\right) \tag{1.229}$$

where we called $\mathbb{F}$ the fully filled matrix. Using the power expansion of $\ln(\mathbb{I}+\mathbb{X})$, and taking the limit $n \to 0$ one can prove

$$\begin{aligned}
\frac{1}{n}\mathrm{Tr}\ln\mathbb{G} &= \ln(g_{\mathrm{diag}} - g_{\mathrm{off}}) + \frac{g_{\mathrm{off}}}{g_{\mathrm{diag}} - g_{\mathrm{off}}} \\
&= \ln\left(\frac{1}{m\omega^2}\right) + \frac{\frac{\beta f_0^2}{(m\omega^2)^2}}{\frac{1}{m\omega^2}} \\
&= -\ln(m\omega^2) + \frac{\beta f_0^2}{m\omega^2}
\end{aligned} \tag{1.230}$$

Going back to

$$\begin{aligned}
[\ln Z_f] &= \frac{1}{2}\ln(2\pi k_B T) + \frac{1}{2}\lim_{n\to 0}\frac{1}{n}\mathrm{Tr}\ln\mathbb{G} = \frac{1}{2}\ln(2\pi k_B T) - \frac{1}{2}\ln(m\omega^2) + \frac{1}{2}\frac{\beta f_0^2}{m\omega^2} \\
&= \frac{1}{2}\ln\left(\frac{2\pi k_B T}{m\omega^2}\right) + \frac{1}{2}\frac{\beta f_0^2}{m\omega^2}
\end{aligned} \tag{1.231}$$

which is the same expression obtained with the direct calculation.

1.8.2 The fully-connected random field Ising model

See the hand-written notes.

1.8.3 The mean-field spin-glass

The disorder averaged free-energy is given by

$$-\beta[\,F_J\,] = -\int dJ\, P(J)\ln Z_J = -\lim_{n\to 0}\frac{1}{n}\left(\int dJ\, P(J) Z_J^n - 1\right)\,, \tag{1.232}$$

where we have exchanged the limit $n \to 0$ with the integration over the exchanges. For *integer* n the *replicated partition function*, Z_J^n, reads

$$Z_J^n = \sum_{\{s_i^a\}} e^{-\beta[H_J(\{s_i^1\})+\cdots+H_J(\{s_i^n\}]}\,. \tag{1.233}$$

Here $\sum_{\{s_i^a\}} \equiv \sum_{\{s_i^1=\pm1\}}\cdots\sum_{\{s_i^n=\pm1\}}$. Z_J^n corresponds to n identical copies of the original system, that is to say, all of them with the same realisation of disorder. Each copy is characterised by an ensemble of N spins, $\{s_i^a\}$. We label the copies with a replica index $a = 1, \dots, n$. For p-spin disordered spin models Z_J^n takes the form

$$Z_J^n = \sum_{\{s_i^a\}} e^{\beta \sum_{a=1}^n \left[\frac{1}{p!} \sum_{i_1 \neq \dots \neq i_p} J_{i_1 \dots i_p} s_{i_1}^a \dots s_{i_p}^a + \sum_i h_i s_i^a \right]} . \tag{1.234}$$

The average over disorder amounts to computing a Gaussian integral for each set of spin indices $i_1 \neq \dots \neq i_p$, with $[J_{i_1 \dots i_p}^2] = J^2 p!/(2N^{p-1})$.[4] One finds

$$[\, Z_J^n \,] = \sum_{\{s_i^a\}} e^{\frac{\beta^2 J^2}{2N^{p-1} p!} \sum_{i_1 \neq \dots \neq i_p} (\sum_a s_{i_1}^a \dots s_{i_p}^a)^2 + \beta \sum_a \sum_i h_i s_i^a} \equiv \sum_{\{s_i^a\}} e^{-\beta F(\{s_i^a\})} . \tag{1.235}$$

The function $\beta F(\{s_i^a\})$ is not random. It depends on the spin variables only but it includes terms that couple different replica indices. Indeed,

$$\sum_{i_1 \neq \dots \neq i_p} \sum_a s_{i_1}^a \dots s_{i_p}^a \sum_b s_{i_1}^b \dots s_{i_p}^b = \sum_{ab} \sum_{i_1 \neq \dots \neq i_p} (s_{i_1}^a s_{i_1}^b) \dots (s_{i_p}^a s_{i_p}^b) \tag{1.236}$$

We first note that all terms are identical to one for $a = b$ since $s_i^2 = 1$. The sum over the spin indices (ignoring the constraint $i_1 \neq \dots \neq i_p$ that will, in any case give a subdominant contribution in the $N \to \infty$ limit) and the sum over the remaining replica index of such terms equal $N^p n$. Focusing then on the cases $a \neq b$ and ignoring the constraint $i_1 \neq \dots \neq i_p$ since, again, the equal site terms give subdominant contributions in the large N limit,

$$\beta F(\{s_i^a\}) \approx -\frac{N\beta^2 J^2}{2p!} \left[\sum_{a \neq b} \left(\frac{1}{N} \sum_i s_i^a s_i^b \right)^p + n \right] - \beta \sum_a \sum_i h_i s_i^a . \tag{1.237}$$

(In complete analogy with what is done for the pure p spin ferromagnet, the dropped terms correspond to adding self-interactions in the Hamiltonian.) The constant term $-Nn\beta^2 J^2/2$ originates in the terms with $a = b$, for which $(s_i^a)^2 = 1$ as argued above.

Hitherto the replica indices act as a formal tool introduced to compute the average over the bond distribution. Nothing distinguishes one replica from another and, in consequence, the 'free-energy' $F(\{s_i^a\})$ is *invariant under permutations* of the replica indices.

The next step to follow is to identify the order parameters and transform the free-energy into an order-parameter dependent expression to be rendered extremal at their equilibrium values. In a spin-glass problem we already know that the order parameter is not the global magnetisation as in a pure magnetic system but the parameter q – or more generally the overlap between states. Within the replica calculation an *overlap between replicas*

$$q_{ab} \equiv N^{-1} \sum_i s_i^a s_i^b \qquad\qquad a \neq b \tag{1.238}$$

[4] We use $\int dJ_\alpha / \sqrt{2\pi\sigma^2}\; e^{-J_\alpha^2/(2\sigma^2) - J_\alpha x} = e^{x^2\sigma^2/2}$.

naturally appeared in eq. (1.237). The idea is to write the free-energy density as a function of these $n \times n$ parameters, which can be though of as the elements of an $n \times n$ matrix, $\mathbb{Q}$, and look for their extreme in complete analogy with what is done for the fully-connected ferromagnet. This is, of course, a tricky business, since the order parameter is here a matrix with number of elements n going to zero! A recipe for identifying the form of the order parameter (or the correct saddle-point solution) was proposed by G. Parisi in the late 70s and early 80s [74]. This solution has been recently proven to be exact for mean-field models by two mathematical physicists, F. Guerra [75] and M. Talagrand [76]. Whether the very rich physical structure that derives from this rather formal solution survives in finite dimensional systems remains a subject of debate.

Let us now focus on the Sherrington-Kirkpatrick model, for which $p = 2$. A way to introduce the parameters q_{ab} in the expression of the disorder averaged partition sum is to introducing them *via* the Gaussian decoupling or Hubbard-Stratonovich transformation

$$\int dq_{ab}\, e^{\beta J q_{ab} \sum_i s_i^a s_i^b - \frac{N}{2} q_{ab}^2} = e^{\frac{N}{2}\left(\beta J \frac{1}{N} \sum_i s_i^a s_i^b\right)^2} \qquad a \neq b \tag{1.239}$$

for each pair of replica indices $a \neq b$. Thus, one indeed decouples the site indices, i and the averaged replicated partition function can be rewritten as

$$[Z_J^n] = \int \prod_{a \neq b} dq_{ab}\, e^{-\beta F(\{q_{ab}\})} \tag{1.240}$$

and

$$\beta F(\{q_{ab}\}) = -\frac{N\beta^2 J^2}{2}\left[-\sum_{a\neq b} q_{ab}^2 + n\right] - N \ln \zeta(\{q_{ab}\}) \,, \tag{1.241}$$

$$\zeta(\{q_{ab}\}) = \sum_{\{s_a\}} e^{-\beta H(\{q_{ab}, s_a\})} \,, \tag{1.242}$$

$$H(\{q_{ab}, s_a\}) = -J \sum_{a \neq b} q_{ab} s_a s_b - h \sum_a s_a \,, \tag{1.243}$$

where for simplicity we set $h_i = h$. The factor N in front of $\ln \zeta$ comes from the decoupling of the site indices. Note that the transformation (1.239) serves to uncouple the sites and to obtain the very useful factor N in front of the exponential. The partition function

$$\zeta(\{q_{ab}\}) = \sum_{\{s_a\}} e^{-\beta H(\{q_{ab}, s_a\})} \tag{1.244}$$

is the one of a fully-connected Ising model with interaction matrix $\mathbb{Q}$ with elements q_{ab}. As it is posed, this problem remains unsolvable. However, important steps forward will be possible taking advantage of the $n \to 0$ limit.

To summarise, we started with an interacting spin model. Next, we enlarged the number of variables from N spins to $N \times n$ replicated spins by introducing n non-interacting copies of the system. By integrating out the disorder and introducing the matrix $\mathbb{Q}$ we *decoupled the sites* but we payed the price of *coupling the replicas*.

Saddle-point evaluation

Having extracted a factor N in the exponential suggests to evaluate the integral over q_{ab} with the saddle-point method. This, of course, involves the *a priori* dangerous exchange of limits $N \to \infty$ and $n \to 0$. The replica theory relies on this assumption. One then writes

$$\lim_{N\to\infty} -[\, f_J \,] \to -\lim_{n\to 0} \frac{1}{n} f(\mathbb{Q}_{\rm sp}) \;, \tag{1.245}$$

where the result is the physical Gibbs free-energy which depends on the control parameters βJ and βh, and searches for the solutions to the $n(n-1)/2$ extremisation equations

$$\left.\frac{\partial f(\{q_{ab}\})}{\partial q_{cd}}\right|_{\mathbb{Q}_{\rm sp}} = 0 \;. \tag{1.246}$$

In usual saddle-point evaluations the saddle-point one should use is (are) the one(s) that correspond to absolute minima of the free-energy density. In the replica calculation the number of variables is $n(n-1)/2$ that becomes negative! when $n < 1$ and makes the saddle-point evaluation tricky. It turns out that in order to avoid unphysical complex results one needs to focus on the saddle-points with positive (or at least semi-positive) definite Hessian

$$\mathcal{H} \equiv \left.\frac{\partial f(\{q_{ab}\})}{\partial q_{cd}\partial q_{ef}}\right|_{\mathbb{Q}_{\rm sp}} \;, \tag{1.247}$$

and these sometimes corresponds to *maxima* (instead of minima) of the free-energy density.

The saddle-point equations are also self-consistency equations

$$q^{\rm sp}_{ab} = \langle s_a s_b \rangle_{H(\{q_{ab}, s_a\})} = [\, \langle s_a s_b \rangle \,] \tag{1.248}$$

where the second member means that the average is performed with the *single site many-replica Hamiltonian* $H(\{q_{ab}, s_a\})$ and the third member is just one of the averages we would like to compute.

The partition function in eq. (1.243) cannot be computed for generic q_{ab} since there is no large n limit to exploit. On the contrary, $n \to 0$. Thus, one usually looks for solutions to eqs. (1.246) within a certain family of matrices $\mathbb{Q}$. We discuss below the relevant parametrizations.

Permutation symmetry & spontaneous symmetry breaking

The order parameter dependent free-energy to be extremised is invariant under replica index permutations.

$$f(\{q_{ab}\}) = f(\{q_{\pi(a)\pi(b)}\}) \;, \tag{1.249}$$

with $\pi(a)$ an arbitrary permutation of replica indices.

In the Curie-Weiss description of the paramagnetic - ferromagnetic transition, we already encountered the problem of extremising an order parameter dependent free-energy density, $f(m)$, with a symmetry, under global spin reversal, $m \to -m$, and finding a spontaneous symmetry breaking with (two) extremes that break that symmetry, and the action of the global spin reversal takes one extreme into the other. Here, we have face the extremisation of a free-energy function with a more abstract symmetry that may also be broken.

Let us see how to proceed considering first a simple example, a function of two variables

$$g(x, y) = g(y, x) \; , \tag{1.250}$$

which is symmetric under the exchange of the two arguments. One can then search for the pair $(x, y)_{\rm sp}$ that extremises this function in two ways:

- The first possibility is that $x_{\rm sp} = y_{\rm sp}$, which means that the extremal point is invariant under the action of the symmetry of the problem, namely the exchange of the two variables. In order to find such an extremal point we could easily restrict to the line $x = y$ and minimise $g(x, x)$ in this one-dimensional subspace.
- The second possibility is that $x_{\rm sp} \neq y_{\rm sp}$. In this case, both $(x_{\rm sp}, y_{\rm sp})$ and $(y_{\rm sp}, x_{\rm sp})$ are extremes of g. If this happens, the symmetry is said to be broken and the action of the symmetry group of the problem transforms one extremal point into the other and viceversa.

Replica symmetry (RS)

In principle, nothing distinguishes one replica from another one. This is the reason why Sherrington and Kirkpatrick looked for solutions that preserve replica symmetry:

$$q_{ab} = q \; , \qquad\qquad \text{for all } a \neq b \; , \tag{1.251}$$

and in terms of the permutation symmetric discussed above satisfy

$$q_{ab} = q_{\pi(a)\pi(b)} \; . \tag{1.252}$$

Inserting this *Ansatz* in eqs. (1.241)-(1.243), one finds

$$-\beta f(q) = -\frac{\beta^2}{4}(1-q)^2 - \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \, e^{-\frac{z^2}{2}} \, \ln[2\cosh(\beta\sqrt{q}z + \beta h)] \tag{1.253}$$

at leading order in n. (The details of this calculation can be found in [62].) Looking for the extreme value of q one finds the saddle-point equation[5]

$$q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \, e^{-z^2/2} \, \tanh^2\left(\beta\sqrt{q}z + \beta h\right) \; . \tag{1.254}$$

[5]In the p-spin case the argument of the $\tanh^2$ is replaced by $\beta\sqrt{\frac{pq^{p-1}}{2}}z + \beta h$.

In order to lighten the notation we have not added a subscript $_{\rm sp}$ to q. This equation resembles strongly the one for the magnetisation density of the Curie-Weiss ferromagnet.

In the absence of a magnetic field, one finds a second order phase transition at $T_c = J$ from a paramagnetic ($q = 0$) to a spin-glass phase with $q \neq 0$. In the presence of a field there is no phase transition. SK soon realized though that there is something wrong with this solution: the entropy at zero temperature is negative, $S(0) = -1/(2\pi)$, and this is impossible for a model with discrete spins, for which S is strictly positive.

A bit later, de Almeida and Thouless showed that the reason for this failure is that the replica symmetric saddle-point is not stable, since the Hessian (1.247) is not positive definite and has negative eigenvalues [77]. The eigenvalue responsible for the instability of the replica symmetric solution is called the *replicon*.

Indeed, the Hessian of order parameter dependent free-energy density reads

$$\mathcal{H}_{ab,cd} = \frac{\partial f(\mathbb{Q})}{\partial q_{ab} \partial q_{cd}} = \beta^2 \delta_{(ab),(cd)} - \beta^4 \left[\langle s_a s_b s_c s_d \rangle_\zeta - \langle s_a s_b \rangle_\zeta \langle s_c s_d \rangle_\zeta \right] \tag{1.255}$$

where the averages $\langle \dots \rangle - \zeta$ are to be taken with the partition function ζ. The Hessian has four indices but from its structure and the fact that one has to evaluate it at a replica symmetric point in which $q_{ab} = q$, these appear in particular combinations

$$\begin{array}{rclll}
\mathcal{H}_1 &=& = \beta^2 - \beta^4 (1-q)^2 & & (ab) = (cd) \; , \\
\mathcal{H}_2 &=& -\beta^2 q (1-q) & & a = c, b \neq d \;\text{ or }\; a \neq c, b = d \; , \\
\mathcal{H}_3 &=& -\beta^4 (r - q^2) & & a \neq c, b \neq d \; ,
\end{array} \tag{1.256}$$

with

$$r = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \; e^{-z^2/2} \; \tanh^4 (\beta \sqrt{q} z + \beta h) \; . \tag{1.257}$$

The eigenvalues of the Hessian matrix are obtained from

$$\sum_{cd} \mathcal{H}_{ab,cd} v_{cd} = \lambda v_{ab} \tag{1.258}$$

and a calculation which can be found, done in detail, in [62] yields the expression of the replicon eigenvalue

$$\lambda_R = \beta^2 - \beta^4 [-(1-q^2) + 2(q - q^2) - (r - q^2)] \; , \tag{1.259}$$

where, again, q is determined by the saddle point equation. This expression vanishes on the curve

$$1 = \beta^2 \int \frac{dz}{\sqrt{2\pi}} \operatorname{sech}^4 (\beta \sqrt{q} z + \beta h) \; , \tag{1.260}$$

the so-called *de Almeida-Thouless* line. Below this line the replica symmetric Ansatz is not valid and replica symmetry must be broken.

Interestingly enough, the numerical values for several physical quantities obtained with the replica symmetric solution do not disagree much with numerical results. For instance,

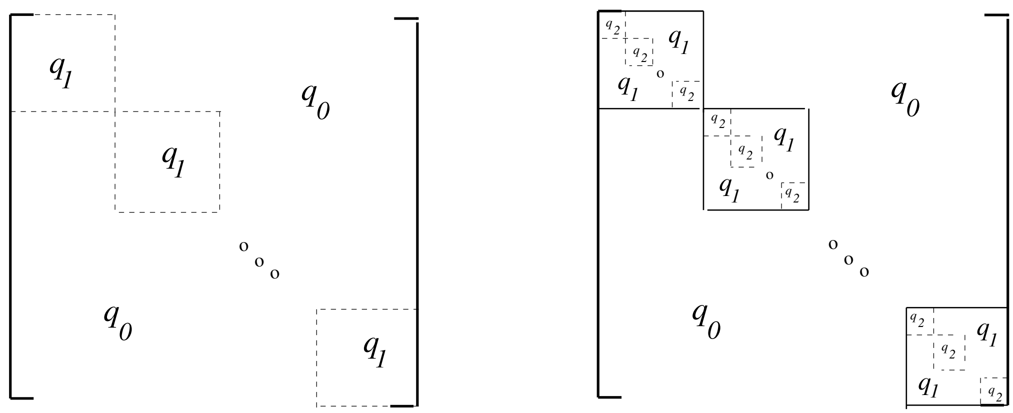


Figure 1.24: Left: a one-step replica symmetry breaking (1RSB) *Ansatz*. Right: a two-step replica symmetry breaking *Ansatz*. The elements on the main diagonal do not appear in the calculation of the partition sum and one can set them to be identical to zero. In the 1RSB case the diagonal blocks have size $m \times m$ (please note that this is a new parameter that is not the magnetisation! this is the name given in the literature to it). In the 2RSB the procedure is repeated and one has blocks of size $m_1 \times m_1$ with smaller diagonal blocks of size $m_2 \times m_2$.

the ground state zero-temperature energy density is $E^{\rm rs}_{\rm gs} = -0.798$ while with numerical simulations one finds $E_{\rm gs} \sim -0.76$.

For the $p > 2$ model one finds that the replica symmetric solution is stable at all temperatures. However, the problem of the negative entropy remains and should be solved by another solution. The transition must then have aspects of a first-order one, with another solution appearing at low temperatures and becoming the most convenient one at the transition.

One step replica symmetry breaking

The next challenge is to device a replica symmetry breaking *Ansatz*, in the form of a matrix q_{ab} that is not invariant under permutations of rows or columns. There is no first principles way of doing this, instead, the structure of the *Ansatz* is the result of trial and error. Indeed, a kind of minimal way to break the replica symmetry is to propose a structure in blocks as the one shown in Fig. 1.24-left. The diagonal elements are set to zero as in the RS case. Square blocks of linear size m (not to be confused with the magnetisation!) close to the main diagonal are filled with a parameter q_1. The elements in the rest of the matrix take a different value q_0 and one takes $0 \leq q_0 \leq q_1$. In the process of taking the $n \to 0$ limit, one also has to promote the group size m, that was originally an integer between 1 and n, to be a real number $m \in [0, 1]$. The matrix $\mathbb{Q}$ depends on three real parameters q_0, q_1, m and one has to find the values such that the free-energy density is a extreme! The conditions are

$$\frac{\partial f(q_0, q_1, m)}{\partial q_0} = \frac{\partial f(q_0, q_1, m)}{\partial q_1} = \frac{\partial f(q_0, q_1, m)}{\partial m} = 0 \qquad (1.261)$$

and the Gibbs free energy reads

$$\tilde{f}(q_1, q_0, m) = -\frac{1}{4}\beta\left[1 + mq_0^2 + (1-m)q_1^2 - 2q_1\right]$$
$$-\frac{1}{m\beta}\int \frac{dz}{\sqrt{2\pi q_0}}\, e^{-\frac{z^2}{2q_0}}\ \ln \int \frac{dy}{\sqrt{2\pi(q_1-q_0)}}\, e^{-\frac{y^2}{2(q_1-q_0)}}\ \left[2\cosh(\beta(z+y))\right]^m \tag{1.262}$$

In the two cases $m = 0$ and $m = 1$ one recovers the RS solution with, respectively, $q = q_0$ or $q = q_1$.

In the SK model ($p = 2$) the 1RSB *Ansatz* yields a second order phase transition ($q_0 = q_1 = 0$ and $m = 1$ at criticality) at a critical temperature $T_c = J$, that remains unchanged with respect to the one predicted by the RS *Ansatz*. The 1RSB solution is still unstable below T_c and in all the low temperature phase. One notices, however, that the zero temperature entropy, even though still negative and incorrect, takes a value that is closer to zero, $S(T = 0) \approx -0.01$, the ground state energy is closer to the value obtained numerically. The replicon eigenvalue even though still negative has an absolute value that is closer to zero: $\lambda_R = -C\tau^2/9$ in the 1RSB compared to $\lambda_R = -C\tau^2$ in the RS, where $\tau = (1-T)$. All this suggests that the 1RSB *Ansatz* is closer to the exact solution. Finally, one notices that the solution to the extremisation conditions found corresponds to a *maximum* of the free-energy density.

Instead, in all cases with $p \geq 3$ the 1RSB *Ansatz* is stable below the static critical temperature T_s and all the way up to a new characteristic temperature $0 < T_f < T_s$. Moreover, one can prove that in this range of temperatures the model is solved exactly by this *Ansatz*. The critical behaviour is quite peculiar: while the order parameters q_0 and q_1 jump at the transition from a vanishing value in the paramagnetic phase to a non-zero value right below T_s, all thermodynamic quantities are continuous since $m = 1$ at T_s and all q_0 and q_1 dependent terms appear multiplied by $1-m$. This is a mixed type of transition that has been baptised *random first-order*. Note that disorder weakens the critical behaviour in the $p \geq 3$-spin models. In the limit $p \to \infty$ the solutions become $m = T/T_c$, $q_0 = 0$ and $q = 1$.

k-step replica symmetry breaking

The natural way to generalize the 1RSB *Ansatz* is to propose a k-step one. In each step the off-diagonal blocks are left unchanged while the diagonal ones of size m_k are broken as in the first step thus generating smaller square blocks of size m_{k+1}, close to the diagonal. At a generic k-step RSB scheme one has

$$0 \leq q_0 \leq q_1 \leq \cdots \leq q_{k-1} \leq q_k \leq 1\ , \tag{1.263}$$
$$n = m_0 \geq m_1 \geq \cdots \geq m_k \geq m_{k+1}\ , \tag{1.264}$$

parameters. In the $n \to 0$ limit the ordering of the parameters m is reversed

$$0 = m_0 \leq m_1 \leq \cdots \leq m_k \leq m_{k+1}\ . \tag{1.265}$$

In the SK model one finds that any finite k-step RSB *Ansatz* remains unstable. However, increasing the number of breaking levels the results continue to improve with, in particular, the zero temperature entropy getting closer to zero. In the $p \geq 3$ case instead one finds that the 2RSB *Ansatz* has, as unique solution to the saddle-point equations, one that boils down to the 1RSB case. This suggests that the 1RSB *Ansatz* is stable as can also be checked with the analysis of the Hessian eigenvalues: the replicon is stricly positive for all $p \geq 3$.

Full replica symmetry breaking

In order to construct the full RSB solution the breaking procedure is iterated an infinite number of times. The full RSB *Ansatz* thus obtained generalises the block structure to an infinite sequence by introducing a function

$$q(x) = q_k \;, \qquad m_{k+1} < x < m_k \tag{1.266}$$

with $0 \leq x \leq 1$. Introducing $q(x)$ sums over replicas are traded by integrals over x; for instance

$$\frac{1}{n} \sum_{a \neq b} q_{ab}^l = \int_0^1 dx \; q^l(x) \; . \tag{1.267}$$

The free-energy density becomes a *functional of the function $q(x)$*. The extremisation condition is then a hard functional equation

$$\frac{\delta f(q(x))}{\delta q(x)} = 0 \; . \tag{1.268}$$

The free energy density in its full-RSB form reads

$$f = -\frac{\beta}{4} \left[1 + \int_0^1 dx \, q^2(x) - 2q(1) \right] - \frac{1}{\beta} \int \mathrm{D}u \; f_0(0, \sqrt{q(0)} u) \, , \tag{1.269}$$

where f_0 is the solution to the non-linear anti-parabolic equation

$$\frac{\partial f_0(x,h)}{\partial x} = -\frac{1}{2} \frac{\mathrm{d}q}{\mathrm{d}x} \left[\frac{\partial^2 f_0}{\partial h^2} + x \left(\frac{\partial f_0}{\partial h} \right)^2 \right] \, , \tag{1.270}$$

with the initial condition

$$f_0(1,h) = \ln(2 \cosh h) \; . \tag{1.271}$$

In spite of the numerical evidence that the solution of eq. (1.270) is unique, it has only recently been proved that the free energy is convex and hence its solution unique. One can recover the particular case of the 1RSB using a $q(x)$ with two plateaux, at q_0 and q_1 and the breaking point at $x = m$.

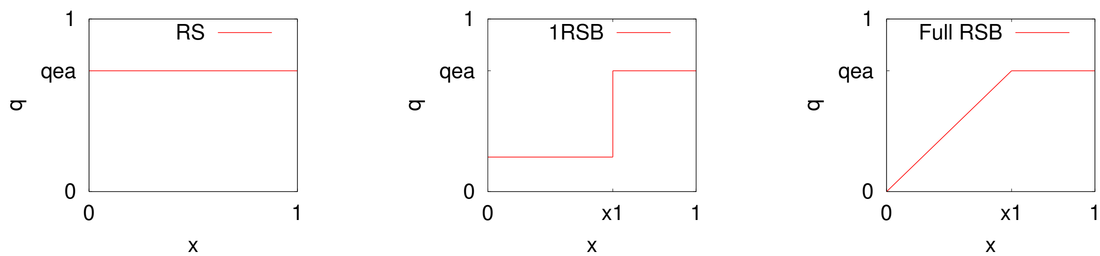


Figure 1.25: The function $q(x)$ for a replica symmetric (left), one step replica symmetry breaking (center) and full replica symmetry breaking *Ansätze*, close to T_c in the latter case.

A Landau expansion – expected to be valid close to the assumed second order phase transition – simplifies the task of solving it. For the SK model one finds

$$q(x) = \begin{cases} \frac{x}{2} \;, & 0 \leq x \leq x_1 = 2q(1) \;, \\ q_{\rm EA} \equiv q_{\max} = q(1) \;, & x_1 = 2q(1) \leq x \leq 1 \;, \end{cases} \tag{1.272}$$

at first order in $|T - T_c|$, with $q(1) = |T - T_c|/T_c$ and $x_1 = 2q(1)$. The stability analysis yields a vanishing replicon eigenvalue signalling that the full RSB solution is *marginally stable*, meaning that the replicon eigenvalue vanishes in the full low-temperature phase.

The RS, 1RSB and FRB (close to T_c) behaviours of $q(x)$ are shown in Fig. 1.25.

Marginality condition

In the discussion above we chose the extreme that *maximize* the free-energy density since we were interested in studying equilibrium properties. We could, instead, use a different prescription, though *a priori* not justified, and select other solutions. For example, we can impose that the solution is *marginally stable* by requiring that the replicon eigenvalue vanishes. In the $p = 2$ this leads to identical results to the ones obtained with the usual prescription since the full-RSB *Ansatz* is in any case marginally stable. In the p-spin models with $p \geq 3$ instead it turns out that the averaged properties obtained in this way correspond to the asymptotic values derived with the stochastic dynamics starting from random initial conditions. This is quite a remarkable result.

Interpretation of replica results

Let us now discuss the implications of the solution to fully-connected disordered models obtained with the, for the moment, rather abstract replica formalism.

The interpretation uses heavily the identification of *pure states*. Their definition is a tricky matter that we shall not discuss in detail here. We shall just assume it can be done and use the analogy with the ferromagnetic system – and its two pure states – and the TAP results at fixed disorder. As we already know, which are the pure states, its properties, number, *etc.* can depend on the quenched disorder realization and fluctuate from sample to sample. We shall keep this in mind in the rest of our discussion.

Let us then distinguish the averages computed within a pure state and over all configuration space. In a ferromagnet with no applied magnetic field this is simple to grasp: at high temperatures there is just one state, the paramagnet, while at low temperatures there are two, the states with positive and negative magnetization. If one computes the averaged magnetization restricted to the state of positive (negative) magnetization one finds $m_{\rm eq} > 0$ ($m_{\rm eq} < 0$); instead, summing over all configurations $m_{\rm eq} = 0$ even at low temperatures. Now, if one considers systems with more than just two pure states, and one labels them with Greeks indices, averages within such states are denoted $\langle O \rangle_\alpha$ while averages taken with the full Gibbs measure are expressed as

$$\langle\, O\,\rangle = \sum_\alpha w_\alpha^J \, \langle\, O\,\rangle_\alpha \; . \tag{1.273}$$

w_α^J is the probability of the α state given by

$$w_\alpha^J = \frac{e^{-\beta F_\alpha^J}}{Z_J} \; , \qquad \text{with} \qquad Z_J = \sum_\alpha e^{-\beta F_\alpha^J} \tag{1.274}$$

and thus satisfying the normalization condition $\sum_\alpha w_\alpha^J = 1$. F_α^J can be interpreted as the total free-energy of the state α. These probabilities, as well as the state dependent averages, will show sample-to-sample fluctuations.

One can then define an *overlap between states*:

$$q_{J\alpha\beta} \equiv N^{-1} \sum_i \langle s_i \rangle_\alpha \langle s_i \rangle_\beta = N^{-1} \sum_i m_i^\alpha m_i^\beta \tag{1.275}$$

and rename the *self-overlap* the Edwards-Anderson parameter

$$q_{J\alpha\alpha} \equiv N^{-1} \sum_i \langle s_i \rangle_\alpha \langle s_i \rangle_\alpha \equiv q_{J{\rm EA}} \tag{1.276}$$

(assuming the result is independent of α). The statistics of possible overlaps is then characterized by a probability function

$$P_J(q) \equiv \sum_{\alpha\beta} w_\alpha^J w_\beta^J \; \delta(q - q_{\alpha\beta}^J) \; , \tag{1.277}$$

where we included a subindex J to stress the fact that this is a strongly sample-dependent quantity. Again, a ferromagnetic model serves to illustrate the meaning of $P_J(q)$. First, there is no disorder in this case so the J label is irrelevant. Second, the high-T equilibrium phase is paramagnetic, with $q = 0$. $P(q)$ is then a delta function with weight 1 (see the left panel in Fig. 1.26). In the low-T phase there are only two pure states with identical statistical properties and $q_{\rm EA} = m^2$. Thus, $P(q)$ is just the sum of two delta functions with weight 1/2 (central panel in Fig. 1.26).

Next, one can consider averages over quenched disorder and study

$$[\, P_J(q)\,] \equiv \int dJ\, P(J) \,\sum_{\alpha\beta} w_\alpha^J w_\beta^J \,\delta(q - q_{\alpha\beta}^J)\;. \tag{1.278}$$

How can one access $P_J(q)$ or $[\, P_J(q)\,]$? It is natural to reckon that

$$P_J(q) = Z^{-2} \sum_{\sigma s} e^{-\beta H_J(\{\sigma_i\})} e^{-\beta H_J(\{s_i\})}\; \delta \left(N^{-1} \sum_i \sigma_i s_i - q \right) \tag{1.279}$$

that is to say, $P_J(q)$ is the probability of finding an overlap q between two *real replicas* of the system with identical disordered interactions in equilibrium at temperature T. This identitiy gives a way to compute $P_J(q)$ and its average in a numerical simulation: one just has to simulate two independent systems with identical disorder in equilibrium and calculate the overlap.

But there is also, as suggested by the notation, a way to relate the pure state structure to the replica matrix q_{ab}. Let us consider the simple case

$$\begin{aligned}
[\, m_i\,] \;&=\; \left[Z_J^{-1} \sum_{\{s_i\}} s_i \; e^{-\beta H_J(\{s_i\})} \right] = \left[\frac{Z_J^{n-1}}{Z_J^n} \sum_{\{s_i^1\}} s_i^1 \; e^{-\beta H_J(\{s_i^1\})} \right] \\
&=\; \left[\frac{1}{Z_J^n} \sum_{\{s_i^a\}} s_i^1 e^{-\beta \sum_{a=1}^n H_J(\{s_i^a\})} \right]
\end{aligned} \tag{1.280}$$

where we singled out the replica index of the spin to average. This relation is valid for all n, in particular for $n \to 0$. In this limit the denominator approaches one and the average over disorder can be simply evaluated

$$[\, m_i\,] \;=\; \sum_{\{s_i^a\}} s_i^1 e^{-\beta H_{\rm eff}(\{s_i^a\})} \tag{1.281}$$

and introducing back the normalization factor $Z^n = 1 = \sum_{\{s_i^a\}} e^{-\beta \sum_{a=1}^n H_J(\{s_i^a\})}$ that becomes $Z^n = [\, \sum_{\{s_i^a\}} e^{-\beta \sum_{a=1}^n H_J(\{s_i^a\})}\,] = e^{-\beta H_{\rm eff}(\{s_i^a\})}$ we have

$$[\, m_i\,] = \langle\, s_i^a\,\rangle_{H_{\rm eff}} \tag{1.282}$$

with a any replica index. The average is taken over the Gibbs measure of a system with effective Hamiltonian $H_{\rm eff}$. In a replica symmetric problem in which all replicas are identical this result should be independent of the label a. Instead, in a problem with replica symmetry breaking the averages on the right-hand-side need not be identical for all a. This could occur in a normal vectorial theory with dimension n in which not all components take the same expected value. It is reasonable to assume that the full thermodynamic average is achieved by the sum over all these cases,

$$[\, m_i \,] = \lim_{n\to 0} \frac{1}{n} \sum_{a=1}^{n} \langle\, s_i^a \,\rangle_{H_{\rm eff}} \ . \tag{1.283}$$

Let us now take a less trivial observable and study the spin-glass order parameter q

$$\begin{aligned} q \ \equiv \ [\, \langle\, s_i \,\rangle^2 \,] &= \left[Z_J^{-1} \sum_{\{s_i\}} s_i \; e^{-\beta H_J(\{s_i\})} \; Z_J^{-1} \sum_{\{\sigma_i\}} \sigma_i \; e^{-\beta H_J(\{\sigma_i\})} \right] \\ &= \left[\frac{Z^{n-2}}{Z^n} \sum_{\{s_i\},\{\sigma_i\}} s_i \sigma_i \; e^{-\beta H_J(\{s_i\}) - \beta H_J(\{\sigma_i\})} \right] \\ &= \left[\frac{1}{Z_J^n} \sum_{\{s_i^a\}} s_i^1 s_i^2 \; e^{-\beta \sum_{a=1}^n H_J(\{s_i^a\})} \right] \end{aligned} \tag{1.284}$$

In the $n \to 0$ limit the denominator is equal to one and one can then perform the average over disorder. Introducing back the normalization one then has

$$q = \langle\, s_i^a s_i^b \,\rangle_{E_{\rm eff}(\{s_i^a\})} \tag{1.285}$$

for any arbitrary pair of replicas $a \neq b$ (since $\langle\, s_i^a s_i^a \,\rangle = 1$ for Ising spins). The average is done with an effective theory of n interacting replicas characterized by $E_{\rm eff}(\{s_i^a\})$. Again, if there is replica symmetry breaking the actual thermal average is the sum over all possible pairs of replicas:

$$q = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} q^{ab} \ . \tag{1.286}$$

A similar argument allows one to write

$$q^{(k)} = [\, \langle\, s_{i_1} \dots s_{i_k} \,\rangle^2 \,] = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} q_{ab}^k \ . \tag{1.287}$$

One can also generalize this argument to obtain

$$P(q) = [\, P_J(q) \,] = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \delta(q - q^{ab}) \tag{1.288}$$

Thus, the replica matrix q_{ab} can be ascribed to the overlap between pure states.

Note that a small applied field, though uncorrelated with a particular pure state, is necessary to have non-zero local magnetizations and then non-zero q values.

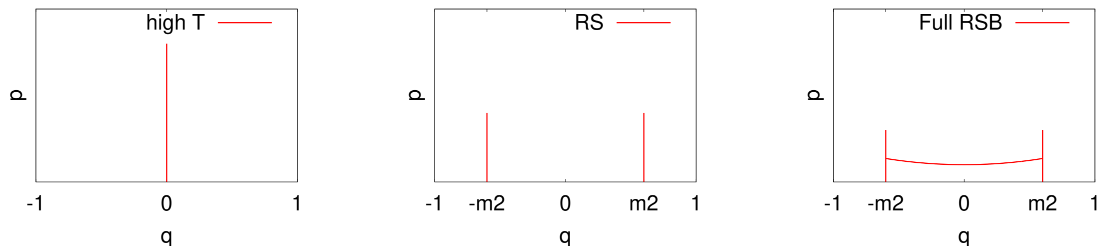


Figure 1.26: $[\, P_J(q)\,]$ in a paramagnet (left), in a ferromagnet or a replica symmetric system (centre) and for system with full RSB (right).

The function $P(q)$ then extends the concept of order parameter to a function. In zero field the symmetry with respect to simultaneous reversal of all spins translates into the fact that $P_J(q)$ must be symmetric with respect to $q = 0$. $[\, P_J(q)\,]$ can be used to distinguish between the droplet picture prediction for finite dimensional spin-glasses – two pure states – that simply corresponds to

$$[\, P_J(q)\,] = \frac{1}{2}\delta(q - q_{\rm EA}) + \frac{1}{2}\delta(q + q_{\rm EA}) \tag{1.289}$$

(see the central panel in Fig. 1.26) and a more complicated situation in which $[\, P_J(q)\,]$ has the two delta functions at $\pm q_{\rm EA}$ plus non-zero values on a finite support (right panel in Fig. 1.26) as found in mean-field spin-glass models.

The linear susceptibility

Taking into account the multiplicity of pure states, the magnetic susceptibility, eq. (1.117), and using (1.273) becomes

$$T\chi = T[\, \chi_J\,] = 1 - \frac{1}{N}\sum_i [\, \langle\, s_i\, \rangle^2\,] = 1 - \sum_{\alpha\beta} [\, w^J_\alpha w^J_\beta\,]\, q_{\alpha\beta} = \int dq\, (1-q)\, P(q)\ . \tag{1.290}$$

There are then several possible results for the susceptibility depending on the level of replica symmetry breaking in the system:

- In a replica symmetric problem or, equivalently, in the droplet model,

  $$\chi = \beta(1 - q_{\rm EA})\ . \tag{1.291}$$

  This is also the susceptibility within a pure state of a system with a higher level of RSB.

- At the one step RSB level, this becomes

  $$\chi = \beta\, [1 - (1-m) q_{\rm EA}]\ . \tag{1.292}$$

- For systems with full RSB one needs to know the complete $P(q)$ to compute χ, as in (1.290).

Note that in systems with RSB (one step or full) the susceptibility is larger than $\beta(1-q_{\rm EA})$.

A system with $q_{\rm EA}=1$ in the full low-temperature phase (as the REM model or $p\to\infty$ limit of the p spin model, see below) has just one configuration in each state. Systems with $q_{\rm EA}<1$ below T_c have states formed by a number of different configurations that is exponentially large in N. (Note that $q_{\rm EA}<1$ means that the two configurations differ in a number of spins that is proportional to N.) The logarithm of this number is usually called the intra-state entropy.

Even if the number of pure states can be very large (exponential in N) only a fraction of them can have a non-negligible weight. This is the case if one finds, for example, $\sum_\alpha w_\alpha^2<+\infty$

Symmetry and ergodicity breaking

In all $p\geq 2$ spin models there is a phase transition at a finite T_s at which the rather abstract *replica symmetry* is broken. This symmetry breaking is accompanied by ergodicity breaking as in the usual case. Many pure states appear at low temperatures, each one has its reversed $s_i\to -s_i$ counterpart, but not all of them are related by real-space symmetry properties.

The one-step RSB scenario

In this case the transition has first-order and second-order aspects. The order parameters q_0 and q_1 jump at the critical point as in a first-order transition but the thermodynamic quantities are continuous.

The full RSB scenario

Right below T_c an exponential in N number of equilibrium states appear. The transition is continuous, the order parameter approaches zero right below T_c. Lowering further the temperature each ergodic component breaks in many other ones. In this sense, the full spin-glass phase, $T<T_c$, is 'critical' and not only the single point T_c.

The pinning field

We can nevertheless choose a possible direction, given by another field $\sigma(x)$, and compute the free–energy of our system when it is weakly pinned by this external quenched field

$$F_\phi\left[\sigma,g,\beta\right]=-\frac{1}{\beta}\log\int d\phi(x)\; e^{-\beta H[\phi]-\frac{g}{2}\int dx(\sigma(x)-\phi(x))^2} \tag{1.293}$$

where $g>0$ denotes the strength of the coupling. This free-energy (1.293) will be small when the external perturbing field $\sigma(x)$ lies in a direction corresponding to the bottom of a well of the unperturbed free-energy. Therefore, we should be able to obtain useful information about the free-energy landscape by scanning the entire space of the configurations $\sigma(x)$ to locate all the states in which the system can freeze after spontaneous

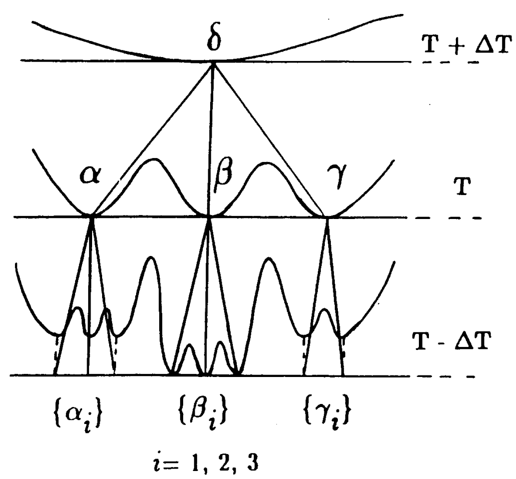


Figure 1.27: The subsequent phase transitions in the SK model.

ergodicity breaking ($g \to 0$). According to this intuitive idea, we now consider the field $\sigma(x)$ as a thermalized variable with the "Hamiltonian" $F_\phi\left[\sigma, g, \beta\right]$. The free-energy of the field σ at inverse temperature βm where m is a positive free parameter therefore reads

$$F_\sigma(m,\beta) = \lim_{g\to 0^+} -\frac{1}{\beta m}\log\int d\sigma(x)\ e^{-\beta m F_\phi[\sigma,g,\beta]} \tag{1.294}$$

When the ratio m between the two temperatures is an integer, one can easily integrate $\sigma(x)$ in Eq.(1.294) after having introduced m copies $\phi^\rho(x)$ ($\rho = 1...m$) of the original field to obtain the relation

$$F_\sigma(m,\beta) = \lim_{g\to 0^+} -\frac{1}{\beta m}\log\int\prod_{\rho=1}^{m} d\phi^\rho(x)\ e^{-\beta\sum_\rho H[\phi^\rho]+\frac{1}{2}\sum_{\rho,\lambda} g^{\rho\lambda}\int dx\phi^\rho(x)\phi^\lambda(x)} \tag{1.295}$$

where $g^{\rho\lambda} = g(\frac{1}{m} - \delta^{\rho\lambda})$. Let us define two more quantities related to the field σ : its internal energy $W(m,\beta) = \frac{\partial(mF_\sigma)}{\partial m}$ and its entropy $S(m,\beta) = \beta m^2\frac{\partial F_\sigma}{\partial m}$. Since the case $m = 1$ will be of particular interest, we shall use hereafter $F_{hs}(\beta) \equiv W(m = 1,\beta)$ and $\mathcal{S}_{hs}(\beta) \equiv S(m = 1,\beta)$ where hs stands for "hidden states". We stress that $S(m,\beta)$ and $\beta^2\frac{\partial F_\phi}{\partial\beta}$ which are respectively the entropies of the fields σ and ϕ are two distinct quantities with different physical meanings.

When the pinning field $\sigma(x)$ is thermalized at the same temperature as $\phi(x)$, that is when $m = 1$, one sees from Eq.(1.295) that $F_\phi(\beta) = F_\sigma(m = 1,\beta)$. The basic idea of this letter is to decompose F_σ into its energetic and entropic contributions to obtain

$$\mathcal{S}_{hs}(\beta) = \beta\Big[F_{hs}(\beta) - F_\phi(\beta)\Big] \tag{1.296}$$

To get some insights on the significance of the above relation, we shall now turn to the particular case of disordered mean-field systems. We shall see how it rigorously gives back some analytical results derived within the mean-field TAP and dynamical approaches. We shall then discuss the physical meaning of identity (1.296) for the general case of glassy systems.

Coupling replicas and the effective potential

Let us take a spin-configuration, $\{s\}$, in equilibrium at temperature T', that is to say, drawn from the canonical probability distribution $P[\{s\}] = \exp(-\beta' H[\{s\}])/Z(T')$. One computes the free-energy cost to keep the system at a fixed overlap $\tilde{p} = q_{s,\sigma}$ with $\{s\}$ at a temperature T (in general different from T'):

$$V_J(\beta, \tilde{p}, \{s\}) = -\frac{T}{N} \ln Z_J(\beta, \tilde{p}, \{s\}) - f_J(T); \tag{1.297}$$

$$Z_J(\beta, \tilde{p}, \{s\}) \equiv \sum_{\{\sigma\}} e^{-\beta H_J[\{\sigma\}]} \, \delta\left(\tilde{p} - q_{s,\sigma}\right) \tag{1.298}$$

$$\beta N f_J(T) = \ln Z_J(\beta) = \ln \sum_{\{s\}} e^{-\beta H_J[\{s\}]} \, . \tag{1.299}$$

($f_J(T)$ is the disorder-dependent free-energy density without constraint.) In this problem the spins s_i are quenched variables on the same footing as the random interactions in the Hamiltonian. One then assumes that V is self-averaging with respect to the quenched disorder and the probability distribution of the reference configuration $\{s\}$. One then computes the two averages:

$$NV(\beta, \beta', \tilde{p}) \equiv N[V_J(\beta, \tilde{p}, \{s\})]_{J,\{s\}} = \left[\sum_{\{s\}} \frac{e^{-\beta' H_J[\{s\}]}}{Z_J(\beta')} \left(-T \ln Z_J(\beta, \tilde{p}, \{s\}) - f_J(T)\right) \right]_J \, . \tag{1.300}$$

This average can be done using the replica method:

$$NV(\beta, \beta', \tilde{p}) = -T \lim_{n \to 0} \lim_{m \to 0} \left[\sum_{\{s\}} e^{-\beta' H_J[\{s\}]} Z_J(\beta')^{n-1} \left(\frac{Z_J(\{s\}; \tilde{p}, \{s\})^m - 1}{m} \right) \right]_J \, . \tag{1.301}$$

The analytic continuation is performed from integer n and m. One then has

$$Z^{(n,m)} = \left[\sum_{\{s^a\}} \sum_{\{\sigma^\alpha\}} \exp\left[\beta' \sum_{a=1}^{n} H[\{s^a\}] + \beta \sum_{\alpha=1}^{m} H[\{\sigma^\alpha\}] \right] \prod_{\alpha=1}^{m} \delta\left(\sum_i s_i^1 \sigma_i^\alpha - N\tilde{p} \right) \right]_J \, . \tag{1.302}$$

After averaging over the disorder strength distribution one introduces the order parameters:

$$Q_{ab} = \frac{1}{N} \sum_i s_i^a s_i^b \, , \qquad R_{\alpha\gamma} = \frac{1}{N} \sum_i \sigma_i^\alpha \sigma_i^\gamma \qquad P_{a\alpha} = \frac{1}{N} \sum_i s_i^a \sigma_i^\alpha \, , \tag{1.303}$$

with $a, b = 1, ..., n$ and $\alpha, \gamma = 1, ..., m$. Combining the order parameters in the single $(n+m) \times (n+m)$ matrix

$$\mathcal{Q} = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix} \tag{1.304}$$

one finds

$$\frac{1}{N} \log Z^{(n,m)} = \frac{1}{2} \left[\beta'^2 \sum_{a=1,b=1}^{n} Q^p_{ab} + \beta^2 \sum_{\alpha=1\gamma=1}^{m} R^p_{\alpha,\gamma}) + 2\beta\beta' \sum_{a=1}^{n} \sum_{\alpha=1}^{m} P^p_{a\alpha}) \right] + \frac{1}{2} \mathrm{Tr} \ln \mathcal{Q}. \tag{1.305}$$

We shall not present the details of the RSB *Ansatz* here.

One studies different ranges of β and β' and analyses the minima of V with respect to $\tilde{p}$.

The effective potential for four different temperatures, $T = T'$ for $p = 3$ is shown in [69] From top to bottom, the curves represent the potential at temperature higher then T_d, equal to T_d between T_d and T_S, and right at T_S. We can see from the figure that for $T > T_d$ the potential is monotonically increasing, and the only extremum of the potential is the minimum at $\tilde{p} = 0$. At the temperature T_d where the dynamical transition happens, the potential develops for the first time a minimum with $\tilde{p} \equiv r \neq 0$. It is interesting to observe that the energy in this flex point is equal to the asymptotic value of the energy in the out-of-equilibrium dynamics. The same is true for the parameter r which turns out to be equal to the dynamical Edward-Anderson parameter.

The condition for the potential of having a flex coincides with the marginality condition. Indeed the flex implies a zero eigenvalue in the longitudinal sector and at $x = 1$ the replicon and the longitudinal eigenvalues are degenerate. The marginality condition is well known to give exact results for the transition temperatures in p-spin spherical models.

We have observed that in general more then one minimum can be present in the potential. In the p-spin model it happens that two minima develop at the same temperature T_d. The rightmost one, that we will call primary is the one with $\tilde{p} = r$, while the other, secondary, has $\tilde{p} < r$. For temperatures smaller than T_d the minima have a finite depth, i.e. are separated by extensive barriers from the absolute minimum.

The primary minimum is easily interpreted. There the system denoted by s is in the same pure state as the system σ. In the region $T_S < T < T_D$ the number of pure states is exponentially large in N: $\mathcal{N} = e^{N\Sigma(T)}$. Consequently the probability of finding two system in the same state is exponentially small and proportional to $e^{-N\Sigma(T)}$. The free energy cost to constrain two systems to be in the same state is then proportional to the logarithm of this probability, namely we have

$$V_{primary} = T\Sigma(T). \tag{1.306}$$

Coherently at the statical transition temperature $T = T_S$ one finds $V_{primary} = 0$. The quantity Σ has been computed for the p-spin model as the number of solution of the

TAP equation with given free energy and coincides with our calculation. The secondary minima, could aso be associated to metastable states, but at present we do not have an interpretation for them. This conclusion on the equivalence of the potential with the number of solution of the TAP equation hold also in the ROM.

The study of the potential for temperatures smaller than T_S would require to take into account RSB effects, which would complicate a bit the analysis. However it is physically clear that the shape of the potential in that region it is not different qualitatively from the one at $T = T_S$. It has a minimum where $r = \tilde{p}$ are equal to the Edwards Anderson parameter and the value of potential is zero.

The study of the effective potential at different gives information about the chaotic properties of the models. We shall not develop it here.

Appendices

1.A Mathematical support

1.A.1 Fourier transform

Finite volume

We define the Fourier transform (FT) of a function $f(\vec{x})$ defined in a volume V as

$$\tilde{f}(\vec{k}) = \int_V d^d x \; f(\vec{x}) \, e^{-i\vec{k}\vec{x}} \tag{1.A.1}$$

This implies

$$f(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} \tilde{f}(\vec{k}) \, e^{i\vec{k}\vec{x}} \tag{1.A.2}$$

where the sum runs over all $\vec{k}$ with components k_i satisfying $k_i = 2m\pi/L$ with m an integer and L the linear size of the volume V.

Infinite volume

In the large V limit these equations become

$$\begin{aligned} \tilde{f}(\vec{k}) &= \int_V d^d x \; f(\vec{x}) \, e^{-i\vec{k}\vec{x}} && (1.A.3) \\ \tilde{f}(\vec{x}) &= \int_V \frac{d^d k}{(2\pi)^d} \; f(\vec{k}) \, e^{i\vec{k}\vec{x}} && (1.A.4) \end{aligned}$$

On a lattice

Take now a function $f_{\vec{x}}$ defined on a lattice. Its Fourier transform is

$$\tilde{f}(\vec{k}) = \sum_{\vec{x}} f_{\vec{x}}\ e^{-i\vec{k}\vec{x}} \tag{1.A.5}$$

with the inverse

$$f_{\vec{x}} = \int \frac{d^d k}{2\pi} f(\vec{k})\ e^{i\vec{k}\vec{x}} \tag{1.A.6}$$

and $\int d^dk/(2\pi)^d = \prod_{i=1}^d \int_{-\pi}^{\pi} dk_1/(2\pi) \cdots \int_{-\pi}^{\pi} dk_d/(2\pi)$ with these integrals running over the *first Brillouin zone* in reciprocal space.

Time domain

The convention for the Fourier transform is the time-domains is

$$f(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{-i\omega\tau}\ f(\omega)\ , \tag{1.A.7}$$

$$f(\omega) = \int_{-\infty}^{\infty} d\tau\ e^{+i\omega\tau}\ f(\tau)\ . \tag{1.A.8}$$

Properties

The Fourier transform of a real function $f(\vec{x})$ satisfies $\tilde{f}^*(\vec{k}) = \tilde{f}(-\vec{k})$.
The Fourier transform of the theta function reads

$$\theta(\omega) = i\mathrm{vp}\frac{1}{\omega} + \pi\delta(\omega)\ . \tag{1.A.9}$$

The convolution is

$$[f \cdot g](\omega) = f \otimes g(\omega) \equiv \int \frac{d\omega'}{2\pi} f(\omega')g(\omega - \omega')\ . \tag{1.A.10}$$

1.A.2 Stirling

Stirling formula for the factorial of a large number reads:

$$\ln N! \sim N \ln N - \ln N\ , \qquad \text{for} \qquad N \gg 1\ . \tag{1.A.1}$$

1.A.3 Moments

Introducing a source h that couples linearly to a random variable x one easily computes all moments of its distribution $p(x)$. Indeed,

$$\langle\, x^k \,\rangle = \frac{\partial^k}{\partial h^k} \int dx\ p(x) e^{hx} \Big|_{h=0}\ . \tag{1.A.1}$$

1.A.4 Gaussian integrals

The Gaussian integral is

$$I_1 \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1 \; . \tag{1.A.1}$$

It is the normalization condition of the Gaussian probability density written in the *normal form*. One has

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, x &= \mu \; , \\ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, x^2 &= \sigma^2 \; . \end{aligned} \tag{1.A.2}$$

From (1.A.1) one has

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}+\frac{\mu x}{\sigma^2}} = e^{\frac{\sigma^2\mu^2}{2}} \; . \tag{1.A.3}$$

The generalization to N variables

$$I_N \equiv \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}\vec{x}^t A \vec{x} + \vec{x}^t \vec{\mu}} \tag{1.A.4}$$

with

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{pmatrix} \; , \qquad \vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_N \end{pmatrix} \; , \qquad A = \begin{pmatrix} A_{11} & \dots & A_{1N} \\ A_{21} & \dots & A_{2N} \\ & \dots & \\ A_{N1} & \dots & A_{NN} \end{pmatrix} \; ,$$

and

$$-\frac{1}{2}\vec{x}^t A \vec{x} + \vec{x}^t \vec{\mu} \tag{1.A.5}$$

is the most generic quadratic form. Note that A plays here the role σ^{-2} in the single variable case. One can keep the symmetric part $(A + A^t)/2$ of the matrix A only since the antisymmetric part $(A - A^t)/2$ yields a vanishing contribution once multiplied by the vectors $\vec{x}$ and its transposed. Focusing now on a symmetric matrix, $A^t = A$, that we still call A we can ensure that it is diagonalizable and all its eigenvalues are positive definite, $\lambda_i > 0$. One can then define $A^{1/2}$ as the matrix such that $A^{1/2}A^{1/2} = A$ and its eigenvalues are the square root of the ones of A. Writing $\vec{x}^t A \vec{x} = (\vec{x}^t A^{1/2})(A^{1/2}\vec{x}) = \vec{y}\vec{y}$, the integral I_N in (1.A.4) becomes

$$I_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dy_i J e^{-\frac{1}{2}\vec{y}^t\vec{y} + \vec{y}^t (A^{-1/2}\mu)} \tag{1.A.6}$$

where $J = \det(A^{1/2})^{-1} = (\det A)^{-1/2}$ is the Jacobian of the change of variables. Calling $\vec{\mu}'$ the last factor one has the product of N integrals of the type I_1; thus

$$I_N = (2\pi)^{N/2}(\det A)^{-1/2} e^{\frac{1}{2}\vec{\mu}^t A^{-1}\vec{\mu}} \tag{1.A.7}$$

Finally, the functional Gaussian integral is the continuum limit of the N-dimensional Gaussian integral

$$\vec{x} \equiv (x_1, \dots, x_N) \to \phi(\vec{x}) \tag{1.A.8}$$

and

$$I = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\int d^dx d^dy \; \phi(\vec{x})A(\vec{x},\vec{y})\phi(\vec{y}) + \int d^dx \, \phi(\vec{x})\mu(\vec{x})} \; . \tag{1.A.9}$$

The sum runs over all functions $\phi(\vec{x})$ with the spatial point $\vec{x}$ living in d dimensions. The first and the second term in the exponential are quadratic and linear in the field, respectively. In analogy with the I_N case the result of the *path integral* is

$$I \propto e^{\frac{1}{2}\int d^dx d^dy \; \mu(\vec{x}) \, A^{-1}(\vec{x},\vec{y}) \, \mu(\vec{y})} \tag{1.A.10}$$

where we ignore the proportionality constant. Indeed, this one depends on the definition of the path-integral measure $\mathcal{D}\phi$. Usually, the actual value of this constant is not important since it does not depend on the relevant parameters of the theory. The inverse A^{-1} is defined by

$$\int d^dy \; A^{-1}(\vec{x},\vec{y})A(\vec{y},\vec{z}) = \delta(\vec{x}-\vec{z}) \; . \tag{1.A.11}$$

1.A.5 Wick's theorem

Take a Gaussian variable x with mean $\langle x \rangle = \mu$ and variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$. Its pdf is

$$p(x) = (2\pi\sigma^2)^{-1/2} \, e^{-(x-\mu)^2/(2\sigma^2)} \; . \tag{1.A.1}$$

All moments $\langle x^k \rangle$ can be computed with (1.A.1). One finds

$$\langle e^{hx} \rangle = e^{\frac{h^2\sigma^2}{2} + h\mu} \tag{1.A.2}$$

and then

$$\langle x^k \rangle = \frac{\partial^k}{\partial h^k} \, e^{\frac{h^2\sigma^2}{2} + \mu h}\bigg|_{h=0} \tag{1.A.3}$$

from where

$$\begin{aligned} &\langle x \rangle = \mu \; , && \langle x^2 \rangle = \sigma^2 + \mu^2 \; , \\ &\langle x^3 \rangle = 3\sigma^2\mu + \mu^3 \; , && \langle x^4 \rangle = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 \end{aligned}$$

etc. One recognizes the structure of Wick's theorem: given k factors x one organises them in pairs leaving the averages μ aside. The simplest way of seeing Wick's theorem in action is by drawing examples.

The generalization to N Gaussian variables is immediate. Equation (1.A.2) becomes

$$\langle\, e^{\vec{h}\,\vec{x}}\,\rangle = e^{\frac{1}{2}\vec{h}A^{-1}\vec{h}+\vec{h}\vec{\mu}} \tag{1.A.4}$$

and the generalization of (1.A.3) leads to

$$\langle\, x_i \,\rangle = \mu_i \;, \qquad\qquad \langle\, x_i x_j \,\rangle = {A^{-1}}_{ij} + \mu_i\mu_j \;, \tag{1.A.5}$$

etc. In other words, whereever there is σ^2 in the single variable case we replace it by ${A^{-1}}_{ij}$ with the corresponding indices.

The generalization to a field theory necessitates the introduction of functional derivatives that we describe below. For completeness we present the result for a scalar field in d dimensions here

$$\langle\, \phi(\vec{x}) \,\rangle = \mu(\vec{x}) \;, \qquad\qquad \langle\, \phi(\vec{x})\phi(\vec{y}) \,\rangle = A^{-1}(\vec{x},\vec{y}) + \mu(\vec{x})\mu(\vec{y}) \;, \tag{1.A.6}$$

etc.

1.A.6 Functional analysis

A functional $F[h]$ is a function of a function $h:\ \vec{x} \to h(\vec{x})$. The variation of a functional F when one changes the function h by an infinitesimal amount allows one to define the functional derivative. More precisely, one defines $\delta F \equiv F[h+\delta h] - F[h]$ and one tries to write this as $\delta F = \int d^dx\ \alpha(\vec{x})\delta h(\vec{x}) + \frac{1}{2}\int d^dx d^dy\ \beta(\vec{x},\vec{y})\,\delta h(\vec{x})\delta h(\vec{y}) + \dots$ and one defines the functional derivative of F with respect to h evaluated at the spatial point $\vec{x}$ as

$$\frac{\delta F}{\delta h(\vec{x})} = \alpha(\vec{x}) \;, \qquad \frac{\delta^2 F}{\delta h(\vec{x})\delta h(\vec{y})} = \beta(\vec{x},\vec{y}) \tag{1.A.1}$$

etc. All usual properties of partial derivatives apply.

1.A.7 The saddle-point method

Imagine one has to compute the following integral

$$I \equiv \int_a^b dx\; e^{-Nf(x)} \;, \tag{1.A.1}$$

with $f(x)$ a positive definite function in the interval $[a,b]$, in the limit $N \to \infty$. It is clear that due to the rapid exponential decay of the integrand, the integral will be dominated by the minimum of the function f in the interval. Assuming there is only one absolute minimum, x_0, one then Taylor expands $f(x)$ upto second order

$$f(x) \sim f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 \tag{1.A.2}$$

and obtains

$$I \sim e^{-Nf(x_0)} \int_a^b dx\; e^{-N\frac{1}{2}f''(x_0)(x-x_0)^2} = e^{-Nf(x_0)}[Nf''(x_0)]^{-1/2} \int_{y_a}^{y_b} dy\; e^{-\frac{1}{2}(y-y_0)^2} \;, \quad (1.A.3)$$

with $y_0 \equiv \sqrt{Nf''(x_0)}x_0$ and similarly for y_a and y_b. The Gaussian integral is just an error function that one can find in Tables.

This argument can be extended to multidimensional integrals, cases in which there is no absolute minimum within the integration interval, cases in which the function f is not positive definite, etc.

1.A.8 Jensen's inequality

Jensen's inequality relates the value of a convex function of an integral to the integral of the convex function. In its simplest form the inequality states that the convex transformation of a mean is less than or equal to the mean applied after convex transformation; it is a simple corollary that the opposite is true of concave transformations.

In probability theory, the Jensen's inequality implies that, for x a random variable and f a convex function, then

$$f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)] \quad (1.A.1)$$

where $\mathbb{E}[\dots]$ is the expectation value of ...

We recall that a function is convex function iff $\forall x_1, x_2$ and $t \in [0,1]$:

$$f(tx_1 + (1-t)x-2) \leq tf(x_1) + (1-t)f(x_2) \;. \quad (1.A.2)$$

1.A.9 The central limit theorem

In probability theory, the central limit theorem (CLT) establishes that, in most situations, when independent random variables are added, their properly normalized sum tends toward a normal (Gaussian) distribution (informally a "bell curve") even if the original variables themselves are not normally distributed. More precisely, for x_i *i.i.d.* with average μ and variance σ^2,

$$X = \frac{1}{N}\sum_i x_i \quad (1.A.1)$$

is a Gaussian distributed with average $[X] = \mu$ and variance $[(X-[X])^2] = \sigma^2/N$.

They all express the fact that a sum of many independent and identically distributed (*i.i.d.*) random variables, or alternatively, random variables with specific types of dependence, will tend to be distributed according to one of a small set of attractor distributions. When the variance of the i.i.d. variables is finite, the attractor distribution is the normal distribution. In contrast, the sum of a number of *i.i.d.* random variables with power law tail distributions decreasing as $|x|^{-\alpha-1}$ where $0 < \alpha < 2$ (and therefore having infinite variance) will tend to an alpha-stable distribution with stability parameter (or index of stability) of α as the number of variables grows.

1.B Classical results in statistical physics

We recall here some classical results in statistical physics.

1.B.1 High temperature expansion

The partition function of the Ising ferromagnet reads

$$Z = \sum_{s_i=\pm 1} e^{\beta J \sum_{\langle ij \rangle} s_i s_j} = \sum_{s_i=\pm 1} \prod_{\langle ij \rangle} e^{\beta J s_i s_j} \tag{1.B.2}$$

Using the identity $e^{\beta J s_i s_j} = a(1+b s_i s_j)$ with $a = \cosh(\beta J)$ and $b = \tanh(\beta J)$ and the fact that b is order β, an expansion if powers of b can be established. The average of products of the spins s's that remains can be non-zero only if each spin appears an even number of times s. The expansion can then be represented as graphs on the lattice, a representation that makes the enumeration of terms easier.

1.B.2 Lee-Yang theorem

The Lee-Yang theorem states that if partition functions of models with ferromagnetic interactions are considered as functions of an external field, then all zeros are purely imaginary (or on the unit circle after a change of variable) [79].

1.B.3 Critical behaviour

Second order phase transitions are characterised by the diverge of the correlation length. In normal conditions, far from the critical point, the correlation function of the fluctuations of an observable decay as an exponential of the distance between the measuring points:

$$C(\vec{r}) \equiv \langle [O(\vec{r}+\vec{r}') - \langle (O(\vec{r}+\vec{r}') \rangle][O(\vec{r}') - \langle (O(\vec{r}') \rangle) \rangle] \rangle \simeq e^{-r/\xi} \ . \tag{1.B.3}$$

ξ is the correlation length that diverges at the critical point as

$$\xi \simeq |T - T_c|^{-\nu} \tag{1.B.4}$$

with ν a critical exponent. A power-law singularities in the length scales leads to power-law singularities in observable quantities. We summarise in Table 2 all the critical exponents associated to various quantities in a second order phase transition. The values of the critical exponents generally do not depend on the microscopic details but only on the space dimensionality and the symmetries of the system under consideration.

The collection of all these power laws characterizes the critical point and is usually called the critical behavior.

	exponent	definition	conditions
Specific heat	α	$c \propto \lvert u\rvert^{-\alpha}$	$u \to 0,\ h = 0$
Order parameter	β	$m \propto (-u)^{\beta}$	$u \to 0-,\ \ h = 0$
Susceptibility	γ	$\chi \propto \lvert u\rvert^{-\gamma}$	$u \to 0,\ h = 0$
Critical isotherm	δ	$h \propto \lvert m\rvert^{\delta}\mathrm{sign}(m)$	$h \to 0,\ u = 0$
Correlation length	ν	$\xi \propto \lvert r\rvert^{-\nu}$	$r \to 0,\ h = 0$
Correlation function	η	$G(\vec{r}) \propto \lvert\vec{r}\rvert^{-d+2-\eta}$	$r = 0,\ h = 0$

Table 2: Definitions of the commonly used critical exponents. m is the order parameter, *e.g.* the magnetization, h is an external conjugate field, *e.g.* a magnetic field, u denotes the distance from the critical point, *e.g.* $|T - T_c|$, and d is the space dimensionality.

Whether fluctuations influence the critical behavior depends on the space dimensionality d. In general, fluctuations become less important with increasing dimensionality.

In sufficiently low dimensions, *i.e.* below the lower critical dimension d_l, fluctuations are so strong that they completely destroy the ordered phase at all (nonzero) temperatures and there is no phase transition. Between d_l and the upper critical dimension d_u, order at low temperatures is possible, there is a phase transition, and the critical exponents are influenced by fluctuations (and depend on d). Finally, for $d > d_u$, fluctuations are unimportant for the critical behavior, and this is well described by mean-field theory. The exponents become independent of d and take their mean-field values. For example, for Ising ferromagnets, $d_l = 1$ and $d_u = 4$, for Heisenberg ferromagnets $d_l = 2$ and $d_u = 4$.

1.C Derivations of the TAP equations

The TAP equations can be derived in different ways. We present two of them here.

1.C.1 Simplest derivation

The simplest one is to propose that the joint pdf of all Ising spins is factorized

$$P[\{s_i\}] = \prod_{k=1}^{N} p_k(s_k) = \prod_{k=1}^{N} \left(\frac{1+m_k}{2} \delta_{s_k,1} + \frac{1-m_k}{2} \delta_{s_k,-1} \right) . \tag{1.C.5}$$

One can readily check that the individual $p_k(s_k)$ are normalized and that

$$m_k = \langle s_k \rangle \tag{1.C.6}$$

where the average is taken over the probability defined above. The m_k are the local order parameters. Note that since the spins are bi-valued, the individual probabilities, once the joint one factorized, is forced to take the form $A_k^+\delta_{s_k,1} + A_k^-\delta_{s_k,-1}$ with $\delta_{a,b}$ the Kronecker delta.

Once this done, the order parameter free-energy is

$$F[\{m_i\}] = U[\{m_i\}] - TS[\{m_i\}] \tag{1.C.7}$$

with

$$U[\{m_i\}] = \langle H \rangle \qquad\qquad S[\{m_i\}] = -k_B \langle \ln P[\{s_i\}] \rangle \; . \tag{1.C.8}$$

These two contributions are easy to calculate. For the SK model under an external field $h_{\rm ext}$,

$$U[\{m_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \sum_i h_i^{\rm ext} m_i \tag{1.C.9}$$

and

$$S[\{m_i\}] = -k_B \sum_i \left[\frac{1+m_i}{2} \ln\left(\frac{1+m_i}{2}\right) + \frac{1-m_i}{2}\ln\left(\frac{1-m_i}{2}\right)\right] \; . \tag{1.C.10}$$

The extremization of the resulting free-energy with respect to the $\{m_i\}$ yields the so-called naive TAP equations

$$m_i = \tanh\left(\beta \sum_{j(\neq i)} J_{ij} m_j + \beta h_i^{\rm ext}\right) = \tanh\left(\beta h_i^{\rm loc}\right) \; . \tag{1.C.11}$$

The analysis of the self-effect of the magnetization of site i on itself yields the Onsager reaction term that we discussed in the main text and corrects this equation.

The choice of the individual probabilities p_k, with the prefactors $A_k^\pm$, may seem bizarre at first sight. However, once the equation (1.C.11) derived, one can simply verify that $p_i(s_i)$ takes the familiar form $p_i(s_i) = e^{\beta h_i^{\rm loc} s_i}/\mathcal{N}$ with $\mathcal{N} = \cosh(\beta h_i^{\rm loc})$ a normalization. The proof goes this way:

$$\begin{aligned} A_i^\pm &= \frac{1\pm m_i}{2} = \frac{1}{2}\left[1 \pm \tanh\left(\beta h_i^{\rm loc}\right)\right] = \frac{\cosh(\beta h_i^{\rm loc}) \pm \sinh(\beta h_i^{\rm loc})}{2\cosh(\beta h_i^{\rm loc})} \\ &= \frac{e^{\beta h_i^{\rm loc}} + e^{-\beta h_i^{\rm loc}} \pm e^{\beta h_i^{\rm loc}} \mp e^{-\beta h_i^{\rm loc}}}{2\cosh(\beta h_i^{\rm loc})} \\ &= \begin{cases} \dfrac{e^{\beta h_i^{\rm loc}}}{\cosh(\beta h_i^{\rm loc})} \\ \dfrac{e^{-\beta h_i^{\rm loc}}}{\cosh(\beta h_i^{\rm loc})} \end{cases} \end{aligned} \tag{1.C.12}$$

Therefore,

$$p_i(s_i = \pm 1) = A_i^\pm \qquad \Leftrightarrow \qquad p_i(s_i = \pm 1) = \frac{e^{h_i^{\rm loc} s_i}}{\cosh \beta h_i^{\rm loc}} \; . \tag{1.C.13}$$

Note that we do not need to make the form of the local field explicit to show eq. (1.C.13).

1.C.2 Cavity method

This method allows one to derive the full TAP equations, including the Onsager reaction term. The idea is the following. Take a system of N spins, with Hamiltonian

$$H_N[\{s_i\}] = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j(\neq i)=1}^{N} J_{ij}s_is_j \tag{1.C.14}$$

in canonical equilibrium at temperature β, hence, with a probability distribution

$$P_N[\{s_i\}] = \frac{1}{\mathcal{Z}_N}e^{-\beta H_N[\{s_i\}]} \; . \tag{1.C.15}$$

Add a spin s_0 to it, connect it to all other N spins in the sample, and consider the ensemble also in equilibrium at the same temperature:

$$H_{N+1}[\{s_i\},s_0] = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j(\neq i)=1}^{N} J_{ij}s_is_j - \sum_{i=1}^{N} J_{i0}s_is_0 = H_N[\{s_i\}] - h_0s_0 \; , \tag{1.C.16}$$

$$P_{N+1}[\{s_i\},s_0] = \frac{1}{\mathcal{Z}_{N+1}}e^{-\beta H_{N+1}[\{s_i\},s_0]} \; . \tag{1.C.17}$$

s_0 is the *cavity* spin and we defined the local field acting on s_0 due to the N-spin system:

$$h_0 \equiv \sum_{i=1}^{N} J_{i0}s_i \tag{1.C.18}$$

Averaged properties at the cavity site

We focus now on the cavity site, the joint probability distribution of the spin s_0 and local field h_0 is

$$p_{N+1}(s_0,h_0) = \sum_{\{s_i=\pm 1\}} \delta\left(h_0 - \sum_{i=1}^{N} J_{i0}s_i\right) \frac{1}{\mathcal{Z}_{N+1}} \, e^{-\beta H_{N+1}[\{s_i\},s_0]} \; . \tag{1.C.19}$$

Let us introduce

$$p_N(h_0) = \sum_{\{s_i=\pm 1\}} \delta\left(h_0 - \sum_{i=1}^{N} J_{i0}s_i\right) \frac{1}{\mathcal{Z}_N} \, e^{-\beta H_N[\{s_i\}]} \; . \tag{1.C.20}$$

Then, it is immediate to show

$$p_{N+1}(s_0,h_0) = \frac{\mathcal{Z}_N}{\mathcal{Z}_{N+1}} \, e^{h_0s_0} \, p_N(h_0) \tag{1.C.21}$$

Now, the thermal average of the cavity spin and the local field in the $N+1$ spin system are

$$\langle s_0 \rangle_{N+1} = \sum_{s_0=\pm 1} \int dh_0 \, s_0 \, p_{N+1}(s_0, h_0) = \frac{\langle \sinh(\beta h_0) \rangle_N}{\langle \cosh(\beta h_0) \rangle_N} \,, \tag{1.C.22}$$

$$\langle h_0 \rangle_{N+1} = \sum_{s_0=\pm 1} \int dh_0 \, h_0 \, p_{N+1}(s_0, h_0) = \frac{\langle h_0 \cosh(\beta h_0) \rangle_N}{\langle \cosh(\beta h_0) \rangle_N} \,, \tag{1.C.23}$$

where $\langle \dots \rangle_N$ represents a thermal average with respect to the N spin system, that is, with respect to $p_N(h_0)$. In order to go further we need to characterize this distribution.

Probability distribution of the cavity field generated by the N spin system

The average and variance of the local cavity field is

$$\langle h_0 \rangle_N = \sum_{i=1}^{N} J_{0i} \langle s_j \rangle_N \tag{1.C.24}$$

$$\langle (\delta h_0)^2 \rangle_N = \sum_{i=1}^{N} \sum_{j=1}^{N} J_{0i} J_{0j} \langle \delta s_i \delta s_j \rangle_N \tag{1.C.25}$$

with $\delta s_i = s_i - \langle s_i \rangle_N$ and $\delta h_0 = h_0 - \langle h_0 \rangle_N$.

We now estimate the sums using the order of magnitud of the interaction strengths. Working with the SK model, $J_{ij} = O(1/\sqrt{N})$ and we expect h_0 to be $O(1)$. The cavity spin s_0 has *no effect* on $\langle \delta s_i \delta s_j \rangle_N$, since the latter is calculated for the N spin system. Moreover, $\langle \delta s_i \delta s_j \rangle_N$ is independent of $J_{0i} J_{0j}$. The double sum runs over $N(N-1)$ terms for $i \neq j$ and N terms for $i = j$. When $i \neq j$, the two factors $\langle \delta s_i \delta s_j \rangle_N$ and $J_{0i} J_{0j}$ are independent and do not have a definite sign. On top, one can expect $\langle \delta s_i \delta s_j \rangle_N$ to be $O(1/\sqrt{N})$. The sum is then of order $\sqrt{N}/N = 1/\sqrt{N}$. Instead, the terms with $i = j$ are

$$\sum_{i=1}^{N} J_{0i}^2 \langle (\delta s_i)^2 \rangle_N \sim \frac{J^2}{N} \sum_{i=1}^{N} (1 - \langle s_i \rangle_N^2) \,. \tag{1.C.26}$$

In the replacement $J_{0i} \mapsto J/\sqrt{N}$ we used a self-averaging argument.

The next step is to assume that, given than h_0 is the result of the sum over many terms, it has Gaussian statistics

$$p_N(h_0) = \frac{1}{\sqrt{2\pi \langle (\delta h_0)^2 \rangle_N}} \, e^{-\frac{1}{2\langle (\delta h_0)^2 \rangle_N} (h_0 - \langle h_0 \rangle_N)^2} \,. \tag{1.C.27}$$

The TAP equations

We are now in a position to obtain the TAP equations. We go back to Eqs. (1.C.22) and (1.C.23). Using the Gaussian pdf of h_0 the averages can be calculated and

$$\langle s_0\rangle_{N+1} = \tanh(\beta\langle h_0\rangle_N) \tag{1.C.28}$$
$$\langle h_0\rangle_{N+1} = \langle h_0\rangle_N + \langle(\delta h_0)^2\rangle_N\langle s_0\rangle_{N+1} \tag{1.C.29}$$

and after substitution of the second line in the first line

$$\langle s_0\rangle_{N+1} = \tanh\left(\beta\sum_{j(\neq i)} J_{ij}\langle s_j\rangle_{N+1} - \frac{\beta^2}{N}\sum_{j(\neq i)} J_{ij}^2\,(1-\langle s_j\rangle_N^2)\,\langle s_0\rangle_{N+1}\right) \tag{1.C.30}$$

If we now take the large N limit and claim that N and $N+1$ systems should give the same averages,

$$m_i = \tanh\left(\beta\sum_{j(\neq i)} J_{ij}m_j - \frac{\beta^2}{N}\sum_{j(\neq i)} J_{ij}^2(1-m_j^2)m_i\right) \tag{1.C.31}$$

the TAP equations with the Onsager reaction term. The "delay" in the Onsager term, though, is at the origin of Bolthausen's argument to iterate this term in a previous time step to search for solutions of these equations with iterative methods.

1.D Solvable disordered models

There are few solvable disordered models. Even if rather far from describing realistic systems in detail, these models are of great help to test several features of disordered systems that we expect to find in more realistic cases.

In this Section we describe the static properties of a family of solvable models that include the spherical ferromagnet and spin-glass. These models illustrate a mechanism for slow relaxation that is due to the existence of saddles and flat directions in phase space.

1.D.1 Spherical spin models with two-body interactions

Spherical spin models are not very realistic but have the advantage of rendering the models easy to solve analytically. In the spherical approximation the Ising constraint is relaxed and the individual spins are taken to be unbounded continuous variables $-\infty \leq s_i \leq \infty$ subject to the global constraint $\sum_{i=1}^N s_i^2 = N$ that is imposed on average. One can then represent the configuration of the system with an N-dimensional vector, $\vec{s} = \{s_1, \ldots, s_N\}$, pointing on an N-dimensional sphere with radius $\sqrt{N}$. The spherical model with generic two-body interactions in a local magnetic field is defined by the quadratic Hamiltonian

$$H_J = -\frac{1}{2}\sum_{i\neq j} J_{ij}s_is_j - \sum_i h_is_i\ . \tag{1.D.32}$$

The first sum is over all distinct pairs of spins and the interactions J_{ij} are symmetric but otherwise arbitrary.

The spherical constraint is enforced on average by an adding an extra term to the energy

$$H_J \to H_J + \frac{z}{2}\left(\sum_{i=1}^{N} s_i^2 - N\right) \tag{1.D.33}$$

with z a complex *Lagrange multiplier*. In this way, the constrained is enforced on average and not strictly, as in the partition sum one sums over all configurations of the spins and not only over the ones on the sphere.

We shall see below that the *density of eigenvalues of the interaction matrix* J_{ij} determines the phase transition and most of the static and dynamic properties of the spherical models. All density of states with a *finite support*, $\rho(\lambda_\mu) \neq 0$ in $[\lambda_{\min}, \lambda_{\max}]$ lead to similar static and dynamic behaviours while the ones with *long tails* yield a rather different phenomenology.

One can now distinguish ordered and disordered spherical spin models. The *spherical ferromagnet* introduced by Berlin and Kac [25] is such that the spins lie on the vertices of a cubic d dimensional lattice with lattice spacing a that one usually sets to one. The interactions are ferromagnetic nearest-neighbour couplings with strength, say, unity. In the limit $N \to \infty$ the density of eigenvalues λ_μ of the corresponding interaction matrix J_{ij} is

$$\rho(\lambda_\mu) = \pi^{-1} \int_0^\infty dy\ \cos(\lambda_\mu y)\, [J_o(2y)]^d\ \ \theta(2d - |\lambda_\mu|)\ , \tag{1.D.34}$$

where $J_o(y)$ is the zero-th order Bessel function. The definition of spherical antiferromagnets is slightly more complicated but is is also possible.

In the *disordered case* the interactions J_{ij} are taken from a probability distribution. Since one is usually interested in describing the spin-glass state its average, $[J_{ij}]$, is set to zero. The scaling of its variance, $[J_{ij}^2]$, is chosen in such a way to have a sensible thermodynamic limit.

If the model is *fully connected*, meaning that all entries J_{ij} are typically different from zero, the variance scales as $J^2/(2N)$. One such model is the one with a Gaussian distribution of exchanges and it was introduced by Kosterlitz, Thouless and Jones [26] as a spherical spin-glass (although, as we will see later, it is not really a spin-glass). When $N \to \infty$ the eigenvalues of a typical member of this Gaussian orthogonal ensemble, that we call λ_μ, with $\mu = 1, \dots, N$, are distributed according to the Wigner semi-circle law [55][6],

$$\rho(\lambda_\mu) = \frac{1}{2\pi J} \sqrt{4J^2 - \lambda_\mu^2}\, \theta(2J - |\lambda_\mu|)\ . \tag{1.D.35}$$

In the following we measure temperature in units of the interaction strength J and thus we set $J = 1$.

[6] The spectrum of a large symmetric random matrix can be evaluated with several methods, including the replica trick, as explained in [84].

A *dilute* system in which each spin interacts with only a finite fraction of other ones in the sample is modelled with

$$P(J_{ij}) = (1-p/N)\ \delta(J_{ij}) + p/N\, \rho(J_{ij})\ . \tag{1.D.36}$$

One can visualize this model as one with the spins occupying the vertices of a random graph with average connectivity p. When $p \to N$ one recovers the complete graph and the fully-connected case. If $\rho(J_{ij})$ has support on positive values of J_{ij} only one has a dilute random ferromagnet. If $\rho(J_{ij})$ is Gaussian centred in zero one has a dilute spin-glass. In this case the density of eigenvalues has a symmetric central band in $[-\lambda_c(p), \lambda_c(p)]$, a crossover extending beyond $|\lambda_c(p)|$ that is not known in detail, and two tails that vanish as $\rho(\lambda_\mu) \sim \exp[-p\lambda_\mu^2 \ln \lambda_\mu^2]$ when $\lambda_\mu \to \pm\infty$. The tails are due to large fluctuations of the local connectivity. For $k \gg 2p$ a site with k neighbors gives rise to an eigenvalue $\lambda_\mu \sim \sqrt{k/p}$ with a localized eigenvector $\vec{v}_\mu$ on it. When $p \to N \to \infty$, $\lambda_c(p) \to \lambda_{\max} = \max\{\lambda_1, \dots, \lambda_N\}$ and the tails disappear [85].

The magnetic field h_i might be quenched and random, uniform and stationary, or time-dependent.

The potential energy landscape

Let us label the eigenvalues of J_{ij} in such a way that they are ordered: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. We call their associated eigenvectors $\pm\vec{v}_\mu$ with $\mu = 1, \dots, N$. (We take orthonormal eigenvectors such that $v_\mu^2 = 1$.) In the absence of a magnetic field, all eigenstates of the interaction matrix are stationary points of the energy hyper-surface,

$$\left.\frac{\partial H_J}{\partial s_i}\right|_{\vec{s}^*} = -\sum_{j(\neq i)}^{N} J_{ij} s_j + z s_i|_{\vec{s}^*} = 0 \quad \forall i\ , \qquad \Rightarrow \qquad \vec{s}^* = \pm\sqrt{N}\vec{v}_\mu \ \text{ and } \ z^* = \lambda_\mu \ \ \forall \mu\ .$$

These stationary points are the metastable states in the models and their number is linear in N, the number of spins.

The Hessian of the potential energy surface on each stationary point is

$$\left.\frac{\partial H_J}{\partial s_i \partial s_j}\right|_{\vec{s}^*} = -J_{ij} + z\delta_{ij}|_{\vec{s}^*, z^*} = -J_{ij} + \lambda_\mu \delta_{ij}\ . \tag{1.D.37}$$

This matrix can be easily diagonalized, one finds $D_{\nu\eta} = (-\lambda_\nu + \lambda_\mu)\delta_{\nu\eta}$. Thus, on the stationary point, $\vec{s}^* = \pm\vec{v}_\mu$, the Hessian has one vanishing eigenvalue (when $\nu = \mu$), $\mu - 1$ positive eigenvalues (when $\nu < \mu$), and $N - \mu$ negative eigenvalues (when $\nu > \mu$). Positive (negative) eigenvalues of the Hessian indicate stable (unstable) directions. This implies that each saddle point labeled by μ has one marginally stable direction, $\mu - 1$ stable directions and $N - \mu$ unstable directions. (In other words, the number of stable directions plus the marginally stable one is given by the index μ labelling the eigenvalue associated to the stationary state.) In conclusion, there are two maxima, $\vec{s}^* = \pm\sqrt{N}\vec{v}_1$, in general two saddles $\vec{s}^* = \pm\sqrt{N}\vec{v}_I$ with $I = \mu - 1$ stable directions and $N - I$ unstable ones, with I running with μ as $I = \mu - 1$ and $\mu = 2, \dots, N$ and finally two (marginally stable) minima, $\vec{s}^* = \pm\sqrt{N}\vec{v}_N$. In the large N limit the density of eigenvalues of the Hessian at each metastable state μ is a translated semi circle law.

The energy of a generic configuration under no applied field is

$$H_J = -\frac{1}{2}\sum_{\mu}(\lambda_\mu - z)s_\mu^2 - \frac{z}{2}N \; . \tag{1.D.38}$$

The zero-temperature energy-density of each stationary point is

$$H_J^* = -\frac{1}{2}(\lambda_\mu - z^*)v_\mu^2 - \frac{z^*}{2}N = -\frac{1}{2}\lambda_\mu N \; . \tag{1.D.39}$$

The energy difference between the absolute minimum and the lowest saddle is $\Delta H_J = -(\lambda_{N-1} - \lambda_N)\, N/2 > 0$ depends on the distribution of eigenvalues.

A magnetic field reduces the number of stationary points from a macroscopic number to just two. Indeed, the stationary state equation now reads

$$\left.\frac{\partial H_J}{\partial s_i}\right|_{\vec{s}^*} = -\sum_{j(\neq i)}^{N} J_{ij}s_j + zs_i - h_i|_{\vec{s}^*} = 0 \; , \; \forall i \; , \qquad \Rightarrow \qquad s_i^* = (z^* - J)^{-1}_{ij}h_j$$

and z^* is fixed by imposing the spherical constraint on $\vec{s}^*$. One then finds two solutions for the Lagrange multiplier that lie outside the interval of variation of the eigenvalues of the J_{ij} matrix: $|z^*| > \lambda_N$. The stability analysis shows that the stationary points are just one fully stable minimum and a fully unstable maximum. The elimination of the saddle-points has important consequences on the dynamics of the system.

The free-energy density

The partition function reads

$$Z_J = \prod_{i=1}^{N}\int_{-\infty}^{\infty} ds_i \; e^{\beta/2\sum_{i\neq j}J_{ij}s_is_j + \beta\sum_i h_is_i} \; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \; e^{-\frac{\beta z}{2}\left(\sum_{i=1}^{N}s_i^2 - N\right)}$$

where c is a real constant to be fixed below.

It is convenient to diagonalise the matrix J_{ij} with an orthogonal transformation and write the exponent in terms of the projection of the spin vector $\vec{s}$ on the eigenvectors of J_{ij}, $s_\mu \equiv \vec{s}\cdot\vec{v}_\mu$ This operation can be done for any particular realisation of the interaction matrix. In the disordered case this means that one uses a fixed realisation of the random exchanges. The new variables s_μ are also continuous and unbounded and the partition function can be recast as

$$Z_J = \prod_{\mu=1}^{N}\int_{-\infty}^{\infty} ds_\mu \; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \; e^{\sum_{\mu=1}^{N}\beta(\lambda_\mu - z)s_\mu^2/2 + \beta\sum_{\mu=1}^{N}h_\mu s_\mu + \beta zN/2} \tag{1.D.40}$$

with $h_\mu \equiv \vec{h} \cdot \vec{v}_\mu$ and $\vec{h} = (h_1, \ldots, h_N)$. Assuming that one can exchange the quadratic integration over s_μ with the one over the Lagrange multiplier, and that c is such that the influence of eigenvalues $\lambda_\mu > c$ is negligible, one obtains

$$Z_J = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \; e^{-N\left[-\beta z/2+(2N)^{-1}\sum_\mu \ln[\beta(z-\lambda_\mu)/2]-\beta N^{-1}\sum_\mu (z-\lambda_\mu)^{-1} h_\mu^2\right]} \; . \tag{1.D.41}$$

In the saddle-point approximation the Lagrange multiplier is given by

$$1 = \langle\langle \; k_B T \, (z_{sp} - \lambda_\mu)^{-1} + h_\mu^2 \, (z_{sp} - \lambda_\mu)^{-2} \; \rangle\rangle \tag{1.D.42}$$

where $\langle\langle \ldots \rangle\rangle$ indicate an average over the density of eigenvalues of the J_{ij} matrix, which replaces the sum over them

$$\frac{1}{N} \sum_\mu g(\lambda_\mu) \mapsto \int d\lambda \, \rho(\lambda) \, g(\lambda) \equiv \langle\langle \, g(\lambda) \, \rangle\rangle \tag{1.D.43}$$

in the large N limit. Equation (1.D.42) determines the different phases in the model. We indicate with double brackets the sum over the eigenvalues of the matrix J_{ij} that in the limit $N \to \infty$ can be traded for an integration over its density:

$$\frac{1}{N} \sum_{\mu=1}^{N} g(\lambda_\mu) = \int d\lambda_\mu \, \rho(\lambda_\mu) \, g(\lambda_\mu) = \langle\langle \, g(\lambda_\mu) \, \rangle\rangle \; . \tag{1.D.44}$$

Let us first discuss the problem in the absence of a magnetic field. The integral in eq. (1.D.42) yields

$$k_B T = \frac{1}{2J^2} \left[z_{\rm sp} - \sqrt{z_{\rm sp}^2 - (2J)^2} \right] \qquad \text{for } \; z_{\rm sp} > 2J \; , \tag{1.D.45}$$

and this can be inverted to express

$$z_{\rm sp} = k_B T + \frac{J^2}{k_B T} \; . \tag{1.D.46}$$

This is the high temperature solution which can be smoothly continued to lower temperatures until the critical value,

$$k_B T_c = J \tag{1.D.47}$$

where $z_{\rm sp}$ reaches $2J = \lambda_{\rm max}$, the value of the maximum eigenvalue of the matrix J_{ij}. $z_{\rm sp}$ sticks to this value for all $T < T_c$:

$$z_{sp} = \lambda_{\rm max} = \lambda_N = 2J = 2k_B T_c \qquad T \leq T_c \; . \tag{1.D.48}$$

If one now checks whether the spherical constraint is satisfied by these saddle-point Lagrange multiplier values, one verifies that it is in the high temperature phase, but it is not in the low temperature phase, where

$$\frac{1}{N}\sum_{\mu=1}^{N}\langle s_\mu^2\rangle \mapsto k_BT\,\langle\langle\frac{1}{z_{\rm sp}-\lambda}\rangle\rangle = \frac{T}{T_c}\;. \tag{1.D.49}$$

The way out is to give a macroscopic weight to the variance of the component $s_N = \vec{s}\cdot v_N$ so that

$$\sum_{\mu=1}^{N}\langle s_\mu^2\rangle = \langle s_N^2\rangle + \sum_{\nu=1}^{N-1}\langle s_\nu^2\rangle \mapsto \left(1-\frac{T}{T_c}\right)\,N + \frac{T}{T_c}N = N\;, \tag{1.D.50}$$

where we chose $\langle s_N^2\rangle = (1-T/T_c)N$ and we took the continuous limit in the sum and hence replaced it by $\frac{T}{T_c}N$. One way to achieve $\langle s_N^2\rangle = (1-T/T_c)N$ is to give a projection of the spin vector in the direction of the eigenvector that corresponds to the largest eigenvalue,

$$s_N = m_0\sqrt{N} + \delta s_N = \sqrt{\left(1-\frac{T}{T_c}\right)\,N} + \delta s_N\;, \tag{1.D.51}$$

with $\langle \delta s_N\rangle = 0$. Like this, the thermal average of the projection of the spin vector on each eigenvalue vanishes in the high temperature phase, while reads

$$\langle s_\mu\rangle \;\;=\;\; \left\{\begin{array}{ll} [N(1-T/T_c)]^{\frac{1}{2}} & \qquad \lambda_\mu = \lambda_{\rm max}\;, \\ 0 & \qquad \lambda_\mu < \lambda_{\rm max}\;, \end{array}\right. \tag{1.D.52}$$

below the phase transition (once we have chosen one of the ergodic components with the spontaneous symmetry breaking of the $\vec{s}\to-\vec{s}$ invariance). The configuration *condenses* on the eigenvector associated to the largest eigenvalue of the exchange matrix that carries a weight proportional to $\sqrt{N}$. Going back to the original spin basis, the mean magnetisation per site is zero at all temperatures but the thermal average of the square of the local magnetisation, that defines the Edwards-Anderson parameter, is not when $T<T_c$:

$$\langle m_i^2\rangle \;\;=\;\; 1-T/T_c \qquad \Rightarrow \qquad q_{\rm EA}\equiv [\langle m_i^2\rangle]_J = 1-T/T_c \;\;\text{with}\;\; T_c = J\;. \tag{1.D.53}$$

The order parameter $q_{\rm EA}$ vanishes at T_c and the static transition is of second order.

The condensation phenomenon occurs for any distribution of exchanges with a finite support. If the distribution has long tails the energy density diverges and the behaviour is more subtle.

The disorder averaged free-energy density can also be computed using the replica trick and a replica symmetric *Ansatz*. This *Ansatz* corresponds to an overlap matrix between replicas $Q_{ab} = \delta_{ab} + q_{\rm EA}\epsilon_{ab}$ with $\epsilon_{ab}=1$ for $a\neq b$ and $\epsilon_{ab}=0$ for $a=b$. When $N\to\infty$ the saddle point equations fixing the parameter $q_{\rm EA}$ yield 0 above T_c and a marginally stable solution with $q_{\rm EA} = 1-T/T_c$ and identical physical properties to the ones discussed above below T_c.

The equilibrium energy is given by

$$e_{\rm eq} = \begin{cases} -\frac{J^2}{2T}\left[1-\left(1-\frac{T}{J}\right)^2\right] = \frac{1}{2}(k_BT-\lambda_{\rm max}) & T<T_c \ , \\ -\frac{J^2}{2T} & T>T_c \ . \end{cases} \tag{1.D.54}$$

The entropy diverges at low temperatures as $\ln T$, just as for the classical ideal gas, as usual in classical continuous spin models.

A magnetic field with a component on the largest eigenvalue, $\vec{h}\cdot\vec{v}_{\rm max}\neq 0$, acts as an ordering field and erases the phase transition.

The disordered averaged free-energy density can also be computed using the *replica trick*. When $N\to\infty$ a replica symmetric *Ansatz* yields a marginally stable solution with identical physical properties to the ones discussed above.

1.D.2 The O($\mathcal{N}$) model

In this case the spins are generalised to have $\mathcal{N}$ components and the large $\mathcal{N}$ limit is taken. More precisely, the Hamiltonian is given by

$$H_J = -\sum_{\langle ij\rangle} J_{ij}\vec{s}_i\cdot\vec{s}_j - \sum_i \vec{h}_i\vec{s}_i \tag{1.D.55}$$

where the spins

$$\vec{s}_i = (s_i,\dots,s_i^{\mathcal{N}}) \tag{1.D.56}$$

have $\mathcal{N}$ components and length $\mathcal{N}^{1/2}$

$$\sum_{a=1}^{\mathcal{N}}(s_i^a)^2 = \mathcal{N} \tag{1.D.57}$$

In the ferromagnetic finite d case, this procedure defines the celebrated O($\mathcal{N}$) model, that becomes fully solvable in the large $\mathcal{N}\to\infty$ limit.

The large $\mathcal{N}$ limit is usually taken in the field theoretical (coarse-grained Ginzburg-Landau) representation of the free-energy

$$F[\phi] = \int d^dx \left[\frac{1}{2}(\vec{\nabla}\vec{\phi})^2 + \frac{m_0}{2}(\vec{\phi})^2 + \lambda_0((\vec{\phi})^2)^2\right] \tag{1.D.58}$$

where $\vec{\phi}=(\phi_1,\dots,\phi_{\mathcal{N}})$,

$$\vec{\phi}^2 = \sum_{\alpha=1}^{\mathcal{N}}\phi_\alpha^2 \ , \qquad\qquad (\vec{\nabla}\vec{\phi})^2 = \sum_{a=1}^{d}\sum_{\alpha=1}^{\mathcal{N}}\frac{\partial\phi_\alpha}{\partial x_a}\frac{\partial\phi_\alpha}{\partial x_a} \ , \tag{1.D.59}$$

and the vector position in d dimensions is $\vec{x}=(x_1,\dots,x_d)$. We have not included randomness here. This can be done by including a random potential $V[\vec{\phi}]$, for example. The cases $\mathcal{N}=2$ and $\mathcal{N}=3$ correspond to the XY and Heisenberg models, respectively. The mean field theory for this model yields critical exponents which are independent of $\mathcal{N}$, but the renormalisation group below $d=4$ gives $\mathcal{N}$-dependent results.

In the $\mathcal{N}\to\infty$ limit the model becomes exactly solvable. Note that by counting powers of $\mathcal{N}$ one easily remarks that the last quartic term is of higher order than the two previous ones. This will be cured with a special scaling of the parameter λ_0.

The simplest way to see that this model is solvable is to notice that, by the central limit theorem, the random variable $\phi^2=\sum_{\alpha=1}^{\mathcal{N}}\phi_\alpha^2$ is a sum over a large number of terms, that by symmetry should be identically distributed, and have then a normal (Gaussian) distribution. This means, in particular, that the fourth cumulant $(\phi^2)^2-3\langle\phi^2\rangle\phi^2$ vanishes, so we can replace the last term in the action by $3\lambda_0\langle\phi^2\rangle\phi^2$. This makes the free-energy Gaussian, with an effective mass

$$m_{\rm eff}^2=\xi_{\rm eq}^{-2}=m_0^2+6\lambda_0\langle\phi^2\rangle\;. \tag{1.D.60}$$

In a spatially homogeneous state, the average $\langle\phi^2\rangle$ should be independent of the space point on which is it is measured and, in particular, it should be identical to its value at the origin, $\langle\phi^2(\vec{x})\rangle=\langle\phi^2(\vec{0})\rangle$. Moreover, since we argued that ϕ^2 is Gaussian distributed, its average can be readily computed, and for any of its components,

$$\langle\phi_\alpha^2(\vec{0})\rangle=\int_{|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m_{\rm eff}^2} \tag{1.D.61}$$

where we have included a (ultra-violet) cut-off Λ that can be related to the inverse lattice spacing of the original microscopic theory. Replacing in (1.D.60)

$$m_{\rm eff}^2=m_0^2+6\lambda_0\mathcal{N}\int_{|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m_{\rm eff}^2} \tag{1.D.62}$$

and the factor $\mathcal{N}$ is due to the sum over α. This equation admits a non-trivial result for $\lambda=6\lambda_0\mathcal{N}$ finite, that is to say, $\lambda_0\propto\mathcal{N}^{-1}$.

This problem can be studied statically within the canonical formalism. If the volume V is kept finite the equilibrium order parameter probability distribution is given by the Gibbs state [87]

$$P_{\rm eq}[\vec{\phi}(\vec{k})]=\frac{1}{Z}\;\exp\left(-\frac{1}{2k_BTV}\sum_{\vec{k}}(k^2+\xi_{\rm eq}^{-2})\vec{\phi}(\vec{k})\cdot\vec{\phi}(-\vec{k})\right) \tag{1.D.63}$$

where $\xi_{\rm eq}$ is the correlation length

$$\xi_{\rm eq}^{-2}=-m_0^2+\frac{\lambda}{\mathcal{N}}\langle\vec{\phi}^2(\vec{x})\rangle_{\rm eq} \tag{1.D.64}$$

with $\langle\cdots\rangle_{\rm eq}$ standing for the average taken with (1.D.63). (In this expression we have not distinguished one vector direction to signal the symmetry breaking [88] but we considered the symmetric measure in which one sums over all such states.) Note that this is, indeed, a *Gaussian measure*.

In order to analyze the properties of $P_{\rm eq}[\vec{\phi}(\vec{k})]$ it is necessary to extract from (1.D.64) the dependence of $\xi_{\rm eq}^{-2}$ on T, m_0, λ_0 and V. Evaluating the average, the above equation yields

$$\xi_{\rm eq}^{-2} = -m_0^2 + \frac{\lambda}{V}\sum_{\vec{k}} \frac{k_B T}{k^2 + \xi_{\rm eq}^{-2}}. \tag{1.D.65}$$

The solution of this equation is well known [82] and here we summarize the main features, as presented in [87]. Separating the $\vec{k} = 0$ term under the sum, for very large volume we may rewrite

$$\xi_{\rm eq}^{-2} = -m_0^2 + \lambda k_B T B(\xi_{\rm eq}^{-2}) + \frac{\lambda k_B T}{V \xi_{\rm eq}^{-2}} \tag{1.D.66}$$

where

$$B(\xi_{\rm eq}^{-2}) = \lim_{V\to\infty} \frac{1}{V}\sum_{\vec{k}} \frac{1}{k^2+\xi_{\rm eq}^{-2}} = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2+\xi_{\rm eq}^{-2}} \tag{1.D.67}$$

regularising the integral by introducing the high momentum (ultra-violet) cutoff Λ. The function $B(x)$ is a non negative monotonically decreasing function with the maximum value at $x = 0$

$$B(0) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2} = (4\pi)^{-\frac{d}{2}} \frac{2}{d-2} \Lambda^{d-2} \quad . \tag{1.D.68}$$

By graphical analysis one can easily show that (1.D.66) admits a finite solution for all $k_B T$. However, there exists the critical value of the temperature T_c defined by

$$-m_0^2 + \lambda k_B T_c B(0) = 0 \tag{1.D.69}$$

such that for $T > T_c$ the solution is independent of the volume, while for $T \leq T_c$ it depends on the volume. Using

$$B(x) = (4\pi)^{-\frac{d}{2}} x^{\frac{d}{2}-1} e^{\frac{x}{\Lambda^2}} \Gamma\left(1 - \frac{d}{2}, \frac{x}{\Lambda^2}\right) \tag{1.D.70}$$

where $\Gamma(1-\frac{d}{2},\frac{x}{\Lambda^2})$ is the incomplete gamma function, for $0 < \frac{T-T_c}{T_c} \ll 1$ one finds $\xi_{\rm eq} \sim (\frac{T-T_c}{T_c})^{-\nu}$, i.e. close but above T_c, where $\nu = 1/2$ for $d > 4$ and $\nu = 1/(d-2)$ in $d < 4$, with logarithmic corrections for $d = 4$. At T_c one has $\xi_{eq} \sim V^{\overline{\lambda}}$ with $\overline{\lambda} = 1/4$ for $d > 4$ and $\overline{\lambda} = 1/d$ for $d < 4$, again with logarithmic corrections in $d = 4$. Finally, below T_c one finds $\xi_{\rm eq}^2 = \frac{M^2 V}{k_B T}$ where $M^2 = \phi_0^2\left(\frac{T_c - T}{T_c}\right)$ and $\phi_0^2 = m_0^2/\lambda$.

Let us now see what are the implications for the equilibrium state. As Eq. (1.D.63) shows, the individual Fourier components are independent random variables, with a Gaussian distribution with zero average. The variance is given by

$$\frac{1}{\mathcal{N}} \langle \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k}) \rangle_{\rm eq} = V S(\vec{k}) \tag{1.D.71}$$

where

$$S(\vec{k}) = \frac{k_B T}{k^2 + \xi_{\rm eq}^{-2}} \tag{1.D.72}$$

is the equilibrium structure factor. For $T > T_c$, all $\vec{k}$ modes behave in the same way, with the variance growing linearly with the volume. For $T \leq T_c$, instead, $\xi_{\rm eq}^{-2}$ is negligible with respect to k^2 except at $\vec{k} = 0$, yielding

$$S(\vec{k}) = \begin{cases} \frac{T_c}{k^2}(1 - \delta_{\vec{k},0}) + cV^{2\overline{\lambda}}\delta_{\vec{k},0} & \text{for } T = T_c \quad , \\ \frac{T}{k^2}(1 - \delta_{\vec{k},0}) + M^2 V \delta_{\vec{k},0} & \text{for } T < T_c \quad , \end{cases} \tag{1.D.73}$$

where c is a constant. This produces a volume dependence in the variance of the $\vec{k} = 0$ mode growing faster than linear. Therefore, for $T \leq T_c$ the $\vec{k} = 0$ mode behaves differently from all the other modes with $\vec{k} \neq 0$. For $T < T_c$ the probability distribution (1.D.63) takes the form

$$P_{\rm eq}[\vec{\phi}(\vec{k})] = \frac{1}{Z} e^{-\frac{\vec{\phi}^2(0)}{2M^2V^2}} e^{-\frac{1}{2k_BTV}\sum_{\vec{k}} k^2 \vec{\phi}(\vec{k})\cdot\vec{\phi}(-\vec{k})} \quad . \tag{1.D.74}$$

Therefore, crossing T_c there is a transition from the usual disordered high temperature phase to a low temperature phase characterized by a macroscopic variance in the distribution of the $\vec{k} = 0$ mode. The distinction between this phase and the mixture of pure states, obtained below T_c when $\mathcal{N}$ is kept finite can be discussed but we will not do it here.

Although the effective Hamiltonian is 'almost' quadratic, the phase transition in the form of a Bose-Einstein-like condensation on the $\vec{k} = \vec{0}$ mode is due to the self-consistent constraint.

1.D.3 Connection between the two models

The behaviour of the large $\mathcal{N}$ $O(\mathcal{N})$ model is very similar to what derived for the spherical spin-glass model [26]. Why is this so? The reason is that the behaviour of both models are pseudo-quadratic models, for which the behaviour is controlled by the way in which the distribution of modes decays to zero. In the field theory these are the wave-vectors modulii k while in the spherical model these are the eigenvalues of the random interaction matrix close to the edge of their distribution.

More precisely, requiring

$$\rho(\overline{\lambda}_\mu) d\overline{\lambda}_\mu = \varrho(k) dk \tag{1.D.75}$$

with $\overline{\lambda}_\mu = 2J - \lambda_\mu$,

$$\begin{aligned} \rho(\overline{\lambda}_\mu) &= \frac{1}{2\pi J}\sqrt{4J^2 - \lambda_\mu^2} \simeq \frac{1}{\pi J}\sqrt{J\overline{\lambda}_\mu} \, , \\ \varrho(k) &= k^{d-1} \, , \end{aligned} \tag{1.D.76}$$

and $k^2 = \overline{\lambda}_\mu$ from the equivalence between the quadratic term in the Hamiltonian of the spherical model and the free-energy of th e field theory, then this implies

$$k^2 \propto k^{d-1} \tag{1.D.77}$$

and the two models are equivalent, in this sense, in $d = 3$.

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