Advanced Statistical Physics

Exam

January 2020

Surname :
Name :
Master :

1. The ergodic hypothesis.

In Fig. 1 four time signals (realisations) of a continuous time random process $x(t)$ are displayed.

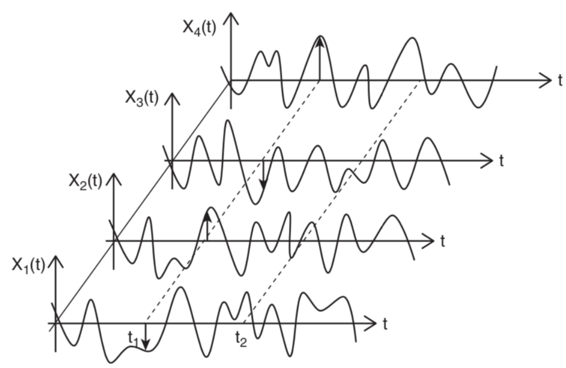

Figure 1: A continuous time stochastic process. Image taken from Ref. [1].

In your opinion, is this process ergodic? Explain.

2. (In) equivalence of ensembles.

Consider a system of N Potts spins, that is to say, variables taking q integer values, $s_i = 1, \ldots, q$, and potential energy given by the sum of a two-body potential that favours coupled spins taking the same value

$$H_J[\{s_i\}] = -J \sum_{ij} \delta_{s_i s_j} \ , \tag{1}$$

with the coupling constant $J > 0$. We consider the model defined on:

(a) a square lattice in two dimensions (the sum should then be interpreted as $\sum_{\langle ij \rangle}$ with $\langle ij \rangle$ first neighbours on the lattice),

(b) a fully connected graph (and the sum being interpreted as $\sum_{i \neq j}$).

When studied in the canonical ensemble, the d dimensional Potts model has a phase transition at a finite T_c that is *second order* for $q < q_c(d)$ and *first order* for $q > q_c(d)$. For example, in the two dimensional model the phase transition is *second order* for $q = 2, 3, 4$ and *first order* for $q > 4$.

Is the energy *extensive* for the model defined as in (a) and/or (b)? Justify the answer with one equation and explain the conclusion for (a) and (b).

Is the energy *additive* in (a) and/or (b)? Justify the answer with one equation and explain the conclusion for (a) and (b).

Explain why the violation of these properties may affect the equivalence of ensembles.

Do you expect inequivalence of ensembles in this model? In which cases?

Mention a physical system (realised in Nature) with inequivalence of ensembles.

3. The two dimensional Ising model.

Consider the two dimensional ferromagnetic Ising model with two-body nearest-neighbour interactions

$$H_J[\{s_i\}] = -J \sum_{\langle ij \rangle} s_i s_j \tag{2}$$

defined on a lattice with coordination number z and the sum $\sum_{\langle ij \rangle}$ is such that each the contribution of each bond is counted once. The variables $s_i = \pm 1$ indicate the Ising spins, the index $i = 1, \ldots N$ labels the lattice sites and therefore the spins placed on them, and the coupling constant is positive, $J > 0$. We will analyse the statistical properties of the model in contact with a thermal bath at temperature T.

Which phase(s) do you expect at $T = 0$?

Write the ground state energy.

Which phase(s) do you expect at $T \to \infty$?

Which are the simplest excitations over the ground state configuration that you can imagine? Make a drawing to explain the answer.

Give an expression for the difference, ΔE, between the energy of the excited state described in the previous question and the energy of the ground state.

Estimate the entropy associated to the excitation that you proposed and explain the reasoning that you followed to reach the estimate.

What do you conclude concerning the possibility of having an ordered state at low (but non-vanishing) temperatures and, therefore, a finite temperature phase transition from the expression found?

Which kind of argument have you applied here?

We consider a domain of flipped spins, in a background of spins with long range order, but now the domain is two-dimensional. Suppose the border between the flipped spins and the up spins contains n bonds. Then the energy difference between the state with a domain and one with complete long range order is $\Delta E \sim 3Jn$. To obtain entropy choose a point on the boundary. If the coordination number of the lattice is z, then an upper bound on the number of configurations of the boundary or domain wall is z^n. The precise number is less than this, because the boundary of just a single domain cannot intersect itself, by definition (otherwise there are two domains!). As a crude first guess, we could argue that at each step, the domain wall can only go in $z-1$ directions, because we should disallow the step that would retrace the previous one. This still allows the boundary to intersect itself, so our estimate of the entropy will be an overestimate. Then, as in $d = 1$, we can compute the entropy difference due to the presence of the domain: $\Delta S \sim k_B n \ln(z-1)$. Thus, the change in free energy due to a domain whose boundary contains n bonds is

$$\Delta F_n = [2J - k_B T \ln(z-1)]n.$$

Thus there exists a critical temperature T_c such that as

$$T \to T_c = \frac{2J}{k_B \ln(z-1)}$$

then $\Delta F \to \infty$ as $n \to \infty$ and the system is unstable towards the formation of domains. Accordingly we anticipate a disordered, para-magnetic phase with $M = 0$. For $0 < T < T_c$, however, ΔF is minimized by $n \to 0$, and the state with long range order is stable. Thus, for $0 < T < T_c$, the system exhibits a net magnetization M, which can be either positive or negative, in the absence of an applied field. This magnetization is often referred to as spontaneous magnetization.

4. Frustration

Define frustration

Mention two effects of frustration on spin systems. Explain, briefly, their origin.

Take now the $\mathcal{O}(n)$ anti-ferromagnetic model

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \; . \tag{3}$$

The sum over pairs of spins ij runs over nearest neighbours on a generic lattice made of corner-sharing plaquettes. Assume that there are q spins on each plaquette (e.g. $q = 3$ for a triangular plaquette) and that the lattice is made by $2N$ such plaquettes attached by their corners (see Fig. 2 (a) for an example, the Kagome lattice). The negative exchange energy $J < 0$ favours antiparallel alignment of the nearest neighbour n component spins $\vec{s}_i = (s_i^1,\ s_i^2, \dots,\ s_i^n)$ with fixed length $s_i = |\vec{s}_i|$.

With its length being fixed, how many degrees of freedom does each spin $\vec{s}_i$ have?

Each spin $\vec{s}_i$ has n components but the modulus is fixed to one, there is one constraint and the total number of d.o.f. per spin is $n - 1$.

(a) (b)

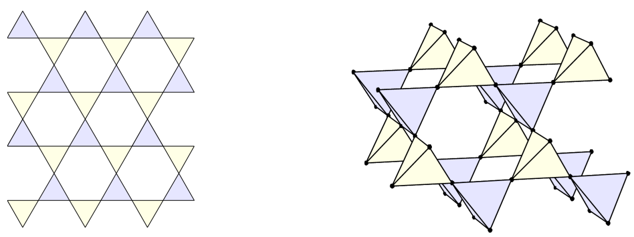

Figure 2: (a) The so-called Kagome lattice, made of corner sharing triangular plaquettes. (b) The pyrochlore lattice of corner sharing tetrahedra. Images taken from Ref. [2].

How many degrees of freedom, that we will call D, are there in the system?

There are $2N$ units in the system and each of them has q spins. Since each spin is shared by two units, one has Nq spins to consider, and each of them has $n-1$ d.o.f. Therefore, $D = Nq(n-1)$.

Which is the condition that the spins on a given plaquette (disconnected from the rest of the system) should satisfy to minimize their contribution to the total energy? Explain.

To minimise the single plaquette P energy one needs to minimise $E_P/J = \sum_{\langle ij\rangle\in P} \vec{s}_i \cdot s_j$. This can be rewritten $E_P/J = (\sum_{i\in P} \vec{s}_i)^2 - \sum_i (\vec{s}_i)^2 = 1/2\ (\sum_{i\in P} \vec{s}_i)^2 - 1/2\ \sum_i (\vec{s}_i)^2 = 1/2\ (\sum_{i\in P} \vec{s}_i)^2 - q/2$. The last term is a constant due to the normalisation of the spins. The first term is positive or at best zero; thus the best one can achieve is to make it vanish. This leads to the condition $\sum_{i\in P} \vec{s}_i = 0$.

Let us call K the number of constraints that must be satisfied to put the system in a ground state. This is very difficult to calculate. So, the idea is to approximate it by the sum over the constraints on each plaquette taken to be independent. What is the value of K under this assumption?

On each plaquette one needs to impose $\sum_{i\in P} \vec{s}_i = 0$, a vectorial relation, with n components. There are $2N$ units. Then, the number of constraints is $K = 2Nn$.

The number of degrees of freedom in the ground state, F, is estimated to be $D - K$ with D and K defined above. Which is the condition to have a macroscopic degeneracy of the ground state?

$F = D - K = Nq(n-1) - 2Nn = N[q(n-1) - 2n]$ should scale with N

Do the Kagome ($n = 3,\ q = 3$) and pyrochlore ($n = 3,\ q = 4$) lattices have macroscopically degenerate ground states?

For the triangular case, $F = D - K = N[q(n-1) - 2n] = [3 \times 2 - 2 \times 3]N = 0$. The constraints and degrees of freedom are perfectly matched and one does not expect macroscopic degeneracy in this case, according to this argument.
For the pyrochlore case, $F = D - K = N[q(n-1) - 2n] = [4 \times 3 - 2 \times 3]N = 6N$ the constraints are definitely not enough and the ground state has macroscopic degeneracy according to this estimate.

5. Random Matrices

In which field of physics were random matrices first used? Which was the qualitative argument put forward to justify the use of random matrices in this field?

In this context, which was the quantity that physicists focused on?

Which are the two "extreme" cases for the probability distribution of the quantity identified in the last question? Explain the answer without reproducing the full calculation.

Consider an $N \times N$ symmetric matrix M with real elements constructed as $m_{ij} = (a_{ij} + a_{ji})/2$ and a_{ij} taken from a probability distribution with zero mean, variance $\sigma^2 < +\infty$ and all higher moments taken finite values. What is the probability distribution of its eigenvalues λ_μ?

5. Quenched randomness

Which are the main effects of quenched randomness on a phase transition?

Explain the difference between "annealed" and "quenched" disorder.

Which is the main difference between "weak" and "strong" quenched disorder?

Mention and state a criterium that allows one to know whether weak disorder may change the critical exponents in a second order phase transition.

Consider the random anisotropy model defined by

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j + \Delta \sum_i (\vec{\nu}_i \cdot \vec{s}_i)^2 \quad (4)$$

The sum runs over nearest neighbours on a finite dimensional lattice. The spins have n components, $\vec{s}_i = (s_i^1, \ldots, s_i^n)$. The vectors $\vec{\nu}_i = (\nu_i^1, \ldots, \nu_i^n)$ have a different quenched random orientation but the same unit module, $|\vec{\nu}_i| = \nu$ on each site.

Do you expect a self-averaging free-energy density for this model? Explain.

Take a d dimensional Ising model defined on a cubic lattice. Correlated random fields with zero mean and variance σ^2 act locally on the spins. These are such that they are identical in one Cartesian direction and different in the $(d-1)$ other ones. For example, in a two dimensional system, they are identical on the vertical direction and different on the horizontal one.

Use and explain a qualitative argument to estimate the lower critical dimension in this problem

References

[1] L. Razdolsky, *Probability-based Structural Fire Load* (Cambridge University Press, Cambridge, 2014).

[2] P. H. Conlon *Aspects of Frustrated Magnetism*, PhD Thesis, The University of Oxford, UK, 2010.