# Advanced Statistical Physics

## Exam

January 2021

Surname :
Name :
Master :

**Write your surname & name in capital letters.**

**Not only the results but especially the clarity and relevance of the explanations will be evaluated.**

**Focus on the questions asked and answer them (and not some other issue).**

**The answers must be written neatly within the boxes.**

**The problems follow the order of the chapters in the Lecture Notes but are not of increasing difficulty.**

## 1. Ergodicity

Figure 1 (a) shows one time series of a stochastic process, $X(i), i = 1, ..., n$, in other words, an indexed set of random variables in which the integer $i$ represents a discrete time. Panel (b) is the histogram of $X(i)$ with $i = 1, \ldots, 100$ over one time series. Panel (c), the histogram of $X$ at a particular instant $i = 25$ measured using $N = 100$ different realizations of the stochastic process.

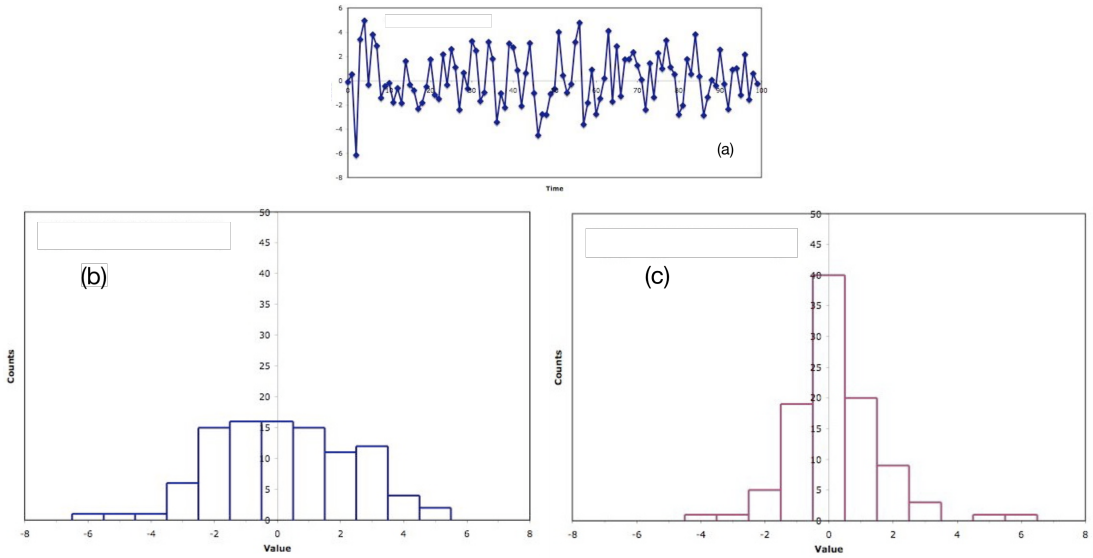



Figure 1: (a) A time series of the stochastic process $X(i)$, with $i = 1, \ldots, 100$. (b) The histogram of $X(i)$ with $i = 1, \ldots, 100$ over one time series. (c) The histogram of $X_a(i = 25)$ over $a = 1, \ldots, N = 100$ realizations of the stochastic process.

In your opinion, is this process ergodic? Give the condition needed to satisfy ergodicity. Explain your conclusion concerning this process. (An argument without an explicit calculation can be accepted.)

In order to have ergodicity, one needs equality between time and ensemble averages for any reasonable observable: $\langle A \rangle = \overline{A}$ with $\langle \ldots \rangle$ the ensemble average and $\overline{\ldots}$ the time-average.

Applied to the data in the figures, we need to ask, for $A = X$: $\overline{X} \equiv \lim_{N \gg 1} \frac{1}{N} \sum_{i=1}^{N} X(i) = \lim_{N \gg 1} \frac{1}{N} \sum_{a=1}^{N} X_a(i = 25) \equiv \langle X \rangle$. But we need the equality for all (non pathological) observables, hence $\overline{X^n} \equiv \lim_{N \gg 1} \frac{1}{N} \sum_{i=1}^{N} X^n(i) = \lim_{N \gg 1} \frac{1}{N} \sum_{a=1}^{N} X_a^n(i = 25) \equiv \langle X^n \rangle$ for any power $n$ if we restrict to $A$s that admit a Taylor expansion. Note that the required equality of averages is a statement about the large $N$ limits.

Of course, since $N = 100$ is not that large even for a trully ergodic process we would see finite $N$ fluctuations. The answer cannot be certain, but we can get an idea of what is going on from the data in the figures.

From the figures, just by looking at the pdfs, the two averages of $X$ seem very close to one another and, at least, they do not exclude ergodicity. At the same time, if we look more carefully at the pdfs, their form suggests that the averages of higher powers of $X$ will differ and thus strongly suggest that the process *is not* ergodic.

Calculating:
$\sum_{i=1}^{N} X(i) =$
$-6(1)-5(1)-4(1)-3(6)-2(15)-1(16)+0(16)+1(15)+2(11)+3(12)+4(4)+5(2) = 20$
Normalization: $N = 1+1+1+6+15+16+16+15+11+12+4+2 = 100$
Then $\frac{1}{N}\sum_{i=1}^{N} X(i) = \frac{20}{100} = 0.2$
On the other hand
$\sum_{a=1}^{N} X_a(i=25) =$
$-4(1) - 3(1) - 2(5) - 1(19) + 0(40) + 1(20) + 2(9) + 3(3) + 4(0) + 5(1) + 6(1) = 22$
Normalization: $N = 1+1+5+19+40+20+9+3+0+1+1 = 100$
Then $\frac{1}{N}\sum_{a=1}^{N} X_a(i=25) = \frac{22}{100} = 0.22$

The relative difference is of the order of $(0.22 - 0.2)/[(0.22 + 0.2)/2]$, say 10%. The difference is rather small and, as said above, from the analysis of the averages of $X$ one cannot exclude ergodicity. However, we can do better by comparing the time and ensemble average of, e.g., $X^2$ given that the histogram in (a) looks more symmetric than the one in (b). If we do it we get $\overline{X^2} = 4.88$ and $\langle X^2 \rangle = 2.08$, quite different values, different by say 80%, suggesting broken ergodicity. The difference will be increased for larger powers $X^n$.

Which kind of improved measurement would you propose to reach a more certain answer to the previous question?

Increase $N$ (use a longer time window for time averages and more samples for the statistical average).
Start sampling (both in time and on different samples) after some transient, to try to better establish stationarity.
If it is impossible to increase $N$, use some smart statistical method to compare the two histograms.

## 2. (In) equivalence of ensembles.

Consider a system of $N$ Ising spins $\{s_i = \pm 1\}$, with $i = 1, \ldots, N$. The Hamiltonian is

$$H(\{s_i\}) = -N_+(\{s_i\}) \ln a + \ln \begin{pmatrix} N \\ N_+(\{s_i\}) \end{pmatrix} , \tag{1}$$

where $N$ is the total number of spins and $N_+(\{s_i\})$ the number of positive spins in the configuration $\{s_i\}$. We will use the notation $n_+ \equiv N_+/N$. The parameter $a$ is real and we will take it to be $0 \leq a \leq 1$. We will work in the limit $N \gg 1$. Note that the second term in (1) is part of the energy function.

For such a Hamiltonian, do you expect to have a well defined thermodynamic limit and equivalence of ensembles for all values of the parameter $a$? Justify your answer.

The case $a = 0$ could be problematic as the first term in $H$ diverges. Still, when we write an Ising model, $-J \sum_{\langle ij \rangle} s_i s_j$, the coupling constant $J$ can also diverge and we do not worry about this limit. Indeed, when we identify the control parameter, we notice that it is $T/J$ and that the zero limit can be reached either by $T \to 0$ or $J \to \infty$.

In this model, we do not worry *a priori* about the possible divergence of the first term for $a \to 0$. We study the model as a function of $a$, and we eventually look at the limit $a \to 0$.

All along this exercise we will use $N \gg 1$ and Stirling $\ln N! \simeq N \ln N - N$.

The second term in the Hamiltonian is then
$\ln(N!/(N_+!(N-N_+)!) = \ln N! - \ln N_+! - \ln(N-N_+)! \sim$
$N \ln N - N - N_+ \ln N_+ + N_+ - (N-N_+)\ln(N-N_+) + N - N_+ =$
$N \ln N - N_+ \ln N_+ - (N-N_+)\ln(N-N_+) =$
$N \ln N - N_+ \ln(N_+/N) - N_+ \ln N - (N-N_+)\ln((N-N_+)/N) - (N-N_+)\ln N =$
$-N_+ \ln(N_+/N) - (N-N_+)\ln((N-N_+)/N)$

The total energy is $H \sim -n_+ N \ln a - n_+ N \ln(n_+ N) - n_- N \ln(n_- N)$, with $n_\pm$ the number density of positive and negative spins, $n_+ + n_- = 1$ and $n_\pm \in [0,1]$. Take the extreme case $n_+ = 1$; then $H(n_+ = 1) = -N \ln a \geq 0$ and the energy is extensive. For the other extreme case $n_+ = 0$, one has $H(n_+ = 0) = 0$. In between, the energy reaches a maximum at $n_+^{\max} = (1+a)^{-1} = n_+^*$ at which it takes the value $H(n_+^{\max}) = -\ln[a/(1+a)]\, N$. All in all, the total energy is *extensive* and we expect a well-defined $N \to \infty$ limit.

We will see later that we can take $n_+$ to be the order parameter.

However, the energy is not *additive*, $H(\{s_i, \sigma_i\}) \neq H(\{s_i\}) + H(\{\sigma_i\})$. The Hamiltonian depends on all spins put together into the variable $n_+$ and there is

no sense of locality that could help up cut the system in two and consider the two sub-systems separately neglecting the interaction between the two (like for the Curie Weiss model for ferromagnetism, in which the energy is written in terms of $m$ the magnetisation density).

Therefore, we may expect in-equivalence of ensembles for some values of the parameter $a$, in some interval or energies (microcanonical) and for some inverse temperature (canonical).

The model is of "long-range" kind.

Treat this problem in the canonical ensemble assuming that the system is coupled to a thermal bath at inverse temperature $\beta \leq 1$, and that $N \gg 1$. Hint: find a Ginzburg-Landau-type free-energy density as a function of a convenient auxiliary variable and the parameters $a$ and $\beta$. Evaluate the average $\langle n_+ \rangle$, express it as a function of $\beta$ and $a$, and discuss its behaviour as a function of these parameters in the intervals $\beta \leq 1$ and $a \leq 1$. Present the dependence using a graphical representation $\langle n_+ \rangle(a)$ at fixed $\beta$.

We will compute the partition sum $Z(\beta, a) = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$ transforming it into a sum over $N_+$, and taking into account the entropic contribution arising from the degeneracy of spin states that yield the same $N_+$ value.

After some simple steps as the ones in the Lecture Notes

$$Z(\beta, a) = \sum_{N_+=0}^{N} e^{-\beta H(N_+) + \ln \mathcal{N}(N_+)}$$

with $\mathcal{N}(N_+) = \begin{pmatrix} N \\ N_+ \end{pmatrix} = \dfrac{N!}{N_+!(N-N_+)!}$

Exploiting now that $n_+ = N_+/N$ becomes a continuous variable $\in [0,1]$ for $N \to \infty$, we have (to leading order in $N$ in the exponential)

$$\begin{aligned} Z(\beta, a) &\to \int_0^1 dn_+ \, e^{-\beta N[-n_+ \ln a - (1-k_B T)(n_+ \ln n_+ + (1-n_+)\ln(1-n_+))]} \\ &= \int_0^1 dn_+ \, e^{-\beta N \phi(n_+;\beta,a)} \qquad \text{with} \\ \phi(n_+;\beta,a) &= -n_+ \ln a - (1-k_B T)(n_+ \ln n_+ + (1-n_+)\ln(1-n_+)) \end{aligned}$$

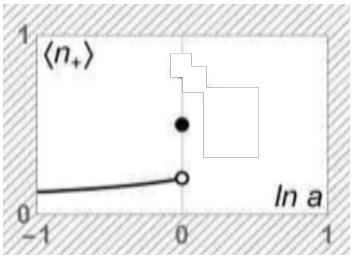
For $\beta \leq 1$ we can now evaluate this integral by saddle-point $\partial\phi(n_+;\beta,a)/\partial n_+ = 0 \Rightarrow$

$$n_+^{\text{ext}}(\beta, a) = \frac{1}{1 + a^{\beta/(\beta-1)}} = \frac{1}{1 + a^{1/(1-k_B T)}} \qquad \text{for} \qquad a \leq 1$$

One can check that $n_+^{\rm ext}$ is a growing function of $a$ taking values $\in [0,1/2]$ for $\beta < 1$. In the limit $\beta \to 1^-$, $n_+^{\rm ext} \to 0$ for all $a < 1$. Finally, for $a = 1$, $n_+^{\rm ext} = 1/2$ for all $\beta \leq 1$. We see a discontinuity of the value of $n_+^{\rm ext}$ as a function of $a$ for $\beta \to 1^-$: it jumps from 0 (for all $a < 1$) to $1/2$ (at $a = 1$).

The average is $\langle n_+ \rangle = \partial \ln Z(\beta,a)/\partial(\beta N \ln a) = -\partial[\beta N \phi(n_+^{\rm ext};\beta,a)]/\partial(\beta N \ln a) = -\partial\phi(n_+^{\rm ext};\beta,a)/\partial \ln a = n_+^{\rm ext}$ and all the special values mentioned above apply to $\langle n_+ \rangle$

The figure below shows $\langle n_+ \rangle$ as a function of $\ln a$ for $\beta \to 1^-$. The black dot is at $(0,1/2)$




$Z(\beta,a) \to e^{-\beta N \phi(n_+^{\rm ext};\beta,a)}$ and the free-energy density is then $f(\beta,a) = \phi(n_+^{\rm ext};\beta,a)$.

One could also work with the local magnetisation density $m$ as an order parameter

$$m = 2n_+ - 1 \qquad \text{or} \qquad n_+ = (1+m)/2$$

The saddle-point equation reads, in terms of $m$,

$$m^{\rm ext} = \tanh a^{\frac{\beta}{2(\beta-1)}} = \tanh a^{\frac{1}{2(1-k_B T)}}$$

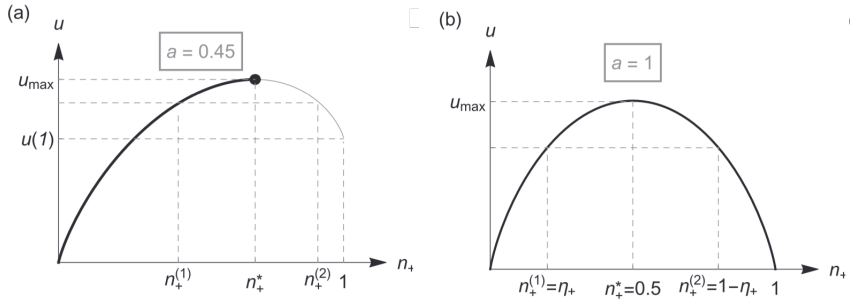
Ref: A. Fronczak, P. Fronczak, and G. Siudem, *Partial equivalence of statistical ensembles in a simple spin model with discontinuous phase transitions*, Phys. Rev. E **101**, 022111 (2020).

Treat this problem in the microcanonical ensemble. Hint: identify the function $u(n_+;a)$ with $u = H/N$ and study its properties for $0 \leq a \leq 1$ (summarize them in two plots of $u(n_+;a)$ for two characteristic values of $a$). For a given $u$, which is the value of $n_+$? Explain your reasoning.

Calculate the microcanonical inverse temperature $\beta_m$ and discuss its dependence on $a$. Invert, when it is possible, the expression to obtain $n_+(\beta_m, a)$. How does it compare to the canonical $\langle n_+\rangle(\beta, a)$? Be careful and think about all possible values of $a \in [0, 1]$.

The energy density is $u(n_+;a) = H/N = -n_+ \ln a - n_+ \ln n_+ - n_- \ln n_-$, and this function vanishes for $n_+ = 0$, equals $-\ln a \geq 0$ at $n_+ = 1$ and reaches a maximum at $n_+^{\max} = (1+a)^{-1} = n_+^*$ at which it takes the value $u_{\max} = u(n_+^{\max};a) = -\ln[a/(1+a)]$. For $a = 1$ the curve becomes symmetric with the maximum at $n_+ = 1/2$.

The figures below show the function $u(n_+;a)$ for $a < 1$ and $a = 1$. We note that there is a range of $-\ln a \leq u \leq -\ln[a/(1+a)]$ in which there are two macro-states (two values of $n_+$) compatible with the same $u$.




The entropy is $s(n_+) = -n_+ \ln n_+ - (1-n_+)\ln(1-n_+)$ independently of $a$.

For the energy values $-\ln a \leq u \leq -\ln[a/(1+a)]$ with two possible $n_+$ for the same $u$, e.g. $n_+^{(1)}$ and $n_+^{(2)}$ in the figures, the one leading to the maximal entropy should be preferred. For $a < 1$ this value is the one on the left branch of the curve in (a). For $a = 1$ the curve is symmetric and the two states are perfectly degenerate, see (b). In the limit $a \to 1^-$, the value on the left branch $n_+^{(1)}$ is selected.

We note that for $a = 1$, $s(n_+) = u(n_+; a = 1)$.

The microcanonical inverse temperature is $\beta_m = \partial s(n_+)/\partial u(n_+;a)$.

For $a = 1$, since $s = u$, it is fixed to $\beta_m = 1$ independently of $u$ and $n_+$.

For $a < 1$, one can compute it as $\beta_m = \partial s(n_+)/\partial n_+ \times \partial n_+/\partial u(n_+;a)$ and this will yield a non-constant $\beta_m(u;a)$.

The first factor is $\partial s/\partial n_+ = -\ln n_+ - 1 + \ln(1-n_+) + 1 = -\ln n_+ + \ln(1-n_+)$.

The second factor is $\partial n_+/\partial u(n_+;a) = (\partial u(n_+;a)/\partial n_+)^{-1} = [-\ln a - \ln n_+ + \ln(1-n_+)]^{-1}$.

Inverting $n_+(u)$ one would then have $\beta_m(u;a)$.

We can instead invert and write

$$n_+(\beta_m, a) = \frac{1}{1 + a^{\beta_m/(\beta_m - 1)}} \qquad \text{for} \qquad a < 1$$

which has the same functional form as the canonical one in this range of $a$. We note that in the micro-canonical ensemble $n_+$ is fixed by the energy density, and one can read it graphically from the figures above. If we fix $u = \langle u \rangle$ from the canonical ensemble at temperature $\beta$ we should get $\beta_m = \beta$ and the same $n_+$.

The difference is at $a = 1$, with non-equivalence of ensembles.

Ref: A. Fronczak, P. Fronczak, and G. Siudem, *Partial equivalence of statistical ensembles in a simple spin model with discontinuous phase transitions*, Phys. Rev. E **101**, 022111 (2020).

## 3. A Heisenberg spin model on the Kagomé lattice

The Hamiltonian of the Heisenberg model is

$$H_J[\{\vec{s}_i\}] = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \tag{2}$$

where $\vec{s}_i$, with $i = 1, \ldots, N$, are three dimensional vectors with unit modulus, $s_i = |\vec{s}_i| = 1$ for all $i$, and the sum runs over nearest neighbours on the Kagomé lattice depicted in Fig. 2. We will consider a strictly negative coupling constant $J < 0$.

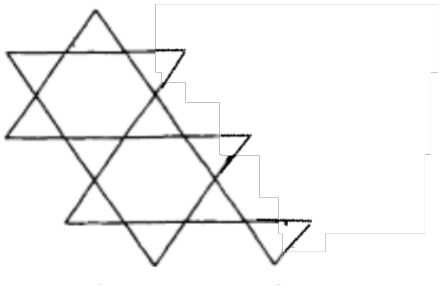


Figure 2: The bidimensional Kagomé lattice, made of corner sharing triangular plaquettes.

Which kind of low temperature ordering does the negative exchange energy $J < 0$ favour? Justify your answer.

The negative $J$ favors anti-alignment of neighbouring spins.
Therefore, this coupling favours antiferromagnetic order.

With its length being fixed, how many degrees of freedom does each spin $\vec{s}_i$ have?

Each spin has three components, but the constant modulus restriction is one constraint, so, the number of degrees of freedom of each spin is $3 - 1 = 2$.

How many degrees of freedom, that we will call $D$, are there in the system?

Since there are $N$ spins in the sample, and we have 2 degrees of freedom per spin, the total number of d.o.f. is $2N$.

Is this model frustrated? Why?

This model is frustrated since each triangular plaquette is frustrated
$\prod_{i \in \triangle} J_{ij} = J^3 < 0$.

We can also note that the hexagons are not frustrated, since they have six (an even number) bonds.

Which kind of partitioning of the Kagomé lattice into plaquettes would you propose to study the ground state properties of this model?

I would consider the full system as the union of up-looking and down-looking triangular plaquettes linked by one of their vertices.

Are these plaquettes independent?

No, they are not since each pair of them share a spin.

Which is the condition that the spins on one of the selected plaquettes (disconnected from the rest of the system) should satisfy to minimize their contribution to the total energy? Explain.

The contribution to the energy of each triangular plaquette is

$E_{\triangle} = -J(\vec{s}_1 \cdot s_2 + \vec{s}_2 \cdot s_3 + \vec{s}_3 \cdot s_1) = -\frac{J}{2}(\vec{s}_1 + \vec{s}_2 + \vec{s}_2)^2 + J(|\vec{s}_1|^2 + |\vec{s}_2|^2 + |\vec{s}_3|^2)$.

The second term is strictly negative (because of $J < 0$) and equal to $3J$ using the normalization of the spin modulus.

One is left with $E_{\triangle} = 3J - \frac{J}{2}(\vec{s}_1 + \vec{s}_2 + \vec{s}_2)^2$. This is minimized by taking

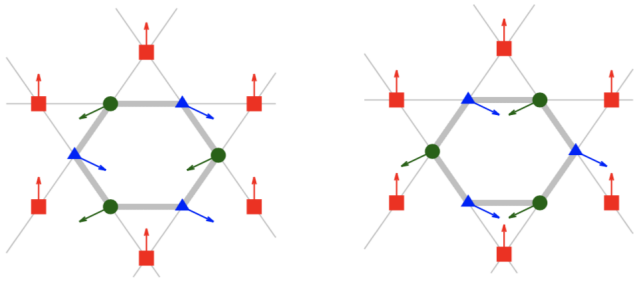
$$\vec{s}_1 + \vec{s}_2 + \vec{s}_3 = \vec{0}$$

If you focus on co-planar spins, re-write this condition as one on the angles formed by neighbouring spins on the plaquette.

The solution to the vectorial sum equal to zero is to place the neighbouring spins at $120^o$ (or $2\pi/3$) angles: $\theta_{ij} = 120^o = 2\pi/3$.

Consider the sub-set of such configurations in which *all* spins are co-planar. How many spin orientations are there in one configuration belonging to this sub-set? Draw the spins on the piece of Kagomé lattice in the figure to explain your answer.

Two examples of ground state configurations. Note that in each of them there are three spin orientations.



From the evaluations above, would you expect to find a macroscopically ground state entropy in this model?

As a hand-waving argument, one would like to compare the number of d.o.f. $D$ to the number of independent constraints, $K$, and see whether $D - K \propto N > 0$, or not. But $K$ is very difficult to estimate! One can just estimate the number of constraints if one considers them to be independent on each plaquette (though they are not).

Let's go back to $D$ and write it in terms of the number of plaquettes, $N_{\rm plaq}$. One has $D = 2N = 2(N_{\rm plaq}\, q)/2 = 3N_{\rm plaq}$, since $N = (N_{\rm plaq} q)/2$ with $q = 3$ the number of spins on each plaquette.

On each plaquette there is a three-component constraint on the sum of the spins. Therefore, if we take the plaquettes to be independent, $K = 3N_{\rm plaq}$.

Therefore, $D = K$ and one could argue that there is no macroscopic degeneracy in this problem. Still, there is, since one is over estimating $K$ with this argument.

The co-planar spin configurations identified in the previous items are a sub-set of all ground states in this model. The full manifold of ground states is disordered in the sense that all correlation functions are short-ranged. However, at low temperatures, $T$ close to zero, one observes the nematic (it is not necessary to know the definition) correlation function $g(r)$ in Fig. 3:

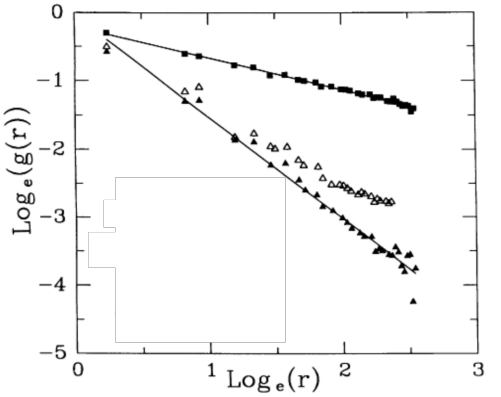



FIG. 3. Distance dependence of nematic correlation function. $T/J = 5\times10^{-3}$, 432 and 768 sites: △, ▲. $T/J = 3\times10^{-3}$, 768 sites: ■.

Identify the main features in this figure.

The correlation function is decaying algebraically. At the higher $T$ one sees finite size effects. At the lower $T$ only data for the larger system size are shown. The exponent depends on $T$ and it is smaller for lower $T$.

An algebraic decay of the correlation function with distance suggests a diverging correlation length and critical behaviour.

What are the data in the figure suggesting? More precisely, which kind of mechanism is at work in this problem? Explain

The correlations at $T = 0$ should be averaged over the ground states and, since these are typically expected to be disordered (see the text), they should decay as exponentials, with only short-range order.

In the fig. we see algebraic decay of nematic correlations at very low $T$.
They suggest critical behaviour at $T \to 0$ and, therefore, an order by disorder mechanism.

The refs. for this section are
J. T. Chalker, P. C. W. Holdsworth, and E. F. Shender, *Hidden order in a frustrated system: properties of the Heisenberg Kagomé antiferromagnet* Phys. Rev. Lett. **68**, 855 (1992).
J. T. Chalker, *Geometrically frustrated antiferromagnets: statistical mechanics and dynamics*, arXiv:0901.3492

## 4. The energy spectrum of a quantum spin chain.

A paper from 2003 studies the level spacing properties of a quantum spin chain with periodic boundary conditions with Hamiltonian

$$\hat{H}_{XXZ}[\{\hat{s}_i\}] = \frac{1}{2}\sum_{i=1}^{N}\left(e^{i\phi N}\hat{s}_i^+\hat{s}_{i+1}^- + e^{-i\phi N}\hat{s}_i^-\hat{s}_{i+1}^+\right) + J_1\sum_{i=1}^{N}\hat{s}_i^z\hat{s}_{i+1}^z + J_2\sum_{i=1}^{N}\hat{s}_i^z\hat{s}_{i+2}^z\,, \quad (3)$$

where hats denote operators and the usual quantum spin notation is used. $\phi$ is a real parameter and represents a flux threading the ring.

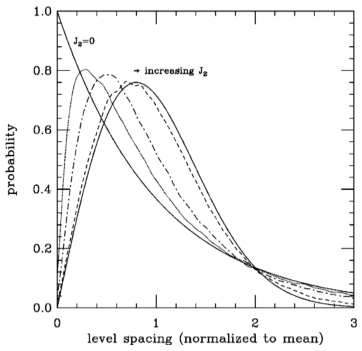



Figure 3: The level spacing, $s$, probability distribution, $P(s)$, of the model defined in Eq. (3) with $N = 18$ spins, $J_1 = 0.2$, $\phi$ fixed, and $J_2$ progressively increasing from $J_2 = 0$ to $J_2 > 0$ as indicated in the figure.

In Fig. 3 the probability distribution of energy level spacings is shown for $J_1$ and $\phi$ fixed and various values of the parameter $J_2$, increasing from $J_2 = 0$. Explain what is shown in the figure and which conclusion can be drawn from the curves.

For $J_2 = 0$ the distribution is of Poisson kind, $P(s) \neq 0$, and the levels are independent.

As $J_2$ increases the pdf leaves the Poisson form and becomes more and more of Wigner type, $P(s) = 0$, and there is level repulsion.

According to the Bohigas-Gianonni-Schmidt hypothesis we go from integrable to chaotic behaviour.

Ref: D. A. Rabson, B. N Narozhny, and A. J. Millis, *Crossover from Poisson to Wigner-Dyson Level Statistics in Spin chains with Integrability breaking*, Phys. Rev. B **69**, 054403 (2004).

## 3. Disordered systems.

In the Lectures we explained and used an argument to prove the self-averageness of certain quantities in systems with finite range interactions. Is this argument applicable to a macroscopic system at a second order phase transition? Discuss.

The argument cannot be applied since the correlation length diverges and one cannot partition the system in boxes with $a \ll \ell \ll L$ considered to be independent.

The relative variance of a random variable $X$ with probability distribution $P(X)$ is defined as

$$R_X \equiv \frac{[X^2] - [X]^2}{[X]^2} \ . \tag{4}$$

with $[\ldots]$ the average over $P(X)$.

Look at the data in Fig. 4. Explain whether the data for $X_p$, $p$, $\chi$, $M$ (four random variables) in (a) and $m$ (another random variable) in (b) are self-averaging or not. Justify your answer.

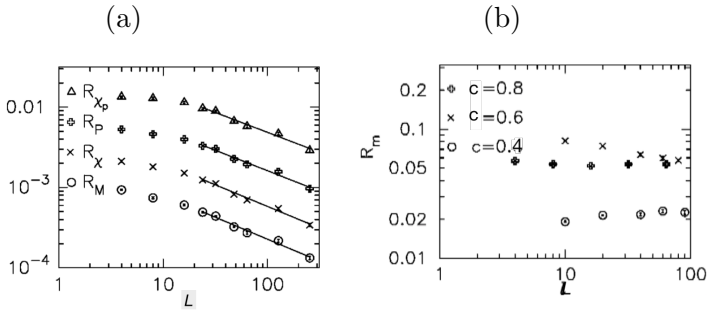



Figure 4: The relative variance defined in Eq. (4) of various observables as a function of the system linear size $L$. The $c$ values given in the key are the ones of a parameter in the model.

In panel (a) the relative variances decrease with system size and seem to tend to zero in the $L \to \infty$ limit. One can then argue that the observables are self-averaging.

In contrast in panel (b) the data-points saturate in the large $L$ limit and one can then conclude that the observable $m$ is not self-averaging.

The figures above have been adapted from

S. Wiseman and E. Domany,
*Self-averaging, distribution of pseudocritical temperatures, and finite size scaling in critical disordered systems*
Phys. Rev. E **58**, 2938 (1998).

Consider an Ising chain with $N$ spins sitting on its sites, with periodic boundary conditions, and interacting via nearest neighbour couplings $J_i$ which are independent identically distributed random variables taking real values and distribution according to

$$p(J_i) = a\,\delta(J_i - K_1) + b\,\theta(K_2 - |J_i|)\ . \tag{5}$$

$\delta$ and $\theta$ are the Dirac and Heaviside functions, $a$ and $b$ are two real parameters, and $K_1 > K_2$ are other two real, finite and non-vanishing parameters.

Find a relation between $a$ and $b$ to make $p(J_i)$ a proper distribution function.

$$\int_{-\infty}^{\infty} dJ_i\, p(J_i) = a + b\,2K_2 = 1 \quad \Rightarrow \quad b = (1-a)/(2K_2)$$

Compute the disorder averaged free-energy density of the model. Give the most compact analytic expression you can.

The first part of the calculation is completely general

$$\begin{aligned}
[\ln Z_J(\beta)] &= \left[\ln \sum_{\{s_i=\pm 1\}} e^{\beta \sum_{i=1}^N J_i s_i s_{i+1}}\right] = \left[\ln \sum_{\{\eta_i=\pm 1\}} e^{\beta \sum_{i=1}^N J_i \eta_i}\right] = \\
&= \left(\prod_{j=1}^N \int_{-\infty}^{\infty} dJ_j\, p(J_j)\right) \ln \prod_{i=1}^N 2\cosh(\beta J_i) \\
&= \left(\prod_{j=1}^N \int_{-\infty}^{\infty} dJ_j\, p(J_j)\right) \sum_{i=1}^N \ln\left(2\cosh(\beta J_i)\right) \\
&= \sum_{i=1}^N \int_{-\infty}^{\infty} dJ_i\, p(J_i) \ln\left(2\cosh(\beta J_i)\right) \\
&= N \int_{-\infty}^{\infty} dJ_1\, p(J_1) \ln\left(2\cosh(\beta J_1)\right)
\end{aligned}$$

The last line uses $J_1$ as any other equivalent $J_i$

We can now use the explicit probability distribution in Eq. (5):

$$[\ln Z_J(\beta K_1, \beta K_2)] = N\left(a \ln\left(2\cosh(\beta K_1)\right) + \frac{1-a}{2\beta K_2} \int_{-\beta K_2}^{\beta K_2} d(\beta J_1)\ \ln\left(2\cosh(\beta J_1)\right)\right)$$