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REPLICAS FOR RFIM

$$H_h = -\frac{J}{N}\sum_{i\neq j} s_i s_j - \sum_i h_i s_i$$

$$P(h_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{h_i^2}{2\sigma^2}}$$

0 - RESCALE $h_i \to h \cdot h_i \Rightarrow P(h_i) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{h_i^2}{2}}$

1 - STARTING FROM THE LATTICE VERSION WHERE $\sum_{\langle ij\rangle}$ AND COUPLING STRENGTH IS $O(1) \Rightarrow$

WE RESCALE $\to \frac{J}{N}$ & TAKE $\sum_{i\neq j}$ TO DO THE MEAN-FIELD CASE.

2 - REPLICAS

$$[-\beta f]_h = \frac{1}{N}[\ln Z_h]_h = \frac{1}{N}\lim_{n\to 0}\left[\frac{Z_h^n - 1}{n}\right]_h$$

$$-\beta N [f]_h = \lim_{n\to 0}\frac{[Z_h^n]_h - 1}{n N}$$

$$[Z_h^n] = \left[\sum_{\{s_i^a\}} e^{-\beta H_h[\{s_i^a\}]}\right]_h$$

CALL THEM η_i

$$= \sum_{\{s_i^a\}} e^{\frac{\beta J}{N}\sum_{i\neq j}\sum_{a=1}^{n} s_i^a s_j^a}\left[e^{-\beta h\sum_i \sum_{a=1}^{n} h_i s_i^a}\right]_h$$

NOTE: SUM OVER REPLICA INDICES IN EXPONENTIAL

$$\left[e^{-\beta h \sum_i \eta_i \sum_{a=1}^{n} s_i^a} \right]_h = \prod_i \int \frac{d\eta_i}{\sqrt{2\pi}}\, e^{-\frac{\eta_i^2}{2} - \beta h \sum_i \eta_i \sum_{a=1}^{n} s_i^a}$$

$$= \prod_i e^{\frac{(\beta h)^2}{2} (s_i^a)^2}$$

$$[Z_h^n] = \sum_{\{s_i^a\}} e^{\frac{\beta J}{N} \sum_{a=1}^{n} \underbrace{\sum_{i \neq j} s_i^a s_j^a}_{\sum_{ij} s_i^a s_j^a - \sum_i (s_i^a)^2} + \frac{(\beta h)^2}{2} \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

NOTE: THE SUM IS WITHIN THE SQUARE $\Rightarrow$ REPLICAS INTERACT ?

$$= \sum_{\{s_i^a\}} e^{\frac{\beta J}{N} \sum_{a=1}^{n} \left(\sum_i s_i^a \right)^2 + \left[\frac{(\beta h)^2}{2} - \frac{\beta J}{N} \right] \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

ALTHOUGH BOTH PROP TO SAME $\sum_i (s_i^a)^2$ FACTOR, THE $1/N$ MAKES 2nd NEGLIGIBLE

$$\xrightarrow[N \gg 1]{} \sum_{\{s_i^a\}} e^{\frac{\beta J}{N} \sum_{a=1}^{n} \left(\sum_i s_i^a \right)^2 + \frac{(\beta h)^2}{2} \sum_i \left(\sum_{a=1}^{n} s_i^a \right)^2}$$

- NOW, USE HUBBARD-STRATONOVICH OR GAUSSIAN DECOUPLING

$$e^{\lambda b^2} = \int \frac{dz}{\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2} z^2 + \sqrt{2\lambda}\, bz - \lambda b^2}_{-\frac{1}{2}\left(z - \sqrt{2\lambda}\, b\right)^2} + \lambda b^2}$$

TO DECOUPLE THE 1st TERM IN THE EXPONENTIAL (i.e, FROM A DOUBLE SUM $\sum_{ij}$ TO A SINGLE ONE.

$$e^{\overset{\lambda}{\frac{\beta J}{N}} \overset{b^2}{\left(\sum_i s_i^a\right)^2}} = \int_{-\infty}^{\infty} \frac{dz_a}{\sqrt{2\pi}} \, e^{-\frac{1}{2} z_a^2 + \sqrt{\frac{2\beta J}{N}} \left(\sum_i s_i^a\right) z_a}$$

FOR EACH
a INDEX
$\Rightarrow$ NEED AN
INDEX a IN Z

AND $[Z^n]$ BECOMES

$$[Z_h^n] = \sum_{\{s_i^a\}} \int \frac{\prod_a dz_a}{\sqrt{2\pi}} \, e^{-\frac{1}{2} z_a^2 + \sqrt{\frac{2\beta J}{N}} \left(\sum_i s_i^a\right) z_a} \times e^{\frac{(\beta h)^2}{2} \sum_i \left(\sum_{a=1}^{n} s_i^a\right)^2}$$

FOR CONVENIENCE, I
FACTORIZED THE EXPONENTIALS

$$= \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{dz_a}{\sqrt{2\pi}} \, e^{-\frac{1}{2} \sum_{a=1}^{n} z_a^2} \sum_{\{s_i^a\}} e^{\sqrt{\frac{2\beta J}{N}} \sum_{a=1}^{n} s_i^a z_a + \frac{(\beta h)^2}{2} \sum_{a=1}^{n} s_i^a \sum_{b=1}^{n} s_i^b}$$

IT'S A MULTIPLE
INTEGRAL, THERE
ARE n OF THEM

SINCE THE i-SITES ARE
DECOUPLED $\Rightarrow$ IT'S A MUTE
INDEX AND ONE CAN ELIMINATE
IT, AND CONSIDER ONLY ONE
SUM $\sum_{s^a}$ TO THE POWER N:

$$\left(\sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{N}} \sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2} \sum_{a=1}^{n} s^a \sum_{b=1}^{n} s^b} \right)^N$$

AND ONE WRITES $[Z^n]$ IN A MORE COMPACT FORM:

$$[Z_h^n] = \left(\frac{1}{2\pi}\right)^{n/2} \int \prod_{a=1}^{n} dz_a \, e^{-\frac{1}{2} \sum_a z_a^2 + N \ln Z_1(\{z_a\})}$$

WITH
$$Z_1(\{z_a\}) = \sum_{\{s^a\}} e^{\sqrt{\frac{2\beta J}{N}} \sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

WE SEEM TO HAVE USELESSLY TRADED THE INT BTW $(ij) \to$ INT BTW (ab) AND NOW WE HAVE ANOTHER PART SUM TO COMPUTE. LET'S GO AHEAD.

- RESCALE $z_a \longrightarrow \sqrt{N}\, z_a$

$$[Z_h^n] = \left(\frac{N}{2\pi}\right)^{n/2} \int \prod_{a=1}^{n} dz_a \; e^{-N\left[\frac{1}{2}\sum_a z_a^2 - \ln Z_1(\{z_a\})\right]}$$

$$Z_1(\{z_a\}) = \sum_{\{s^a\}} e^{\sqrt{2\beta J} \sum_{a=1}^{n} s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

- NOW, $\to N \to \infty$ AND ONE CAN EVALUATE THE INT BY STEEPEST DESCENT (EVEN THOUGH THERE ARE n VARIABLES OVER WHICH OPTIMISE). ASSUME THAT THE EXTREMA WE ARE LOOKING FOR ARE SUCH THAT

$$z_a^{EXT} = z^{EXT} \quad \text{INDEP. OF } a \quad \Longrightarrow$$

$$z_a^{EXT} = \left.\frac{\frac{\partial Z_1(\{z_a\})}{\partial z_a}}{Z_1(\{z_a\})}\right|_{z_a^{EXT}}$$

SOLUTION INDEP OF a

$$= \frac{\sum_{\{s^a\}} \left(\sqrt{2\beta J}\, s^a\right) e^{\sqrt{2\beta J} \sum_a s^a z_a + \frac{(\beta h)^2}{2} \sum_{ab} s^a s^b}}{Z_1(\{z_a\})}$$

WE SEE THAT $\quad z_a^{EXT} = \sqrt{2\beta J} \quad \langle s^a \rangle_{Z_1(\{z_a^{EXT}\})}$

IF, MOREOVER, $\quad z_a^{EXT} = z^{EXT} \quad \Rightarrow$

$$z^{EXT} = \sqrt{2\beta J} \quad \langle s^a \rangle_{Z_1(\{z^{EXT}\})} \qquad \text{INDEP OF } a.$$

REPLACE NOW z_a BY z^{EXT} IN $[Z_h^n]$ AND, FURTHERMORE, IN $[f_R]$:

$$-\beta\,[f_R] = \lim_{\substack{n\to 0\\ N\to\infty}} \frac{1}{nN}\left\{ \underbrace{\left(\frac{N}{2\pi}\right)^{n/2}}_{\text{GOES TO 1 FOR } n\to 0} e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln Z_1\left(z^{EXT}\right)\right]} - 1 \right\}$$

$$= \lim_{\substack{n\to 0\\ N\to\infty}} \frac{1}{nN}\left\{ e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln Z_1\left(z^{EXT}\right)\right]} - 1 \right\}$$

ONE CAN DECOUPLE NOW THE REPLICA INDICES IN $Z_1(z^{EXT})$ TO COMPUTE THE SUM $\sum_{\{s^a\}}$ EXPLICITLY.

$$Z_1(z^{EXT}) = \sum_{\{s^a\}} e^{\sqrt{2\beta J}\, z^{EXT} \sum_a s^a + \frac{(\beta h)^2}{2} \sum_{ab=1}^{n} s^a s^b}$$

$$= \sum_{\{s^a\}} e^{\sqrt{2\beta J}\, z^{EXT} \sum_a s^a} \int \frac{d\omega}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\omega^2} \prod_a e^{\omega\beta h s^a}$$

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NOW WE HAVE DECOUPLED THE $\{s^a\}_s$ AND WE CAN COMPUTE THE $\sum_{\{s^a\}}$:

$$\sum_{\{s^a\}} e^{\sqrt{2\beta J}\, z^{EXT} s^a + w\beta h s^a}$$

FOR EACH $a = 1 \ldots n$ IDENTICAL PART. SUMS, LEADING TO

$$\Longrightarrow \left[2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + w\beta h\right)\right]^n \Longrightarrow$$

$$Z_n(z^{EXT}) = \int \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{1}{2}w^2} \underbrace{\left[2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + w\beta h\right)\right]^n}_{e^{n \ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + w\beta h\right)\right)}}$$

AND, FINALLY,

$$-\beta\,[f_{en}] = \lim_{\substack{n\to 0\\ N\to\infty}} \frac{1}{nN}\left\{ e^{-N\left[\frac{1}{2}n\left(z^{EXT}\right)^2 - \ln\int \frac{dw}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} \underbrace{e^{n\ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + w\beta h\right)\right)}}_{A}\right]} - 1 \right\}$$

AND WE WILL USE $\langle\langle \cdot \rangle\rangle$ TO INDICATE THE AVERAGE OVER THE GAUSSIAN WEIGTH OF THE NEW AUXILIARY VARIABLE w.

FIRST WE TAKE $n \to 0$ IN THE EXPONENTIAL:

$$A \equiv e^{n\ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EST} + w\beta h\right)\right)} \simeq 1 + n\ln\left(2\,\mathrm{ch}\left(\sqrt{2\beta J}\, z^{EXT} + w\beta h\right)\right)$$

$\rightarrow$ THE 1 GOES BELOW $\langle\langle \cdot \rangle\rangle$ $\nearrow$ $\langle\langle 1 \rangle\rangle = 1$

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→ THE SECOND TERM REMAINS AND WE HAVE

$$\ln \langle\langle \Big[\overbrace{1 + n \ln \Big(2\, ch \big(\sqrt{2\beta J}\; z^{EST} + \beta h w \big) \Big)}^{A} \Big] \rangle\rangle$$

$$\xrightarrow[n \to 0]{} n \langle\langle \ln \Big(2\, ch \big(\sqrt{2\beta J}\; z^{EXT} + \beta h w \big) \Big) \rangle\rangle$$

WHERE WE USED $\ln(1+\epsilon) \simeq \epsilon$

SO, WITH THE $n \to 0$ WE GOT RID ON ONE $\ln$ FUNCTION

NOW, THE FULL EXPONENT IS PROP TO n AND WE TAKE AGAIN $n \to 0$:

$$e^{-Nn \text{ Exponent}} \simeq 1 - Nn \text{ Exponent}$$

THE 1 CANCELS THE (-1) AND THE Nn SIMPLIFY WITH THE $\frac{1}{n}$:

$$-\beta [f_h] = \underbrace{\lim_{\substack{n \to 0 \\ N \to \infty}}}_{} - \left\{ + \frac{1}{2} \left(z^{EXT} \right)^2 - \langle\langle \ln \Big(2\, ch \big(\sqrt{2\beta J}\; z^{EXT} + \beta h w \big) \Big) \rangle\rangle \right\}$$

THERE'S NO MORE DEPENDENCE ON (n, N)
⟹ IRRELEVANT

$$\beta [f_h] = \frac{1}{2} \left(z^{EXT} \right)^2 - \langle\langle \ln \Big(2\, ch \big(\sqrt{2\beta J}\; z^{EST} + \beta h w \big) \Big) \rangle\rangle$$

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RECALLING THAT $z^{EXT} = \sqrt{2\beta J}\, m$, WE REPLACE AND

$$[f_J] = Jm^2 - \frac{1}{\beta} \langle\langle \ln \left(2\, ch \left(2\beta J m + \beta h w \right) \right) \rangle\rangle$$

AVERAGED FREE-ENERGY DENSITY

cfr W/ ISING MODEL IN A FIELD.

$$\frac{\partial [f_J]}{\partial m} = 0 \Rightarrow 0 = \cancel{2}Jm - \frac{1}{\cancel{\beta}} \langle\langle \frac{\cancel{2}\, sh(2\beta J m + \beta h w)}{\cancel{2}\, ch(2\beta J m + \beta h w)} \cancel{2\beta J} \rangle\rangle$$

$$m_{EXT} = \langle\langle th \left(2\beta J m_{EXT} + \beta h w \right) \rangle\rangle$$

SADDLE-POINT EQ.

- $m_{EXT} = 0$ IS A SOLUTION $\forall\, (\beta J, \beta h)$

  INDEED $0 = \langle\langle th(\beta h w) \rangle\rangle$ AND THE RHS $= 0$ SINCE $P(w)$ IS SYMM WRT 0 WHILE $th\, \beta h w$ IS NOT.

$\boxed{m_{PM} = 0}$ PARAM SOLUTION.

- ASSUME A 2nd ORDER PHASE TRANSITION, IE, WITH $m_{FM} \simeq 0$ CLOSE TO T_c & EXPAND THE th

$$m_{EXT} \approx \left\langle\!\left\langle \, th\,(\beta h w) + \frac{ch^2 \beta h w - sh^2 \beta h w}{ch^2\, \beta h w}\, 2\beta J\, m_{EXT} \right\rangle\!\right\rangle$$

$$m_{EXT} = \left\langle\!\left\langle \cancel{th\,\beta h w} \right\rangle\!\right\rangle^{0} + 2\beta J\, m_{EXT} \left\langle\!\left\langle \frac{1}{ch^2\, \beta h w} \right\rangle\!\right\rangle$$

$$\Rightarrow \quad \boxed{1 = 2\beta_c J \left\langle\!\left\langle \frac{1}{ch^2\, \beta_c h w} \right\rangle\!\right\rangle}$$

IT'S A CONDITION ON β_c
GIVEN J AND σ_h THAT IS
IMPLICIT IN THE $p(h) \to p(w)$ WITH UNIT VARIANCE BUT
THAT WE HAVE TO USE
h FACTOR IS $h = \sigma_h$ OUTSIDE

PARAMETERS $(\beta_c J)$, $(\beta_c h)$
IN THIS WRITING

OR $\quad x_c = \beta_c J \qquad y_c = \beta_c h \quad$ OR $\quad y_c = \frac{\beta_c}{J} J h$

$$= (\beta_c J)\left(\frac{h}{J}\right)$$

$$= x_c \left(\frac{h}{J}\right)$$

$$\boxed{1 = 2\beta_c J \left\langle\!\left\langle \frac{1}{ch^2 (\beta_c J)\left(\frac{h_c}{J}\right)} \right\rangle\!\right\rangle}$$