Advanced Statistical Physics

TD3 The Random field Ising model

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The random field Ising model (RFIM) is defined as

$$H_h[\{s_i\}] = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i h_i s_i \ , \tag{1}$$

with s_i Ising spin variables, $J > 0$, h_i i.i.d. random fields taken from the Gaussian probability distribution

$$P[\{h_i\}] = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_h^2}} \, e^{-h_i^2/(2\sigma_h^2)} \ , \tag{2}$$

and the first sum running over first neighbours on a lattice.

General properties

1. Do you expect the free-energy density of this problem to be self-averaging?
2. It is convenient to rescale the random fields *via* the definition $h_i = \sqrt{\sigma_h^2}\,\eta_i$. What is the distribution of the η_i? In the following we will use this rescaling and study the model

$$H_h[\{s_i\}] = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i \eta_i s_i \ , \tag{3}$$

   with the η_i distributed in the way determined at the beginning of this question.
3. Consider the particular case $h = 0$, which phases do you expect? And in the opposite limit $h \to \infty$? Guess a topology of the phase diagram $(T/J, h/J)$ based on these two limits.
4. Recall the Imry-Ma argument explained in the lectures. For which dimension d do you expect the phase diagram guessed in the previous item?
5. Consider the mean-field limit in which we modify the first sum and we make it run over all pairs of spins $\sum_{i \neq j}$. How does one have to rescale the ferromagnetic coupling strength J to have an extensive first term?

The free-energy density according to the replica method

We will now calculate the disorder averaged free-energy density of the model defined on the fully-connected graph with the *replica method* [1].

1. Write $-\beta N[f_h] = [\ln Z_h]$ using the Taylor expansion of the $\ln Z$, where the square brackets indicate the average over the quenched random variables.
2. Write the replicated partition function Z^n for integer n.
3. Compute its average over the random fields $\{h_i\}$.
4. Use the Hubbard-Stratonovich identity or Gaussian decoupling

$$e^{\lambda b^2} = \int \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2+(2\lambda)^{1/2}bz} \tag{4}$$

   to decouple the site interaction (that is to say, to render the ferromagnetic term in the exponential one in which a sum over a single site index appears). Note that the site interaction has been traded for a replica interaction.
5. Rewrite the result as

$$\begin{aligned}[Z_h^n] &= \left(\frac{1}{2\pi}\right)^{n/2} \prod_{a=1}^{n} \int dz_a\, e^{-\frac{1}{2}\sum_{a=1}^{n} z_a^2 + N \ln \mathcal{Z}_1(\{z_a\})} \\ \mathcal{Z}_1(\{z_a\}) &= \sum_{\{s^a=\pm 1\}} e^{\sqrt{\frac{2\beta J}{N}}\sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2}\sum_{a,b=1}^{n} s^a s^b}\end{aligned} \tag{5}$$

6. Rescale the auxiliary variables z_a in such a way to get an overall N factor in the exponential and to render the single site partition sum $\mathcal{Z}_1(\{z_a\})$ independent of N.
7. Now, we have rewritten $[Z_f^n]$ is a form that is apt for evaluation using the saddle-point or steepest descent method, in the limit $N \to \infty$. Note that this step implies an exchange of limits $\lim_{N\to\infty} \lim_{n\to 0} = \lim_{n\to 0} \lim_{N\to\infty}$. Write the saddle-point equation for the n auxiliary variables z_a and prove that these are proportional to the spin average over the partition sum $\mathcal{Z}_1$.
8. Assume that z_a^{ext} is independent of a. Decouple the replica indices in $\mathcal{Z}_1(z)$ introducing another Gaussian decoupling.
9. Compute the sum over the replicated spin variables s^a and deduce the form taken by $\mathcal{Z}_1(z)$.
10. Now, we arrived at a point at which the limit $n \to 0$ can be easily taken. Do it.
11. Replace the z^{ext} by $\sqrt{2\beta J} m$, as from one of the items above we know it is proportional to the averaged spin.
12. Show that the disorder averaged free-energy density is given by

$$[f_h] = Jm^2 - k_B T \int dh\, p(h) \ln(2\cosh\beta(2Jm + h)) \tag{6}$$

   with the magnetisation density m determined by

$$m = \int dh\, p(h) \tanh(\beta(2Jm + h)) \ , \tag{7}$$

   with $p(h)$ the Gaussian probability given in Eq. (2) (for any of the local fields).
13. Find the paramagnetic solution. Determine the range of parameters for which it is stable.

14. Is there another solution of ferromagnetic kind, that is to say, with $m_{\rm ext} \neq 0$? For which parameters?
15. Find the phase transition between paramagnetic and ferromagnetic solutions in terms of the dimensionless variables $(T/J, T/\sigma_h)$.
16. What happens at zero temperature? Is there an ordered low-temperature phase?
17. Deduce the order parameter critical exponent $m(T) \simeq (T_c - T)^\beta$. Is it modified with respect to the mean-field value for the clean system?

The naive mean-field equations

Let us come back to the finite dimensional model defined on a regular lattice and study it with a mean-field approximation [2]. Assume that there are N local order parameters, $m_i = \langle s_i \rangle$ and that the joint probability function of the spin configurations factorises and takes the form

$$p(\{s_i\}) = \prod_{i=1}^{N} p_i(s_i) \qquad\qquad p(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1} \ . \tag{8}$$

1. Verify that $p_i(s_i)$ is normalised.
2. Compute the expected value of the Hamiltonian (1) and write it as a function of the local order parameters $\{m_i\}$.
3. Express the entropy $S = \sum_{\{s_i=\pm 1\}} P(\{s_i\}) \ln P(\{s_i\})$ as a function of the local order parameters $\{m_i\}$.
4. Combine the two expressions derived above and build the random-field dependent free-energy density $f_h(\{m_i\})$.
5. Write the N equations that extremise the free-energy density.

Références

[1] T. Schneider and E. Pytte, *Random-field Instability of the Ferromagnetic State* Phys. Rev. B **15**, 1519 (1977).

[2] D. Lancaster, E. Mariani and G. Parisi, *Weighted Mean Field Theory for the Random Field Ising Model*, J. Phys. A **28**, 3359 (1995).