Advanced Statistical Physics

Exam

January, 2022

Surname :

Name :

Master :

Write your surname & name in CAPITAL LETTERS.

Not only the results but especially the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

The answers must be written neatly within the boxes.

The problems follow the order of the chapters in the Lecture Notes but are not of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you find difficult.

1. Ergodicity

Figure 1 shows the numerical evaluation of the solution of a stochastic (Langevin) equation, ruling the time ($\boldsymbol{t}$) evolution of a real variable $\boldsymbol{X}$. We do not need to specify this equation but only remark that it depends on a parameter $\mathbf{a}$. The evolution of $\boldsymbol{X}$ for different initial conditions $\boldsymbol{X}(0)$ and different random noise realisations is the one in the panel above for $\mathbf{a} > 0$ and the one in the panel below for $\mathbf{a} = 0$.

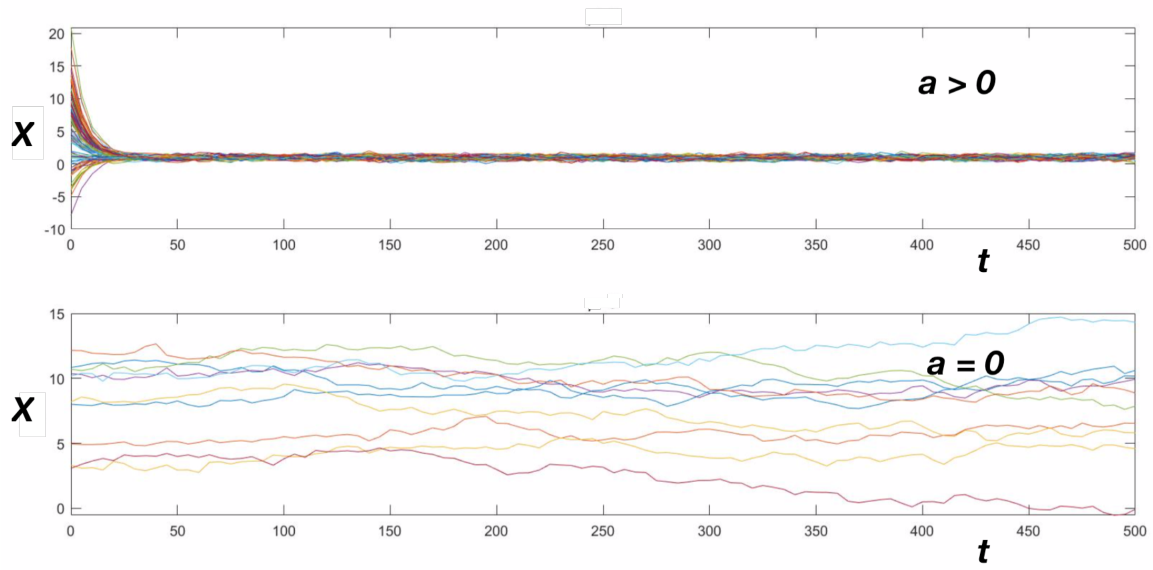

Figure 1: The solution of a stochastic equation with $\mathbf{a} > 0$ above and $\mathbf{a} = 0$ below. The different curves correspond to different initial conditions and different noise realizations.

What do you conclude about the ergodic properties of the system in the two cases? Give the condition needed to satisfy ergodicity and discuss whether it holds or not in the two cases.

2. Phase transitions.

Consider the classical Heisenberg model in three dimensions in canonical equilibrium with a bath at temperature T. This model is defined by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \ , \tag{1}$$

where $J > 0$, $\vec{s}_i$ are placed at the vertices of a three dimensional lattice with unspecified geometry and they are vectors with three components, $\vec{s}_i = (s_i^1, s_i^2, s_i^3)$, each of them taking real values, $-\infty < s_i^a < \infty$ for $a = 1, 2, 3$, but constrained to have unit modulus, $|\vec{s}_i|^2 = (s_i^1)^2 + (s_i^2)^2 + (s_i^3)^2 = 1$. The sum in Eq. (1) runs over nearest-neighbours on the lattice and there are N spins in the system.

1 – In the absence of any phase transition consideration, which is the canonical average of $\vec{s}_i$ at a generic temperature T? Justify your answer with a mathematical proof.

2 – Do you expect a finite temperature phase transition in this problem? Under which conditions on the number of spins?

3 – Which would be the phases? Describe them.

4 – Identify an order parameter and give its mathematical expression.

5 – Which is the mechanism whereby the order parameter just defined would acquire a non vanishing value? Explain its origin in an experimental situation.

6 – How is this mechanism imposed mathematically?

7 – Explain the way in which you have contoured the answer to question 1 – with the mathematical approach proposed in the answer to question 6 –.

8 – Do you expect ergodicity breaking in this problem? Discuss similarities and differences with the Ising cases that we discussed in the Lectures.

9 – Consider now the fully-connected model in which each spin interacts, via the same scalar product, with all other spins, $\sum_{i\neq j} \vec{s}_i \cdot \vec{s}_j$. How do you render the model well-defined in the thermodynamic limit? Justify your answer.

10 – Go back to the model in Eq. (1) defined on a finite three dimensional lattice which we will take to be a cubic one, with either free or periodic boundary conditions, a distinction which is not important in the infinite size limit. Establish the mean-field analysis, determine the phase diagram and sketch the behaviour of the order parameter. (Hint: you can exploit the answer to question 6 – to simplify the vectorial treatment.)

3. (In) equivalence of ensembles.

Consider a system of N spins placed on the vertices of a lattice. The potential energy is given by the sum over all pairs of the elementary constituents of a two-body energy $u(s_i, s_j)$.

1– Explain and illustrate, with one equation, the *extensivity* property of the energy.

2 – Discuss qualitatively the conditions under which this potential is *additive*. Illustrate this property with one equation. You can support your argument with a sketch (drawing).

3 – Explain why the violation of these properties may affect the equivalence of ensembles. Focus on micro-canonical and canonical measures and expand your answer with a mathematical argument.

4. Geometric frustration

1 – Define geometric frustration.

2 – Give a simple spin model example in which frustration is realized.

3 – Which are the main consequences of frustration discussed in the lectures?

4 – Is there an everyday life system, though at extreme conditions, with these unusual properties? Say which is this system and in which kind of phase it is.

5 – Which is the argument used to estimate (quite successfully) the entropy of the ground state? Use the example proposed as an answer to the previous question to explain this argument.

5. The energy spectrum of a quantum spin chain.

A paper from 2003 studies the level spacing properties of a quantum spin chain with periodic boundary conditions and Hamiltonian

$$\hat{H}[\{\hat{s}_i\}] = J \sum_{i=1}^{L} \left(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z \right) + \sum_i h_i \hat{s}_i^z \ , \tag{2}$$

where hats denote operators and the usual quantum spin notation is used. The magnetic fields h_i are uncorrelated random numbers with a Gaussian distribution with zero mean and finite variance h^2.

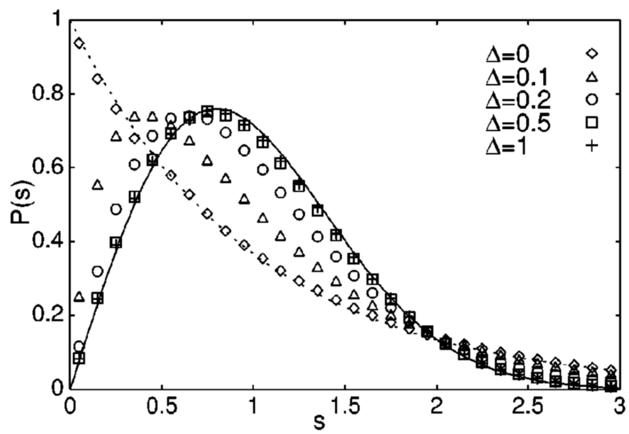

Figure 2: The level spacing, s, probability distribution, $P(s)$, of the model defined in Eq. (2) with $L = 14$ spins, J/h fixed, and values of the parameter Δ given in the key. Figure extracted from K. Kudo and T. Deguchi, *Level statistics of XXZ spin chains with a random magnetic field*, Phys. Rev. B **69**, 132404 (2004)..

In Fig. 2 the probability distribution of energy level spacings is shown for a system with $L = 14$ sites and different values of the parameter Δ. Explain what is shown in the figure and which conclusion can be drawn from the curves.

6. Disordered systems.

1 – Explain the difference between "annealed" and "quenched" disorder.

2 – What is the main difference between "weak" and "strong" quenched disorder?

Take the d-dimensional Ising model with random local fields

$$H = -J \sum_{\langle ij \rangle} s_i s_j + \sum_i h_i s_i \tag{3}$$

where $s_i = \pm 1$ are located at the sites of a hyper-cubic lattice, the sum runs over nearest neighbours on the lattice, and $J > 0$. The fields h_i are random variables taken from a joint probability distribution $P(\{h_i\})$.

In the Lectures we explained that, under a number of conditions, certain macroscopic quantities such as the free-energy density should be self-averaging.

3 – Consider a $P(\{h_i\})$ such that the random fields are independent identically distributed, so that $P(\{h_i\}) = \prod_{i=1}^{N} p(h_i)$ with $p(h_i)$ a Gaussian distribution with zero mean and variance h^2. Do you expect the free-energy density to be self-averaging? Justify your answer.

4 – Choose another $P(\{h_i\})$ such that the random fields still have vanishing average $[h-i]$ but are correlated $[h_i h_j] = (\prod_k \int dh_k)\, P(\{h_k\})\, h_i h_j = C(r_{ij})$ with $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and $\vec{r}_i$ the position of the site i. Is self-averageness ensured in this case? Discuss.

5 – The relative variance of a random variable X with probability distribution $P(X)$ is defined as

$$R_X \equiv \frac{[X^2] - [X]^2}{[X]^2} \ . \tag{4}$$

with $[\ldots]$ the average over $P(X)$.
In Fig. 3 the relative variance of the magnetization density of a system with quenched disorder is plotted. The two curves correspond to system with different kinds of disorder, quantified by the parameter b. Explain whether the data for R_m are self-averaging or not. Justify your answer.

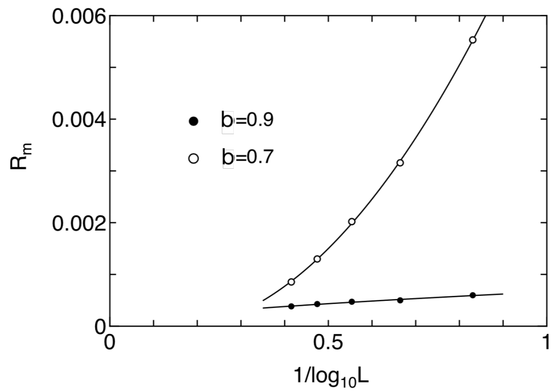

Figure 3: The relative variance defined in Eq. (4) of the magnetization density of a system with quenched randomness as a function of the system's linear size L.

6 – What is(are) the order parameter(s) in the model in Eq. (**??**)?

7 – Write the mean-field equation(s) for the order parameter(s). Explain briefly how these equations are obtained.

8 – Which kind of phases do you expect?

9 – Propose a numerical procedure to identify the phases that you proposed.

10 – Draw a schematic phase diagram.

11 – If you wanted to calculate that disorder averaged free-energy density, which method would you try to apply?