# Advanced Statistical Physics

## TD Frustration - Maxwell

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## Degeneracy of frustrated magnets

The first exercise is an application of Maxwell's counting argument for elastic systems [1] to estimate the ground state degeneracy of the Heisenberg anti-ferromagnet [2, 3]

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \ . \tag{1}$$

The sum over pairs of spins $ij$ runs over nearest neighbours on a generic lattice. The negative exchange energy $J < 0$ favours antiparallel alignment of the nearest neighbour three component Heisenberg spins $\vec{s} = (s^x,\ s^y,\ s^z)$ of fixed length $|\vec{s}|$.

The number of degrees of freedom in the ground state, $F$, is estimated to be $D - K$, where $D$ is the total number of degrees of freedom of the spins and $K$ the number of constraints that must be satisfied to put the system into a ground state. The idea essentially comes from linear algebra, in which a system of $K$ equations for $D$ variables is expected to have a solution space of dimension $F = D - K$. Here, as there, the expectation may be wrong because the constraints imposed by the equations may not be independent (like $2x = 4$ and $4x = 8$, for example), or because they may be mutually exclusive (like $2x = 4$ and $4x = 7$)

1. With its length being fixed, how many degrees of freedom does each spin have?
2. How can one parametrise these degrees of freedom?
3. Take a set of $q$ mutually interconnected spins, for example a plaquette on a triangular planar lattice or a tetrahedron in three dimensional one, with, say, $q$ spins on it. Prove that, thanks to the constant modulus constraint on the spins, its energy can be rewritten as

$$H_{\text{plaq}} = -J \left( \sum_{i \in \text{plaq}}^{q} \vec{s}_i \right)^2 \tag{2}$$

up to a constant.

4. Which are the spin configurations that minimise this energy?
5. What is the energy of the plaquette ground states?

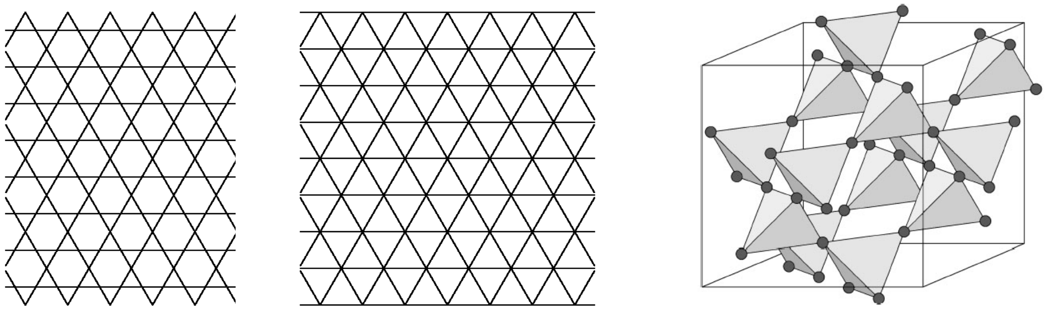

Figure 1: The Kagome and triangular two-dimensional lattices and the three-dimensional pyrochlore lattice.

6. Which are the constraints that this condition supplies? How may are them?
7. Using $F = D - K$, what is the value of $F$ for the plaquette?
8. Three of these degrees of freedom correspond to global rotations. How many remain?
9. Consider the case of a cluster of three spins. How many degrees of freedom does it have? And a tetrahedron?
10. Within the Maxwellian counting, which is the strategy to maximise the number of degrees of freedom $F$ of a set of interconnected spins? (notice that we are reasoning à la Pauling, focusing on a single lattice unit)
11. A lattice is made up of vertex-sharing clusters. Some two dimensional lattices are shown in Fig. 1. In the Kagome case each site belongs to only two triangles; in the triangular lattice, each site belongs to six. By this counting method, the most degenerate – and thus most frustrated or less constrained – lattice readily realisable in three dimensions or less is the one made up of vertex-sharing tetrahedra, the pyrochlore lattice in the same figure.

# References


[1] J. C. Maxwell, *On the calculation of the equilibrium and stiffness of frames*, Philosophical Magazine **27**, 294 (1864). The quote is "A frame of $s$ points in space requires in general $3s - 6$ connecting lines to render it stiff." For a two-dimensional pin-jointed structure, Maxwell's statement would be that a structure with $j$ joints would require, in general, $2j - 3$ bars to be rigid.

[2] J. Chalker and R. Moessner, *Low-temperature properties of classical geometrically frustrated antiferromagnets*, Phys. Rev. B **58**, 12049 (1998).

[3] R. Moessner and A. P. Ramírez, *Geometrical frustration*, Physics Today **59**, 2, 24 (2006).