Advanced Statistical Physics

TD3 Frustration & disorder

November 2017

Degeneracy of frustrated magnets

The first exercise is an application of Maxwell's counting argument for elastic systems [1] to estimate the ground state degeneracy of the Heisenberg anti-ferromagnet [2, 3]

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \ . \tag{1}$$

The sum over pairs of spins ij runs over nearest neighbours on a generic lattice. The negative exchange energy $J < 0$ favours antiparallel alignment of the nearest neighbour three component Heisenberg spins $\vec{s} = (s^x,\ s^y,\ s^z)$ of fixed length $|\vec{s}|$.

The number of degrees of freedom in the ground state, F, is estimated to be $D - K$, where D is the total number of degrees of freedom of the spins and K the number of constraints that must be satisfied to put the system into a ground state. The idea essentially comes from linear algebra, in which a system of K equations for D variables is expected to have a solution space of dimension $F = D - K$. Here, as there, the expectation may be wrong because the constraints imposed by the equations may not be independent (like $2x = 4$ and $4x = 8$, for example), or because they may be mutually exclusive (like $2x = 4$ and $4x = 7$)

1. With its length being fixed, how many degrees of freedom does each spin have?
2. How can one parametrise these degrees of freedom?
3. Take a set of q mutually interconnected spins, for example a plaquette on a triangular planar lattice or a tetrahedron in three dimensional one, with, say, q spins on it. Prove that, thanks to the constant modulus constraint on the spins, its energy can be rewritten as
$$H_{\rm plaq} = -J \left(\sum_{i \in \, \rm plaq}^{q} \vec{s}_i \right)^2 \tag{2}$$
up to a constant.
4. Which are the spin configurations that minimise this energy?
5. What is the energy of the plaquette ground states?

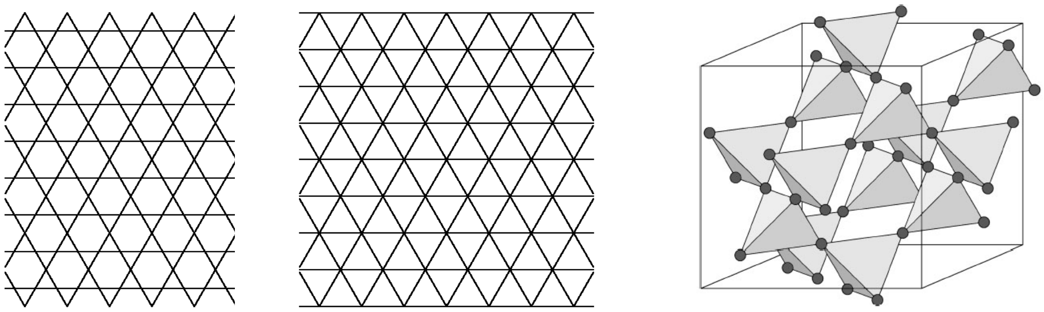

Figure 1: The Kagome and triangular two-dimensional lattices and the three-dimensional pyrochlore lattice.

6. Which are the constraints that this condition supplies? How may are them?

7. Using $F = D - K$, what is the value of F for the plaquette?

8. Three of these degrees of freedom correspond to global rotations. How many remain?

9. Consider the case of a cluster of three spins. How many degrees of freedom does it have? And a tetrahedron?

10. Within the Maxwellian counting, which is the strategy to maximise the number of degrees of freedom F of a set of interconnected spins? (notice that we are reasoning à la Pauling, focusing on a single lattice unit)

11. A lattice is made up of vertex-sharing clusters. Some two dimensional lattices are shown in Fig. 1. In the Kagome case each site belongs to only two triangles; in the triangular lattice, each site belongs to six. By this counting method, the most degenerate – and thus most frustrated or less constrained – lattice readily realisable in three dimensions or less is the one made up of vertex-sharing tetrahedra, the pyrochlore lattice in the same figure.

The Random Field Ising model in the mean-field approach

The second exercise is an application of the naive mean-field method to the study of the random field Ising model. Indeed, following Schneider & Pytte [4] you can evaluate the disordered free-energy density of the random field Ising model in finite dimensions and study the phase diagram at it predicts.

Correlation functions

We now analyse the distribution of the spatial correlation of Ising chains with random interactions [5]. We saw in the lectures that

$$\langle s_i s_{i+R} \rangle = \prod_{j=i}^{i+R-1} \tanh(\beta J_j) \ , \tag{3}$$

a product of *i.i.d.* random variables.

1. Similarly to what we have done in the analysis of the free-energy density, take the logarithm, and use the resulting form to relate the disorder average to the most probable or typical value.

2. Compute the disorder averaged correlation $[\langle s_i s_{i+R} \rangle]$. Is the result the same as the one derived in the previous item?

3. Take a Gaussian probability distribution of the interaction strengths J_i with mean J_0 and variance J^2 and give some concrete values to these parameters. Compute the two correlation functions numerically as a function of the spin distance R. Conclude.

Depending on the physical quantities one wants to study, the correlation function has to be calculated following one or the other procedure. For example, if the spin s_i is fixed to be $+1$ and one asks how the magnetisation $\langle s_{i+R} \rangle$ decreases as a function of the distance R, one must use the first definition. On the other hand, if one wants to calculate the average magnetic susceptibility by summing the correlation functions, one has to use the second one. The difference between the two can be a huge factor.

Note the implication of the observation made in this example for numerical simulations: in Monte Carlo simulations it is impossible to measure $[\langle s_i s_{i+R} \rangle^2]$, almost all the data being concentrated around $\langle s_i s_{i+R} \rangle^2_{\text{typ}}$.

References

[1] J. C. Maxwell, *On the calculation of the equilibrium and stiffness of frames*, Philosophical Magazine **27**, 294 (1864). The quote is "A frame of s points in space requires in general $3s-6$ connecting lines to render it stiff." For a two-dimensional pin-jointed structure, Maxwell's statement would be that a structure with j joints would require, in general, $2j-3$ bars to be rigid.

[2] J. Chalker and R. Moessner, *Low-temperature properties of classical geometrically frustrated antiferromagnets*, Phys. Rev. B **58**, 12049 (1998).

[3] R. Moessner and A. P. Ramírez, *Geometrical frustration*, Physics Today **59**, 2, 24 (2006).

[4] T. Schneider and E. Pytte, *Random-field instability of the ferromagnetic state*, Phys. Rev. B **15**, 1519 (1977).

[5] B. Derrida and H. Hilhorst, *On correlation functions in random magnets*, J. Phys. C: Solid State Phys. **14**, L539 (1981).