Advanced Statistical Physics

Exam

January 2019

Surname :
Name :
Master :

1. The ergodic hypothesis.

Have a look at Fig. 1. In each panel, thirty trajectories of a one dimensional stochastic process are displayed (with the same data points and colour). The data correspond to $x(t)$, the time dependence of the stochastic variable x, and the processes are different in (a) and (b). In each case, one trajectory is signalled out with a different colour (green).

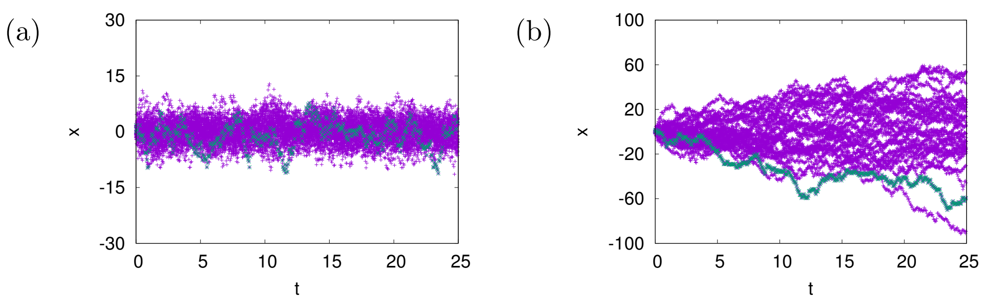


Figure 1: Thirty trajectories of one one-dimensional stochastic process in (a) and other thirty trajectories of a different one-dimensional stochastic process in (b), in all cases starting from $x(0) = 0$, and evolving in the course of time. All trajectories are shown with the same colour and in each panel one is highlighted with a different colour.

What would you conclude about the *ergodic hypothesis* for the data in panel (a)? Explain.

What would you conclude about the *ergodic hypothesis* for the data in panel (b)? Explain.

2. (In) equivalence of ensembles.

Consider a system of N Ising spins, and potential energy given by the sum over pairs of spins of a two-body potential of ferromagnetic kind

$$H_J[\{s_i\}] = -J \sum_{i \neq j} s_i s_j \tag{1}$$

with the coupling constant $J > 0$.

Is this energy *extensive*? Justify the answer with one equation and explain.

Is this energy *additive*? Justify the answer with one equation and explain.

What would you conclude in the case $J < 0$?

Explain why the violation of these properties may affect the equivalence of ensembles.

Do you expect in equivalence of ensembles in this model?

3. A one dimensional Ising model.

Consider the one dimensional ferromagnetic Ising model with two-body interactions decaying algebraically with distance:

$$H_J[\{s_i\}] = -\sum_{i=1}^{N}\sum_{j=1}^{N-i} J_{i\,i+j}\, s_i s_{i+j} \qquad\qquad J_{i\,i+j} = J/|r_{i\,i+j}|^{1+\sigma} = J/|aj|^{1+\sigma} \ . \tag{2}$$

The variables $s_i = \pm 1$ indicate the Ising spins, the index $i = 1, \dots N$ labels the lattice sites and therefore the spins placed on them, the coupling constant is positive, $J > 0$, $r_{i\,i+j}$ is the distance between the sites i and $i + j$, a is the lattice spacing, and σ is a real parameter that controls the decay of the interaction strength that we will take to be $\sigma > -1$. We will analyse the statistical properties of the model in contact with a thermal bath at temperature T.

Which phase(s) do you expect at $T = 0$?

Write the ground state energy

Which phase(s) do you expect at $T \to \infty$?

Which are the simplest excitations over the ground state configuration that you can imagine? Make a drawing to explain the answer.

Give an expression for the difference, ΔE, between the energy of a configuration with one domain wall placed between the sites n and $n+1$ and the energy of the ground state (choose the case in which the two configurations coincide on the *left* of the broken link).

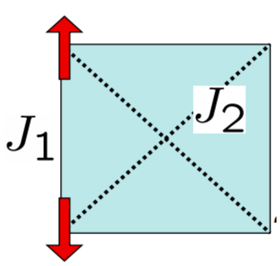
Evaluate the energy gap ΔE in a continuum limit such that $a \sum_k \to \int dx$ and $L = aN$ the length of the chain diverges, $L \to \infty$.

What do you conclude concerning the possibility of having an ordered state at low (but non-vanishing) temperatures from the expression found?

Which kind of argument have you applied here?

4. Fustration

Define frustration



$$H = -J_1(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_1 s_4) - J_2(s_1 s_3 + s_2 s_4)$$

Figure 2: A square plaquette with Ising spins placed on the vertices and nearest and next-nearest interactions with strengths J_1 and J_2, respectively. Two Ising spins at the left-most sites, that we label s_1 and s_2, are shown as an example. Other two Ising spins are placed on the upper right (s_3) and lower right (s_4) vertices. The plaquette's energy is given in the expression for H next to it.

Determine J_1, J_2 so that the plaquette shown in Fig. 2 is not frustrated. Give its ground state configuration, energy and entropy.

Give the simplest choice of J_1 and J_2 so that the plaquette is frustrated and give its ground state configuration, energy and entropy.

Which is the main effect of frustration in this example? Is this general?

5. Random Matrices

Take a two by two Hermitian matrix H. Assume that real and imaginary parts of all its entries are independent identically distributed random variables and take their probability distribution function to be Gaussian with zero mean and the same variance.

Find an expression for the probability distribution of the level spacings. Write in the box the main steps in the derivation.

Estimate its behaviour at small values of the argument.

Does the form found conform with the Wigner surmise?

What is the meaning of this statement concerning the level spacing of large random Hermitian matrices?

How does the result compare to the one known for symmetric real matrices (the GOE)?

5. A model for hetero proteins

A popular model of a (hetero)protein mimics it as a chain (a polymer) of equally spaced pair-interacting monomers that sit on the vertices of a regular cubic lattice (see the left panel in Fig. 3 for an example in two dimensions). More precisely, the Hamiltonian is

$$H = -\sum_{i\neq j} B_{ij}\, \Delta(|\vec{r}_i - \vec{r}_j|) \ , \tag{3}$$

where $\vec{r}_i$ is the position of the ith monomer, the sum runs over all pairs of monomers in the protein, $\Delta(|\vec{r}_i - \vec{r}_j|)$ is a rapidly decaying function of distance (and one can even take it to be different from zero only on neighbouring sites on the lattice), and B_{ij} are quenched random variables taken from, say, a Gaussian probability distribution with zero mean and variance properly normalised.

The numerical evaluation of the fluctuations induced by the quenched randomness on the properties of the free-energy are shown in Fig. 3. The notation here is such that angular brackets denote average over the quenched randomness and $\langle \delta F^2 \rangle = \langle (F - \langle F \rangle)^2 \rangle$.

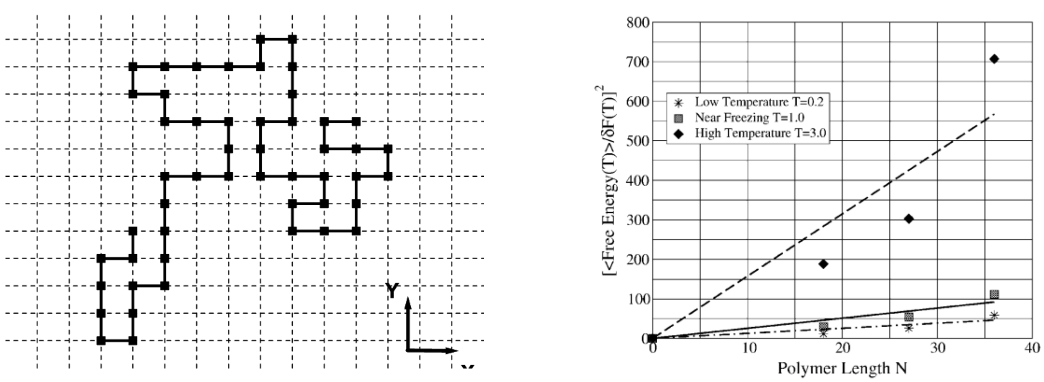


Figure 3: The averaged free-energy density over its standard deviation, the ratio squared, as a function of the length of the chain, given in terms of the number of monomers, for three temperatures given in the key. Figure taken from J. Chuang, A. Yu. Grosberg, and M. Kardar, Phys. Rev. Lett. 87, 078104 (2001).

What do you conclude on the self-averaging property of the free-energy density based on these data? Explain.

Which are the main effects of quenched randomness on a phase transition?

Explain the difference between “annealed” and “quenched” disorder.

Which is the main difference between “weak” and “strong” quenched disorder?

Mention a criterium to know whether weak disorder may change the critical exponents in a second order phase transition.