Advanced Statistical Physics

TD2 Random matrix Theory

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We shall study some properties of random symmetric matrices with elements taking real values.

1. Consider a 2×2 symmetric matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \tag{1}$$

   (a) Compute the eigenvalues λ_1 and λ_2.

   (b) Compute their difference $s = \lambda_1 - \lambda_2$.

   (c) Find the conditions on the matrix elements to have degenerate eigenvalues, in other words, vanishing level spacing, $s = 0$.

   (d) Write the level spacing s as a distance on a two dimensional plane. Suppose that the joint probability distribution of the "coordinates" x and y on this plane does not diverge close to the origin. Prove that the probability distribution of the level spacing vanishes close to zero. This is a manifestation of the level spacing repulsion.

2. Imagine now that the matrix M has been constructed as $M = A + A^T$ with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \Rightarrow \qquad M = \begin{pmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{pmatrix} \tag{2}$$

   and take the real elements of A as independent identically distributed random variables with probability $p_A(a_{ij})$.

   (a) Compute the probability distribution of the four elements m_{ij} that we will call $p_{M_{ij}}(m_{ij})$. What do you note in these distributions?

   (b) Draw the real elements of A from a Gaussian probability distribution with zero mean and unit variance. Compute the probability distributions of the elements of the matrix M in this case. (Hint: as you shall use the probability of a linear combination of two Gaussian random variables in several steps of this and the following questions, first establish the form of $p(z)$ for $z = ax + by$, with a and b real parameters and x and y i.i.d. Gaussian random variables with zero mean and variance σ^2.)

(c) Calculate $p(s)$.

(d) Prove that $\tilde{p}(s/\langle s\rangle)$ with $\langle s\rangle = \int ds\, s\, p(s)$ satisfies the Wigner surmise.

3. Generate an ensemble with $\mathcal{N} = 1000$ Gaussian Orthogonal Eensemble (GOE) matrices constructed as $M = A + A^T$, and A matrices of size $N = 2, 4, 10$ with elements drawn from a Gaussian probability distribution for zero mean and unit variance.
   (a) For $N = 2$ verify numerically the results of the previous item.
   (b) In general, find the N eigenvalues λ_n, with $n = 1, \ldots, N$ of each matrix.
   (c) Sort them in increasing order.
   (d) Find the difference between neighbouring eigenvalues $s_n = \lambda_{n+1} - \lambda_n$.
   (e) Calculate the mean splitting $\langle s\rangle \equiv N^{-1} \sum_{n=1}^{N} s_n$.
   (f) Plot a histogram of the eigenvalue splittings divided by the mean splitting, $\tilde{s}_n = s_n/\langle s\rangle$ with bin size small enough to see some of the fluctuations. (Hint: Debug your work with $M = 10$, and then change to $M = 1000$.)

Références

[1] P. Chau Huu-Tai, N. A. Smirnova and P. Van Isacker, *Generalised Wigner surmise for* 2×2 *random matrices*, J. Phys. A: Math. Gen. **35**, L199 (2002).

[2] M. V. Berry and P. Shukla, *Spacing distributions for real symmetric* 2×2 *generalized Gaussian ensembles* J. Phys. A: Math. Gen. **42**, 485102 (2009).