Advanced Statistical Physics

TD2 Random matrix Theory

November 2019

In this TD we will study some properties of random symmetric matrices with elements taking real values.

1. Consider a 2×2 symmetric matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \tag{1}$$

   (a) Compute the eigenvalues λ_1 and λ_2.
   (b) Compute their difference $s = \lambda_1 - \lambda_2$.
   (c) Find the conditions on the matrix elements to have degenerate eigenvalues, in other words, vanishing level spacing, $s = 0$.
   (d) Write the level spacing s as a distance on a two dimensional plane. Suppose that the joint probability distribution of the "coordinates" x and y on this plane does not diverge close to the origin. Prove that the probability distribution of the level spacing vanishes close to zero. This is a manifestation of the level spacing repulsion.

2. Imagine now that the matrix M has been constructed as $M = A + A^T$ with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \Rightarrow \qquad M = \begin{pmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{pmatrix} \tag{2}$$

   and take the real elements of A as independent identically distributed random variables with probability $p_A(a_{ij})$.

   (a) Compute the probability distribution of the four elements m_{ij} that we will call $p_M(m_{ij})$. What do you note in these distributions?
   (b) Draw the real elements of A from a Gaussian probability distribution with zero mean and unit variance. Compute the probability distributions of the elements of the matrix M in this case. (Hint: as you shall use the probability of a linear combination of two Gaussian random variables in several steps of this and the following questions, first establish the form of $p(z)$ for $z = ax + by$, with a and b real parameters and x and y i.i.d. Gaussian random variables with zero mean and variance σ^2.)

(c) Calculate $p(s)$.

(d) Prove that $\tilde{p}(s/\langle s\rangle)$ with $\langle s\rangle = \int ds\, s\, p(s)$ satisfies the Wigner surmise.

3. Generate an ensemble with $\mathcal{N} = 1000$ Gaussian Orthogonal Ensemble (GOE) matrices constructed as $M = A + A^T$, and A matrices of size $N = 2, 4, 10$ with elements drawn from a Gaussian probability distribution for zero mean and unit variance.
   (a) For $N = 2$ verify numerically the results of the previous item.
   (b) In general, find the N eigenvalues λ_n, with $n = 1, \dots, N$ of each matrix.
   (c) Sort them in increasing order.
   (d) Find the difference between neighbouring eigenvalues $s_n = \lambda_{n+1} - \lambda_n$.
   (e) Calculate the mean splitting $\langle s\rangle \equiv N^{-1}\sum_{n=1}^{N} s_n$.
   (f) Plot a histogram of the eigenvalue splittings divided by the mean splitting, $\tilde{s}_n = s_n/\langle s\rangle$ with bin size small enough to see some of the fluctuations. (Hint: Debug your work with $\mathcal{N} = 10$, and then change to $\mathcal{N} = 1000$.)
   (g) Plot also the probability density of a single eigenvalues λ_n, $\rho(\lambda_n)$, and compare it to the semi-circle law.
   (h) Plot the probability density of the largest eigenvalue, λ_N, sampled over many realisations of the random matrices and compare it to the Tracy-Widom result (`https://en.wikipedia.org/wiki/Tracy%E2%80%93Widom_distribution` [3]).

4. Let us call, again, M an element of the GOE such that its elements are $M_{ij} = (A_{ij} + A_{ji})/2$ and A_{ij} are i.i.d. Gaussian random variables with zero mean and variance σ^2. R is a rotation matrix such that $R^T R = \mathbb{1}$, with $\mathbb{1}$ the identity matrix.
   (a) Show that the trace of $M^T M$ with M^T the transpose of M is equal to the sum of the squares of all the elements of M.
   (b) Write $M^T M$ as a sum over diagonal elements M_{ii} and the elements of, say, the upper half of the matrix, M_{ij} with $j > i$.
   (c) Recall how are the variance of the diagonal and off-diagonal elements of the matrix M related.
   (d) Show that this trace is invariant under orthogonal transformations, $M \mapsto R^T M R$.
   (e) Write the joint probability distribution of the elements of a GOE matrix M in terms of $\mathrm{Tr} M^T M$.
   (f) Prove that this measure is invariant under orthogonal transformations: $\rho(R^T M R) = \rho(M)$.

5. In this exercise we develop an algebraic proof of Wigner's semi-circle law for GOE matrices.
   Define $\rho_N(\lambda) = \frac{1}{N}\sum_{\mu=1}^{N} \delta(\lambda - \lambda_\mu)$, to be the probability density of the eigenvalues of the $N \times N$ (finite size) square real and symmetric random matrix M. Note that since the λ_μ are random, this is a random probability distribution function. The idea here is to prove that, in the $N \to \infty$ limit, the kth moment of this finite N distribution function approaches
$$\lim_{N\to\infty} \frac{1}{N}\sum_{\mu=1}^{N} \lambda_\mu^k = \int d\lambda\, \rho(\lambda)\lambda^k \tag{3}$$

with $\rho(\lambda)$ the semi-circle law, $\rho(\lambda) = 1/(4\pi)\,\sqrt{4-\lambda^2}$.

(a) Relate $\frac{1}{N}\sum_{\mu=1}^{N}\lambda_\mu^k$ to the trace of a power of the random matrix M.

(b) Once written as a trace, order the various terms in the sum over diagonal indices according to their N-dependence. For convenience, scale the elements of M by a convenient factor $\sqrt{N}$.

(c) Prove that the leading (in powers of N) term in the odd moments, k odd, vanish, and hence also does the moment itself.

(d) Prove that the even moments are given by the Catalan numbers $C_{k/2}$, where

$$C_n \equiv \frac{1}{n+1}\left(\begin{array}{c} 2n \\ n \end{array}\right) = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n}\frac{n+k}{k} = \int_0^4 dx\; x^n \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,, \qquad (4)$$

and that these are also the moments of the semi-circle law, as shown in the last member of the equation above.

Références

[1] P. Chau Huu-Tai, N. A. Smirnova and P. Van Isacker, *Generalised Wigner surmise for* 2×2 *random matrices*, J. Phys. A: Math. Gen. **35**, L199 (2002).

[2] M. V. Berry and P. Shukla, *Spacing distributions for real symmetric* 2×2 *generalized Gaussian ensembles* J. Phys. A: Math. Gen. **42**, 485102 (2009).

[3] C.A. Tracy and H. Widom, *Distribution functions for largest eigenvalues and their applications*, Proc. International Congress of Mathematicians (Beijing, 2002), 1, Beijing: Higher Ed. Press, pp. 587-596, MR 1989209.