RANDOM MATRICES

SYMMETRIC 2x2 MATRIX W/ $\mathbb{R}$ ENTRIES.

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

EIGENVALUES $\lambda_{1,2}$:

$$(a-\lambda)(c-\lambda) - b^2 = 0$$

$$ac + \lambda^2 - (a+c)\lambda - b^2 = 0$$

$$\lambda^2 - (a+c)\lambda - (b^2 - ac) = 0$$

$$\lambda_{1,2} = \frac{a+c \pm \sqrt{(a+c)^2 + 4(b^2 - ac)}}{2}$$

$$= \frac{a+c \pm \sqrt{a^2 + c^2 + \cancel{2}\,{}^{2}ac + 4b^2 - \cancel{4}\,{}^{2}ac}}{2}$$

$$= \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

$$s = \lambda_1 - \lambda_2 = \sqrt{(a-c)^2 + 4b^2}$$

NOTE THAT $s = 0 \iff a = c$ AND $b = 0$.

WHAT IS THE PROBABILITY DISTRIBUTION OF s, CLOSE TO $s \simeq 0$, KNOWING THE PROB DIST OF a, b, c?

FIRST ARGUMENT s IS LIKE A RADIUS ON A 2-DIM PLANE, IF I INTERPRET $a-c = x$ $2b = y$, SAY.

IF I CONSIDER $p(x,y)$ AS BEING FINITE AT $x \simeq 0$ $y \simeq 0$ (WHICH IS WHAT WE NEED TO GET CLOSE TO $s \simeq 0$)

WE CAN WRITE $p(s)$ AS

$$p(s) = \int dx\, dy\, \delta\left(s - \sqrt{x^2+y^2}\right) p(x,y)$$

IF WE GO TO POLAR COORDINATES

$$\rho = \sqrt{x^2+y^2}$$

$$x = \rho \cos\theta$$

$$y = \rho \sin\theta$$

THE INTEGRAL BECOMES

$$p(s) = \int_0^{2\pi} d\theta \int_0^{\infty} d\rho\, \rho\, \delta(s-\rho)\, p(\rho\cos\theta, \rho\sin\theta)$$

NOW, IF I USE THE ASSUMPTION $p(\rho\cos\theta, \rho\sin\theta) \simeq$ CTE. FOR $\rho \simeq 0$ TO EVALUATE THE INTEGRAL FOR $\boxed{s \simeq 0}$

$\Rightarrow$

$p(s \simeq 0) \propto s$ AND VANISHES CLOSE TO ZERO.

THIS IS A REALISATION OF THE

LEVEL REPULSION

WIGNER SURMISE

$$M = A + A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 2a_{11} & a_{12}+a_{21} \\ a_{12}+a_{21} & 2a_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

AND $m_{12} = m_{21}$.

WHAT IS THE pdf OF THE m_{ij}'s ?

$$p_M(m_{11}) = \int da_{11} \; \delta(m_{11} - 2a_{11}) \; p_A(a_{11})$$

CHANGE OF VARIABLES $2a_{11} = \overline{a}_{11}$

$$= \int \frac{d\overline{a}_{11}}{2} \; \delta(m_{11} - \overline{a}_{11}) \; p_A\left(\frac{\overline{a}_{11}}{2}\right)$$

$$p_M(m_{11}) = \frac{1}{2} \, p_A\left(\frac{m_{11}}{2}\right)$$

NOTATION: I CALL $p_A(y)$ THE PROB OF THE ELEMENTS OF THE A MATRIX AND $p_M(z)$ THE ONE OF THE ELEMENTS OF THE M MATRIX.

SIMILARLY FOR m_{22}:

$$p_M(m_{22}) = \frac{1}{2} \, p_A\left(\frac{m_{22}}{2}\right)$$

NOW, LET'S COMPUTE $p_M(m_{12})$:

$$p_M(m_{12}) = \int da_{12} \, da_{21} \; \delta\left(m_{12} - (a_{12}+a_{21})\right) \; p_A(a_{12}) \, p_A(a_{21})$$

$$p_M(m_{12}) = \int da_{12} \; p_A(a_{12}) \; p_A(m_{12} - a_{12})$$

NOTE THAT THE ELEMENTS OF M ARE NO LONGER <u>iid</u>.

SINCE, WHILE $p_M(m_{11}) = p_M(m_{22})$ (THEIR FCT DEP ON p_A)

$p_M(m_{12}) \neq p_M(m_{11})$.

TAKE NOW $P_A(a_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{a_{ij}^2}{2\sigma^2}}$ WITH $\sigma = 1$.

AND COMPUTE THE P_M's.

$$P_M(m_{11}) = \frac{1}{2} P_A\left(\frac{m_{11}}{2}\right)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{m_{11}^2}{2\cdot 4}}$$

$$= \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{m_{11}^2}{2\cdot 4}} \Rightarrow \sigma_{m_{11}}^2 = 4.$$

IT'S STILL A GAUSSIAN pdf THOUGH w/ A DIFF σ^2.

$$P_M(m_{12}) = \int \frac{da_{12}}{(\sqrt{2\pi})^2} e^{-\frac{a_{12}^2}{2}} e^{-\frac{(m_{12}-a_{12})^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{m_{12}^2}{2}} \int \frac{da_{12}}{\sqrt{2\pi}} e^{-\frac{2a_{12}^2}{2} + m_{12}a_{12}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{m_{12}^2}{2}} \int \frac{da_{12}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[2a_{12}^2 - 2m_{12}a_{12} + \frac{m_{12}^2}{2} - \frac{m_{12}^2}{2}\right]}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{m_{12}^2}{2}} e^{\frac{m_{12}^2}{4}} \int \frac{da_{12}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\sqrt{2}\,a_{12} - \frac{m_{12}}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{m_{12}^2}{4}} \int \frac{dy}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}y^2}$$

$$y = \sqrt{2}\,a_{12} - \frac{m_{12}}{\sqrt{2}}$$

$$dy = \sqrt{2}\,da_{12}$$

$$= \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{m_{12}^2}{2\cdot 2}}$$

$$\Rightarrow \sigma_{m_{12}}^2 = 2$$

THEREFORE WE HAVE

$$P_{M_{11}}(m_{11}) = \text{GAUSSIAN} = P_{M_{22}}(m_{22}) \qquad \text{WITH } \sigma^2 = 4$$

$$P_{M_{12}}(m_{12}) = \text{GAUSSIAN} = P_{M_{21}}(m_{21}) \qquad \text{WITH } \sigma^2 = 2$$

CHECK FROM CONVOLUTION OF GAUSSIANS

$m_{11} = 2\,a_{11}$ $\qquad z = ax + yb \qquad b = 0 \quad a = 2$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{4}} e^{\frac{-z^2}{2\sigma^2 4}} \qquad \text{WITH } \sigma = 1$$

$$p(z) = \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{z^2}{2\underset{1}{\overset{\shortparallel}{\sigma^2}} \cdot 4}} \implies \sigma^2 = 4$$

$m_{12} = a_{12} + a_{21}$ $\qquad a = 1 \quad b = 1$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{1+1}} e^{\frac{-z^2}{2\sigma^2(1+1)}} \qquad \sigma = 1$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} e^{-\frac{z^2}{2 \cdot 2}} \qquad \sigma^2 = 2$$

IT'S OK.

NOW, WHAT WE REALLY WANT IS

$$m_{11} - m_{22} = a - c = x$$
$$2\,m_{12} = 2b = y$$

FIRST REMARK x AND y DEPEND ON DIFFERENT ELEMENTS OF M

$\implies p(xy) = p(x)\,p(y)$

WHAT ARE $p(x)$ AND $p(y)$?

$$p(x) = p(x = m_{11} - m_{22})$$
$$= p(x = 2a_{11} - 2a_{22})$$

LIKE CONVOLUTION OF GAUSSIANS WITH $a=2$ AND $b=2$

$$p(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4+4}} e^{-\frac{x^2}{2(4+4)}}$$
$$= \frac{1}{\sqrt{2\pi \, 16}} e^{\frac{-x^2}{2\cdot 16}} \qquad \sigma^2 = 16$$

$$p(y) = p(y = 2m_{12}) = p(y = 2a_{12} + 2a_{21})$$

$$p(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4+4}} e^{-\frac{y^2}{2(4+4)}}$$
$$= \frac{1}{\sqrt{2\pi \, 16}} e^{-\frac{y^2}{2\cdot 16}} \qquad \sigma^2 = 16$$

NOW, WE HAVE TO COMPUTE

$$p(s) = \int dx\, dy\ \delta\left(s - \sqrt{x^2+y^2}\right) p(x,y)$$
$$= \int dx\, dy\ \delta\left(s - \sqrt{x^2+y^2}\right) p(x)\, p(y)$$
$$= \int dx\, dy\ \delta\left(s - \sqrt{x^2+y^2}\right) \frac{1}{\sqrt{2\pi\cdot 16}} e^{-\frac{x^2}{2\cdot 16}} \frac{1}{\sqrt{2\pi\cdot 16}} e^{-\frac{y^2}{2\cdot 16}}$$
$$= \frac{1}{2\pi\cdot 16} \int dx\, dy\ \delta\left(s - \sqrt{x^2+y^2}\right) \exp\left[-\frac{(x^2+y^2)}{2\cdot 16}\right]$$
$$= \frac{1}{2\pi\cdot 16} \int_0^{2\pi} d\theta \int_0^{\infty} d\rho\ \rho\ \delta(s-\rho)\ e^{-\frac{\rho^2}{2\cdot 16}}$$

$$p(s) = \frac{1}{\cancel{2\pi}\cdot 16}\ \cancel{2\pi}\ s\ e^{-\frac{s^2}{2\cdot 16}}$$

$$\boxed{p(s) = c \cdot s \cdot e^{-\frac{s^2}{2\cdot 16}}}$$

IF WE WANT TO WRITE THE PROB OF $s/\langle s \rangle$ WE FIRST NEED TO COMPUTE $\langle s \rangle$

$$\langle s \rangle = \int_0^\infty ds \; c\, s^2 e^{-\frac{s^2}{2\cdot 16}}$$

$$= -c \cdot \frac{d}{d\lambda} \int_0^\infty ds \; e^{-\frac{s^2}{2}\left(\frac{1}{16}+2\lambda\right)} \Big|_{\lambda=0} \qquad \sigma^2 = \frac{1}{\frac{1}{16}+2\lambda}$$

$$= -c \; \frac{d}{d\lambda} \sqrt{2\pi \frac{1}{\left(\frac{1}{16}+2\lambda\right)}} \Big|_{\lambda=0} \; 2$$

$$= -c \sqrt{2\pi} \left(-\cancel{\tfrac{1}{2}}\right) \left(\frac{1}{16}+2\lambda\right)^{-3/2} \cdot \cancel{2} \Big|_{\lambda=0} \; 2$$

$$= -c \sqrt{2\pi} \, (-1) \left(\frac{1}{16}\right)^{-3/2} 2$$

$$= c \sqrt{2\pi} \; (+1) \; 4^{+3} \cdot 2$$

$$= c \sqrt{2\pi} \; \frac{(+1)}{64^{-1}} \; 2 \; = 128 \frac{c\sqrt{2\pi}}{\cancel{2\cdot 16}} = \frac{c\sqrt{2\pi}}{\cancel{32}} \; 128$$

NORMALISATION CONSTANT

$$1 = \int_0^\infty ds \; p(s) = \int_0^\infty ds \; c \cdot s \; e^{\frac{-s^2}{2\cdot 16}} = 16\, c \implies \boxed{c = 1/16}$$

$$y = \frac{s^2}{2\cdot 16} \implies dy = \frac{\cancel{2} s \, ds}{\cancel{2}\cdot 16} = \frac{s \cdot ds}{16}$$

WHICH THE VALUE WE HAD OBTAINED $\rightarrow$ ok

$$1 = \int_0^\infty ds \; p(s) = \int_0^\infty dy \; c \; 16 \; e^{-y} = 16\, c \, (-1) e^{-y} \Big|_0^\infty$$

$$= 16\, c$$

THIS IMPLIES $\langle s\rangle = c\sqrt{2\pi}\;128 = \frac{1}{\cancel{16}_{\cancel{8}\,1}}\,\cancel{128}^{\cancel{64}\,8}\sqrt{2\pi}$

$$\langle s\rangle = 8\sqrt{2\pi}$$

$$p(s) \longrightarrow p\left(s/\langle s\rangle\right)?$$

$$z = s/\langle s\rangle \implies s = \langle s\rangle z$$

$$p(z) = \int ds\;\delta\left(s/\langle s\rangle - z\right)\,p(s)$$

$$y = \frac{s}{\langle s\rangle} - z$$

$$= \int \langle s\rangle\, d\cancel{z}y\;\delta(y)\;p\left((y+z)\langle s\rangle\right)$$

$$dy = \frac{ds}{\langle s\rangle}$$

$$(y+z)\langle s\rangle = s$$

$$= \langle s\rangle\, p(z\langle s\rangle)$$

$$= \langle s\rangle^2 c\; z\; e^{-\frac{z^2\langle s\rangle^2}{2\cdot 16}}$$

$$= 8^2\sqrt{2\pi}^2\,\frac{1}{16}\,z\;e^{-\frac{z^2\;64\cdot\cancel{2}\pi}{\cancel{2}\cdot 16}}$$

$$= \frac{\cancel{8}^{4}\,2\pi}{\cancel{2}}\,z\;e^{-\pi z^2 4}$$

$$\boxed{p(z) = 8\pi\, z\, e^{-4\pi z^2}}$$

$$z = s/\langle s\rangle$$

$$p(z) = \frac{\pi}{2}\,z\,e^{-\frac{\pi z^2}{4}}$$

THIS FORM SATISFIES $\langle z\rangle = 1$ & $\langle 1\rangle = 1$

4 a) $\mathrm{Tr}\, H^T H = \sum_i \sum_j H_{ji} H_{ji} = \sum_{ij} H_{ij}^2$

EITHER CHANGING NAMES $i \leftrightarrow j$
OR USING $H_{ij} = H_{ji}$ SYMM OF MATRIX

b) $\mathrm{Tr}\, H^T H = \sum_{ij} H_{ij}^2 = \sum_i H_{ii}^2 + \sum_{i \neq j} H_{ij}^2$

$= \sum_i H_{ii}^2 + 2 \sum_{i<j} H_{ij}^2$ (FACTOR 2 IN NUMERATOR)

c) $P(\{H_{ij}\}) \Rightarrow$ iid DIAGONAL ELEMENTS GAUSSIAN W/ VARIANCE σ^2 $\quad p(H_{ii}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{H_{ii}^2}{2\sigma^2}}$

$\Rightarrow$ NON-DIAG ELEMENTS CONST AS

$$H_{ij} = \frac{M_{ij} + M_{ji}}{2}$$

TO IMPOSE SYMMETRY CHOOSE TO INCLUDE 1/2 OR NOT ?

WITH M A MATRIX THAT IS NOT SYMM BUT $\mathbb{R}$. ELEM, GAUSSIAN DIST

W/ $\langle M_{ij}^2 \rangle = \sigma^2$

$$\langle H_{ij}^2 \rangle = \left\langle \frac{(M_{ij} + M_{ji})^2}{4} \right\rangle = \frac{1}{4}\left[\langle M_{ij}^2 \rangle + \langle M_{ji}^2 \rangle + 2 \langle M_{ij} M_{ji} \rangle \right]$$

$i \neq j$

$$= \frac{1}{4}\left[\sigma^2 + \sigma^2 + \underbrace{\langle M_{ij} \rangle}_{=0} \underbrace{\langle M_{ji} \rangle}_{=0} \right]$$

$$= \frac{1}{2} \sigma^2$$

HAVING DEFINED $H_{ij} = \frac{M_{ij} + M_{ji}}{2} \Rightarrow H_{ij}$ $i \neq j$ GAUSSIAN pdf W/ VARIANCE $\sigma^2/2$.

d) TRANSF. $H \to R^T H R$ $\quad H^T \to R^T H^T R$

$$Tr(H^T H) = Tr(R^T H^T R \, R^T H R)$$
$$= Tr(R R^T H^T H)$$
$$= Tr(H^T H)$$

e) $p(\{H_{ij}\}) = \prod_i p_i(H_{ii}) \prod_{i<j} p_{ij}(H_{ij})$

$$= \prod_i \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{H_{ii}^2}{2\sigma^2}} \prod_{i<j} \frac{1}{\sqrt{2\pi\,\sigma^2/2}} e^{-\frac{H_{ij}^2}{2\cdot\cancel{2}\sigma^2/\cancel{2}}\left(\cancel{\sigma^2/2}\right)}$$

$$= e^{-\frac{1}{2\sigma^2} Tr(H^T H)}$$

CHECK LAST LINE

$$\frac{1}{2\sigma^2} Tr(H^T H) = \frac{1}{2\sigma^2}\left[\sum_i H_{ii}^2 + 2\sum_{i<j} H_{ij}^2\right]$$

$$= \frac{1}{2\sigma^2}\sum_i H_{ii}^2 + \frac{1}{\sigma^2}\sum_{i<j} H_{ij}^2 \qquad \underline{\underline{ok}}.$$

f) ONCE $Tr\, H^T H$ WAS SHOWN TO BE INVARIANT

$\Rightarrow e^{-\frac{1}{2\sigma^2} Tr H^T H}$ IS INVARIANT TOO.

5) $$\rho_N(\lambda) = \frac{1}{N}\sum_{\mu=1}^{N} \delta(\lambda - \lambda_\mu)$$

b) SHOW $\frac{1}{N} Tr\, H_N^k = \frac{1}{N}\sum_{\mu=1}^{N} \lambda_\mu^k$

SIMPLY BECAUSE H IS DIAGONALISABLE VIA

$$P^{-1} H_N P = D_N$$

AND ONE CAN INSERT FACTORS $P^{-1}P$ IN BTW EACH OF THE k-FACTORS H TO OBTAIN

$$Tr\, H_N^k = Tr\, \underbrace{P^{-1} H_N P \cdots P^{-1} H_N P}_{k-\text{FACTORS}}$$

$$= Tr\, D^k = Tr \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

$$= Tr \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_N^k \end{pmatrix} = \sum_{\mu=1}^{N} \lambda_\mu^k$$

c) $\frac{1}{N}\sum_{\mu=1}^{N} \lambda_\mu^k = \frac{1}{N} Tr\, H_N^k$ SHOWN IN b)

= 'CAUSE OF Tr

NOW, STUDY $$Tr\, H_N^k = \sum_{i_1 \cdots i_k = 1}^{N} \underbrace{H_{i_1 i_2} H_{i_2 i_3} \cdots H_{i_k i_1}}_{k \text{ FACTORS}}$$

if $H_{ij} \to \frac{B_{ij}}{\sqrt{N}}$

THERE ARE k SUMS GOING FROM $i: 1 \to N$.
THE MULTI-SUM COULD GO AS N^k
(VERY DIVERGENT!) BUT IT DOESN'T.

$$Tr\, H_N^k = N^{-k/2} \sum_{i_1 \cdots i_k = 1}^{N} B_{i_1 i_2} \cdots B_{i_k i_1}$$

SINCE THE H_{ij} AND THE B_{ij} WILL BE GAUSSIAN RANDOM VARIABLES $\Rightarrow$ EACH INDEX SHOULD APPEAR AN EVEN # TIMES TO YIELD A NON-VANISHING RESULT. $\Rightarrow$ NOT SO MANY DIFF INDICES TO SUM OVER AND THE SUMS WILL BE NORMALISED TO YIELD A FINITE RESULT IN THE END.

THEN FROM $k \to k/2$ SUMS TO DO.

COMBINATORIAL FACTOR COUNTING # WAYS TO SET ALL TWO INDICES EQUAL AMONG k

$\Downarrow$

CATALAN # $C_{k/2}$.

CONVOLUTION OF GAUSSIANS

APPENDIX ON GAUSSIANS

$$z = ax + by \implies y = \frac{z - ax}{b}$$

$$p(z) = \int dx\, dy\; \delta(z - ax - by)\, p(x)\, p(y)$$

$$= \int \frac{dx}{b}\, p(x)\, p\left(\frac{z-ax}{b}\right)$$

IF $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$

$$p(z) = \int \frac{dx}{b} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}\left(\frac{z-ax}{b}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}\, b} \int \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\, e^{-\frac{z^2}{2\sigma^2 b^2}}\, e^{-\frac{a^2x^2}{2\sigma^2 b^2}}\, e^{\frac{axz}{b^2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}\, b} \int \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2} x^2 \left(1 + \frac{a^2}{b^2}\right)}\, e^{-\frac{z^2}{2\sigma^2 b^2}}\, e^{\frac{axz}{b^2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}\, b} \int \frac{dx}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}\left[x\sqrt{1+\frac{a^2}{b^2}} + \frac{z}{b^2}\frac{a}{\sqrt{1+\frac{a^2}{b^2}}}\right]^2}$$

$$e^{-\frac{z^2}{2\sigma^2 b^2}}\, e^{\frac{1}{2\sigma^2}\frac{z^2a^2}{b^4\left(1+\frac{a^2}{b^2}\right)}}$$

$$y = x\sqrt{1+\frac{a^2}{b^2}} - \frac{za}{b^2\sqrt{1+\frac{a^2}{b^2}}} \implies dy = \sqrt{1+\frac{a^2}{b^2}}\, dx$$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}\, b} \frac{1}{\sqrt{1+\frac{a^2}{b^2}}} \underbrace{\int \frac{dy}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}y^2}}_{1}\, e^{-\frac{z^2}{2\sigma^2 b^2}\underbrace{\left(1 - \frac{a^2}{b^2\left(1+\frac{a^2}{b^2}\right)}\right)}_{\frac{b^2 + \cancel{a^2} - \cancel{a^2}}{b^2+a^2}}}$$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}\, b\sqrt{1+\frac{a^2}{b^2}}}\, e^{-\frac{z^2}{2\sigma^2}\frac{1}{b^2+a^2}}$$

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}\,\cancel{b}} \frac{\cancel{b}}{\sqrt{a^2+b^2}} \; e^{-\frac{z^2}{2\sigma^2}\frac{1}{a^2+b^2}}$$

VARIANCE $\sigma^2(a^2+b^2)$ OF THE GAUSSIAN VARIABLE z.

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{a^2+b^2}} \; e^{-\frac{z^2}{2\sigma^2(a^2+b^2)}}$$