Advanced Statistical Physics: Random matrices

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1 Introduction

A *random matrix* is a matrix some or all of whose elements are random variables, drawn from a probability distribution. Random matrix theory is a branch of mathematics but it is also applied to describe numerous physical systems, some of which we will discuss here.

Random matrices were first used in the early 1900's in the study of the *statistics of population characteristics* by Wishart [1] (for example, the analysis of correlations between pairs of features of the population, say height and income..., were set in matricial form). More details on these applications can be found in [2]. (This paper was published in a journal called Biometrika, devoted to the statistical studies of the physical traits of living organisms. If you are interested in history, have a look at the role played by some of the articles published in this journal in what is called *scientific racism*.)

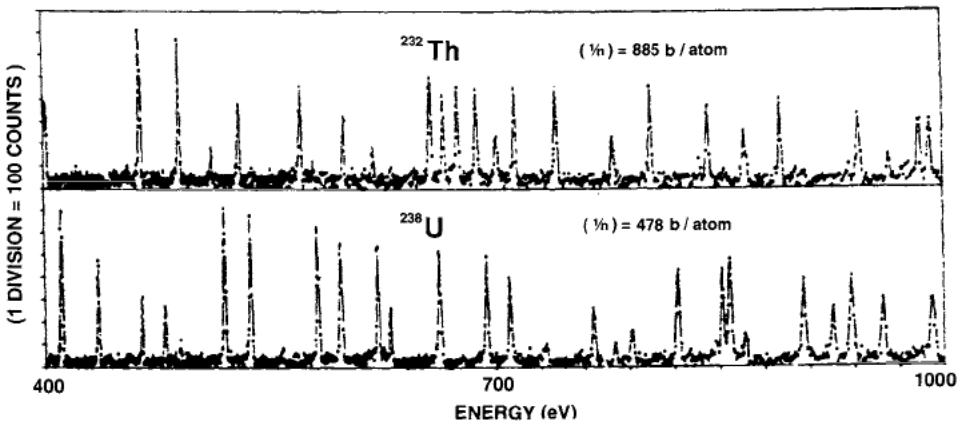

Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Figure 1.1: This figure and caption were copied from Mehta's book [4].

The upsurge of physicists' interest in Random Matrix Theory came with its very successful application in the theory of *heavy nuclei*. Take Uranium. It is an atom with 92 protons, 92 electrons, and common isotopes have more than 140 neutrons. Its nucleus is definitely very heavy, with over 200 protons and neutrons, each subject to and contributing to complex forces. If we completely understood the theory of the nucleus, we could predict all the energy levels of the spectrum; this, of course is not feasible.

Some insights into the nuclear structure of heavy nuclei were obtained with *scattering experiments* that consist in shooting neutrons (or protons) into the nucleus. The large numbers of energy levels of the nuclei appear in experimental data as peaks of the diffusion rate of the neutrons as a function of the energy, see Fig. 1.1. In the 50s data of this kind called the attention of theoreticians who tried to understand not really the low lying levels but the structure of the high energy spectra.

Since it appears to be impossible to derive the spectra of heavy nuclei one by one in full detail, the paradigm changed completely and the "revolutionary" idea put forward by Wigner and Dyson, mainly, was to construct a *statistical theory of energy levels*. In Dyson's words: "such theory will not predict the detailed sequence of levels in any one nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus which is too complicated to be understood in detail." More still, the idea is

> to develop a "new kind of statistical mechanics in which we renounce exact knowledge not of the state of a system but of the nature of the system itself. We picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem is then to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable".

This is the program initiated by Wigner [3] in the 50s. Since the overall energy scale is set by details of the nucleus at hand, the most generic or universal question to ask is what is the statistics of the separations between adjacent energy levels on a scale small compared to the energy. This is the *level spacing statistics*. The main and quite surprising content of this approach is that the statistical properties of the level statistics is related to the one between eigenvalues of conveniently chosen *random matrices*.

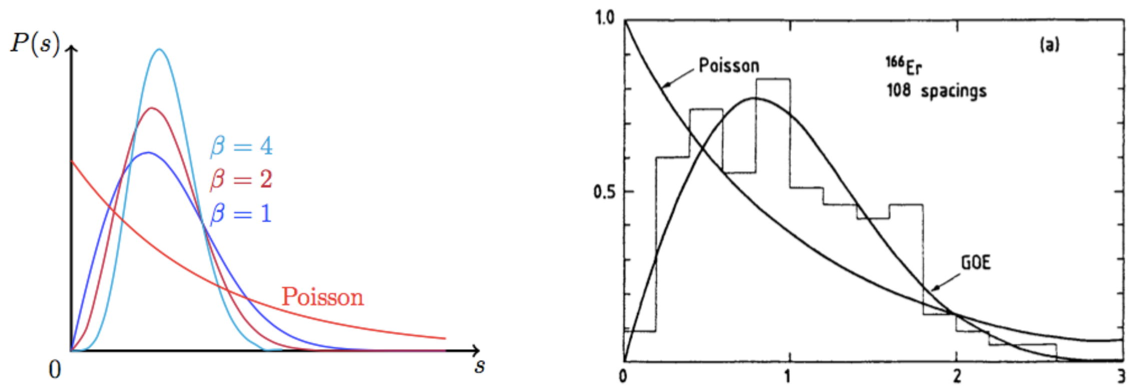

Figure 1.2: Left: the Poisson and Wigner distribution functions for the level spacings (the parameter β distinguished different random matrix ensembles). Figure taken from [5] Right: experimental data for the 108 level spacings in ^{166}Er (Erbium) favourably compared to the one of the Gaussian Orthogonal Ensemble and unfavourably compared a Poisson distribution.

While independent random energy levels would yield a *Poisson distribution* of the distances between neighbouring energy levels, the experimental data were better described by what is called *Wigner's surmise* [3]. The two probability densities are

$$p_{\text{Poisson}}(s) = e^{-s} \qquad\qquad p_{\text{Wigner}}(s) = c_\beta s^\beta e^{-a_\beta s^2} \tag{1.1}$$

where $\beta = 1,\ 2,\ 4$, see Fig. 1.2, depending on the symmetry properties of the nucleus under time-reversal and spin rotation and the parameters a_β and c_β are fixed by $\int ds\, p(s) = \int ds\, s\, p(s) = 1$. (Note that, *a priori* β has nothing to do with the inverse temperature in this Chapter!) The main difference between the two forms is in the two extremes $s \simeq 0$ and $s \gg 1$, see Fig. 1.2. While Poisson's pdf is finite for $s \to 0$, Wigner's pdf has a dip close to $s = 0$, a feature that is associated with what is called *level repulsion*. In the large s limit, both distribution functions decay fast, with Wigner's decaying faster than Poisson's [4, 5, 6, 7]. We will discuss how one goes from the quantum mechanical problem to the diagonalisation of random matrices under certain conditions, and show a proof of these results below and in a particularly simple case in TD 2.

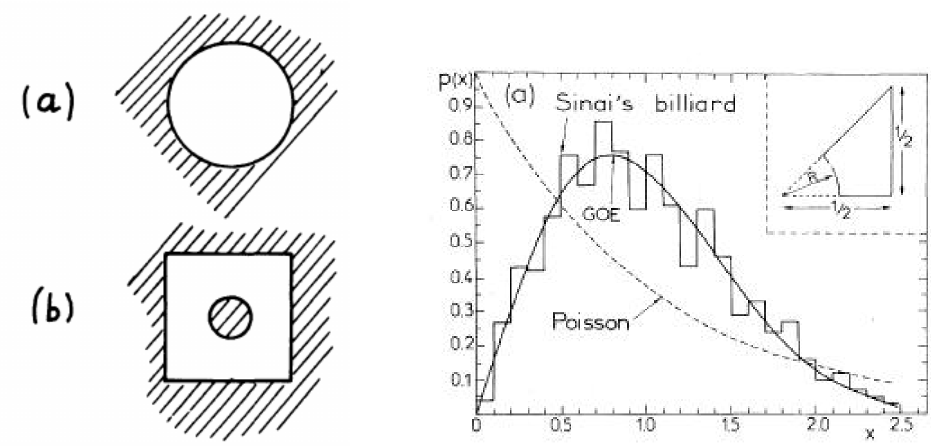

Figure 1.3: Left: an integral (a) and a chaotic (b) billiard. Right: numerical check of the BGS hypothesis in Sinai's billiard. The two figures are taken from [8].

The great success of random matrix theory in the description of level spacings of heavy nuclei suggested that the same approach could be applied to other problems with complicated energy spectra. The quantum mechanics of a particle in a *2d chaotic billiard* is one such case. A classical particle performs free motion and it is simply reflected at the boundaries, it performs chaotic motion for many choices of the boundaries (like for Sinai's or the stadium shaped spaces), and the problem is therefore ergodic, see Fig. 1.3. A remarkable conjecture, that uses random matrices ideas to describe the quantum behaviour of the same problems, is due to Bohigas, Giannoni and Schmit (BGS) [8]:

- if a billiard is classically integrable (for example a rectangle or an ellipse), the spacings between eigenvalues of the corresponding quantum Hamiltonian follow a Poisson law, and there are many level crossings.
- If it is chaotic, then the eigenvalues have the same statistical properties as those of random matrices, and they do not cross, repelling each other instead.

The numerical and experimental evidence for the validity of the BGS conjecture is overwhelming. However, the connection between *quantum chaos* and random matrix theory is not very well understood yet, and there is no satisfying analytic proof of the BGS conjecture.

Quantum chromodynamics (QCD) is a gauge theory with a gauge field that describes gluons and can be set in the form of a matrix of size N_c with N_c the number of colors. The effective coupling constant in quantum chromodynamics becomes large at large distances where the perturbation theory is not applicable. Although $N_c = 3$ in QCD, 't Hooft proposed to study a generalization of QCD in the limit $N_c \to \infty$ [9], where important simplifications occur, and then recover $N_c = 3$ by perturbation theory in a $1/N_c$ expansion. The motivation was an expansion in the inverse number of field components N in statistical mechanics that allows to solve the $O(N)$ model (recall the Phase Transition Lectures) in the large N limit exactly. Following the line of reasoning of Wigner, the $N_c \times N_c$ matrix is random and the $N_c \to \infty$ generalization of QCD becomes a theory of large random matrices. We will give some details on what this simplification brings about diagrammatically.

In the early 1970's a remarkable connection was unexpectedly discovered between two very different fields, nuclear physics and *number theory*, when it was noticed that random matrix theory accurately modelled many problems in the latter as well. The theory of numbers deals with the arithmetic properties of integers $1, 2, 3, \dots$, in particular of prime numbers. The spacing between prime numbers can be related to the one of the zeroes of a mathematical object, Riemann's zeta function, and the statistical properties of these can be set in contact with the ones of the eigenvalues of a certain class of random matrices.

Random matrices also appear in disordered condensed-matter systems, to describe *transport in disordered systems* [10]. In particular, random diffusion matrices can provide a microscopic explanation for the mesoscopic conductance at low temperatures, a regime in which the quantum decoherence length becomes larger than the conductor's size.

Last but not least, random matrices also appear in *string theory*, *2d gravity* [11], *conformal field theory*, *integrable systems*, *RNA folding* [12], *glass theory* [13] and *stochastic processes* [14]. They are at the basis of many practical uses in *finance* [15] as well.

There are numerous excellent books and review articles that treat random matrix theory from different perspectives. Mehta's book is the great classic in this field [4]. A more recent book is [7]. Lectures by Eynard at IPhT [5] and Fyodorov [16] complement the physicists approach, I would say. More mathematical treatments can be found in the books [17, 18] and the lecture notes [19, 2]. Details on the numerical implementation of random matrix studies are in [20]. The connection with stochastic processes is discussed in [14] and a nice description of the historical development of the field is given in the first Chapter written by Bohigas and Weidenmüller.

2 Four emblematic applications

We here briefly present three problems that lead to discrete energy spectra to be analysed with random matrix theory and the connection with QCD.

2.1 Vibrations of a membrane

A membrane is a perfectly flexible and infinitely thin lamina of solid matter. Let us assume that it is made of uniform material and thickness, and that it is stretched in all directions by a tension so great as to remain unaltered during its vibrations. The principal subject in this field is the investigation of the transverse vibrations of membranes of different shapes, the boundaries of which are held fixed. One considers then a membrane with area Ω and boundary Γ. Imagine that the rest position of the membrane defines the zero displacement plane (x, y) and that, during its motion the time-dependent membrane's displacement relative to this plane is $\psi(x, y, t)$. This function obeys the *wave equation*

$$\frac{\partial^2 \psi(x,y,t)}{\partial t^2} = c^2 \nabla^2 \psi(x,y,t) \tag{2.2}$$

with c a velocity that depends on the physical properties of the membrane as well as on the tension imposed on the membrane. The wave equation admits *harmonic solutions*

$$\psi(x,y,t) = \psi(x,y)\ e^{i\omega t} \tag{2.3}$$

or *normal modes*. The resulting equation for ψ (with an abuse of notation we keep the same symbol for the time-independent field) is an eigenvalue equation

$$\nabla^2 \psi(x,y) = -\frac{\omega^2}{c^2}\ \psi(x,y) = -k^2 \psi(x,y) = -E\psi(x,y) \tag{2.4}$$

with the *boundary condition* $\psi(x, y) = 0$ for $(x, y) \in \Gamma$. This equation possesses an infinite number of real positive eigenvalues with no accumulation point. One has

$$0 \le E_1 \le E_2 \le E_3 \dots \qquad\qquad \lim_{n\to\infty} E_n = \infty \tag{2.5}$$

The question is what are the properties of the energy spectrum. This problem can be generalised to higher dimensions and it can be made quantum with $\hbar k = \sqrt{2mE}$. It has attracted the attention of celebrated scientists such as Rayleigh, Sommerfeld, Lorentz, and Kac who asked the question "Can one hear the shape of a drum?" [21] More details on how to concretely analyse an energy spectrum are given with the help of the next example.

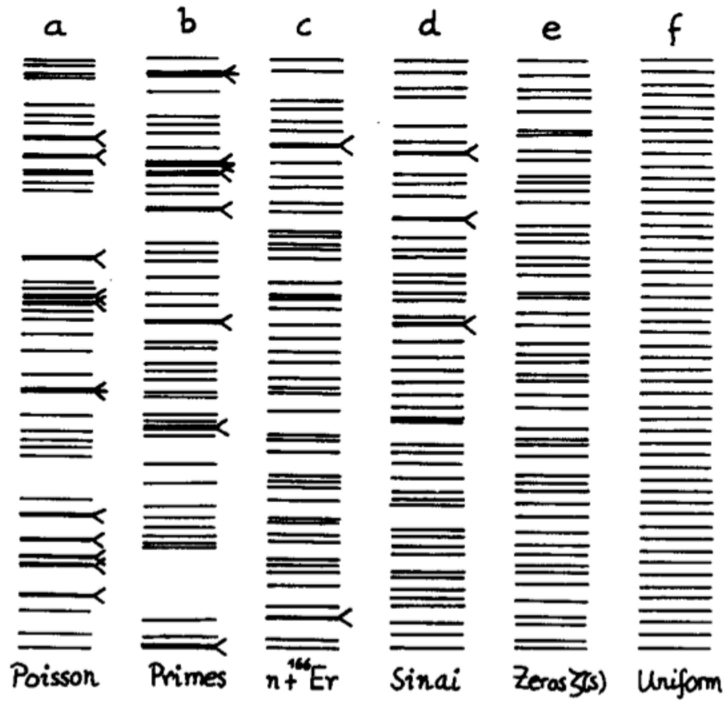

Figure 2.4: Some examples of level spectra taken from [22].

2.2 Energy splitting in a quantum system

Consider a quantum system with Hamiltonian $\hat{H}$. Its eigenstates are

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \tag{2.6}$$

with the (joint set of) quantum number(s) n labelling the levels, that we consider to be discrete. In the matrix representation of quantum mechanics this equation corresponds to the eigenvalue problem

$$H\psi_n = E_n\psi_n \; . \tag{2.7}$$

In general, the Hamiltonian is then represented by a *complex Hermitian matrix*. When there are some exact quantum numbers corresponding to exact integrals of motion, like angular momentum and parity, if the basis states are labelled by these exact quantum numbers, the Hamiltonian matrix splits into blocks, and the matrix elements connecting different blocks vanish. Once such a basis is chosen one can restrict attention to one of the diagonal blocks, an $N \times N$ Hermitian matrix in which N is a large integer, for a system with many levels. The theoretical results are often derived in the limit $N \gg 1$.

The focus is then set on the energy spectrum. As the excitation energy increases, the nuclear energy levels occur on the average at smaller and smaller intervals. In other words, level density increases with the excitation energy. More precisely, let us focus on an interval of energy ΔE centred at E. This interval is much smaller compared to E whereas it is large enough to contain many levels since it is taken to be much larger than the mean distance between neighbouring levels D:

$$D \ll \Delta E \ll E \tag{2.8}$$

In practice these scales can be very different from one nucleus to another. It then makes sense to normalise the level distances by their average (in the interval ΔE of interest). After normalisation, the average distance between the neighbouring levels is unity in all cases.

In the case of the nucleus we do not know the Hamiltonian and, on top, even if we did, the eigenvalue problem would be too difficult to solve. The random matrix approach relies on the assumption that one can take H to be a random matrix with elements restricted only by the symmetry properties of the problem. Imagine that the relevant kind of matrices to use were real symmetric. A conjecture allows one to believe the results obtained in this way will be general:

> Let H be an $N \times N$ real symmetric matrix, with off-diagonal elements h_{ij} with $i \leq j$ being independent identically distributed (i.i.d.) random variables with mean zero and variance σ^2, i.e. $\langle H_{ij} \rangle$ and $\langle H_{ij}^2 \rangle = \sigma^2$ with $0 < \sigma^2 < \infty$. In the limit of large N the statistical properties of n eigenvalues of H become independent of the probability density of the H_{ij} i.e. when $N \to \infty$ the joint probability density of arbitrarily chosen n eigenvalues of H tends, for every finite n, with probability one, to the point correlation function of the *Gaussian orthogonal ensemble*.

This statement means that the local statistical properties of a few eigenvalues of a large random matrix are independent of the distribution of the individual elements. (This statement has a flavour of the central limit theorem.) The same applies to other ensembles of random matrices.

The underlying space-time symmetries obeyed by the system put important restrictions on the admissible matrix ensembles. If the Hamiltonian is time-reversal invariant and invariant under rotations the Hamiltonian matrices can be chosen real symmetric. If the Hamiltonian is not time-reversal invariant then irrespective of its behaviour under rotations, the Hamiltonian matrices are complex Hermitian. Finally, if the system is time-reversal invariant but not invariant under rotations, and if it has half-odd-integer total angular momentum the matrices are "quaternion real".

In these lectures we will focus on the case of real symmetric matrices. The real symmetry property is preserved under orthogonal transformations. Moreover, since we have already argued that for elements with finite variance the actual pdf used to draw them is immaterial one typically chooses to work with Gaussian probabilities. For these two reasons, this set is usually called the *Gaussian orthogonal ensemble* (GOE). The parameter that is commonly used to distinguish the three classes is called β and it is equal to $\beta = 1$ in the GOE class, see Fig. 1.2.

2.3 Riemann's zeta function

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ primes}} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{2.9}$$

for complex s with real part greater than 1.

The *prime number theorem* states that the number of primes smaller than x is $Li(x) + o(Li(x))$, where $Li(x) = \int_2^x dt/\ln t$ and for x large, $Li(x) \simeq x/\ln x$. Riemann observed that the error term in the prime number theorem [23] is expressed in terms of a sum over the zeros of the Riemann zeta function.

The zeta function defined in eq. (2.9) has so-called trivial zeros at all negative even integers $s = -2, -4, -6, \ldots$, and "nontrivial zeros" at complex values $s = \sigma + it$ for s in the *critical strip* $0 < \sigma < 1$. The *Riemann hypothesis* dates from 1859 and asserts that the nontrivial zeros of $\zeta(s)$ all have real part $\sigma = R[s] = 1/2$, a line called the *critical line*.

Riemann stated "...it is very probable that all roots are real. Of course one would wish for a rigorous proof here; I have for the time being, after some fleeting vain attempts, provisionally put aside the search for this, as it appears dispensable for the immediate objective of my investigation." (He was discussing a version of the zeta function, modified so that its roots (zeros) are real rather than on the critical line.) The hypothesis still remains unproven and is one of the millennium prize problems stated by the Clay Mathematics Institute, see `http://www.claymath.org/millennium-problems`.

Assuming the validity of Riemann's hypothesis, the non-trivial zeroes lie on a straight line along the imaginary direction in the complex plane. It therefore makes sense to study the distribution of distances between adjacent zeros. It was noticed by Montgomery and Dyson [24] that the agreement between the pair correlation of the zeroes of the ζ function and the eigenvalues of complex Hermitian random matrices is amazingly good. Since then the numerical checks of this hypothesis have been numerous and the accuracy of the agreement between the two kinds of correlation functions became just amazing.

2.4 QCD

In QCD the there is a gauge field (represing gluinos) that lives in the Lie algebra of the group $G = U(N_c)$. Therefore, A is an anti-Hermitian matrix. Although there is no small parameter in this theory, the number of colours is taken to infinity to allow for the control of the Feynam diagram perturbative expansion. Indeed, the expansion of QCD in the inverse number of colours $1/N_c$ (known as the $1/N_c$ expansion) is such that only planar diagrams of the type in Fig. 2.5 survive in the perturbative expansion when the large-N_c limit is taken. Moreover, the diagrams of the perturbation theory grouped by the power of $1/N_c$ are also grouped according to their topology (e.g. genus nember). As in statistical physics models, there is a large $N_c \to \infty$ limit with a proper choice of the scaling of coupling constants, in which the remaining theory is non-trivial. For the simplest case of the Hermitian one-matrix model, which is related to the problem of enumeration of graphs, an explicit solution at large N_c was first obtained by Brézin, Itzykson, Parisi & Zuber [25]. The problem consists in computing

$$Z = \int dM \; e^{-N_c \operatorname{Tr} V(M)} \tag{2.10}$$

with M an $N_c \times N_c$ Hermitian matrix and a proper choice of the potential $V(M)$. The problem to solve is similar to the computation of Wigner's semicircle law for the GOE although perturbed by the interactions in $V(M)$.

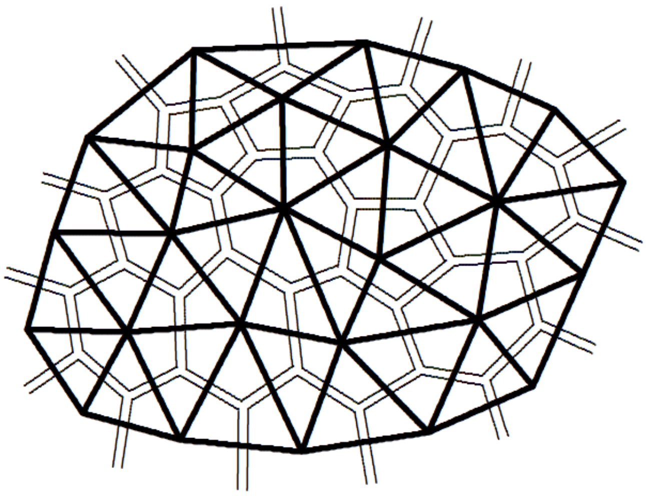

Figure 2.5: A planar graph in large N_c matrix model theory with cubic interactions. The double lines correspond to the two indices in the square matrix M. Its dual graph (depicted by bold lines) is constructed from equilateral triangles.

3 Distribution of level spacings

In this Section we explain the origin of the Poisson and Wigner distribution functions for the energy spacings. The problem is set as follows. Consider an energy interval ΔE, measure (or compute if possible!) the energy levels that falls in this interval $E_1 \leq E_2 \leq \ldots$ and let S_n be the consecutive splittings $S_n = E_{n+1} - E_n$. The average value $D = \langle S_n \rangle$ is the mean splitting over the energy interval ΔE and one considers the relative quantity $s_n = S_n/\langle S_n \rangle = S_n/D$. The probability density of s is called $p(s)$.

3.1 Independent levels: Poisson distribution

One can imagine the energies as being points placed on a real axis. If these positions were independent and not correlated, the probability that any E_n will fall between E and $E + dE$ is independent of E and is simply ρdE, where $\rho = D^{-1}$ is the average number of levels in a unit interval of energy.

Let us determine the probability of a spacing S; that is, given a level at E, what is the probability of having no level in the interval $I = (E, E + S)$ and one level in the interval $dI = (E + S, E + S + dS)$. A sketch of this interval is shown in Fig. 3.6 (where $x_0 = E$, $x = E + S$ and $x + dx = E + S + dS$).

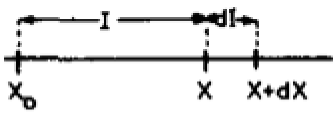

Figure 3.6: The interval partition, from [22].

We first look at the first interval $I = (E, E + S)$ and we investigate the condition of having no level in it. We divide the interval I into m equal parts:

$$I = (E, E + S) = (E, E + S/m] \cup (E + S/m, E + 2S/m] \cup \ldots$$
$$\cup (E + (m-2)S/m, E + (m-1)S/m] \cup (E + (m-1)S/m, E + S) \quad (3.11)$$

Since the levels are independent, the probability of having no level in $(E, E + S)$ is the product of the probabilities of having no level in any of these m parts

$$P(\text{no level in } I = (E, E + S)) = \left(1 - \frac{\rho S}{m}\right)^m \quad (3.12)$$

and in the large m limit

$$P(\text{no level in } I = (E, E + S)) = e^{-\rho S} \quad (3.13)$$

Instead, the probability of having a level in the remaining infinitesimal interval going from $E+S$ to $E+S+dS$ is ρdS. Therefore, the searched probability is

$$p(S)dS = \rho e^{-\rho S} dS \tag{3.14}$$

that, in terms of the normalised variable $s = S/D = \rho S$, becomes

$$p(s)ds = e^{-s} ds \; . \tag{3.15}$$

This is known as the Poisson distribution or the spacing rule for random independent levels.

3.2 Correlated levels: Wigner's argument

We now revisit the same question. The probability we are searching is the one of occurrence of the two events, no-level in $(E, E+S)$ U one level in $(E+S, E+S+dS)$. Therefore, we need to find this joint probability that, for brevity, we call AUB, and $P(A\mathrm{U}B) = P(A/B)P(B)$, where $P(A/B)$ is the *conditional probability* of A given B. Going back to the notation of the problem at hand, $p(S)dS = P(A\mathrm{U}A)$ and the level spacing probability density $p(S)$ is such that

$$p(S)dS = \text{Prob(one level in } dI/ \text{ no level in } I) \text{ Prob(no level in } I) \tag{3.16}$$

Now,

$$\text{Prob(no level in } I) = \int_S^\infty dS' \; p(S') \tag{3.17}$$

so we need to have all level spacing larger than S not to have a level in the interval $(E, E+S)$ We call

$$\mu(S)dS \equiv \text{Prob(one level in } dI/ \text{ no level in } I) \; . \tag{3.18}$$

Therefore, Eq. (3.16) reads

$$p(S) = \mu(S) \int_S^{S_{\max}} dS' \; p(S') \tag{3.19}$$

and can be solved to yield

$$p(S) = C' \, \mu(S) e^{-\int_S^{S_{\max}} dS' \, \mu(S')} \; . \tag{3.20}$$

From here one can obtain the two limiting results, Poisson and Wigner.

For a Poisson case, the two events A and B are independent, $\mu(S) = \mu$ and

$$p_{\text{Poisson}}(S) = C \, \mu e^{-\mu S} \tag{3.21}$$

The normalisation of the probability density $\int_0^{S_{\max}} dS\, p_{\text{Poisson}}(S) = 1$ and the fact that we want to deal with the splittings measured with respect to their mean, $\int_0^{S_{\max}} dS\, S\, p_{\text{Poisson}}(S) = \langle S \rangle = D$, fix $\mu = 1$ and $C' = 1/\mu$. Finally,

$$p_{\text{Poisson}}(s) = e^{-s} \; . \tag{3.22}$$

If, instead, one assumes $\mu(S) = a_\beta S$, mimicking level repulsion, $p(S)$ takes the Wigner form

$$p_{\text{Wigner}}(S) = C a_\beta \, S \, e^{-\int_S^{S_{\max}} dS' \, a_\beta S'} = C' a_\beta \, S \, e^{\frac{a_\beta}{2} S^2} \; . \tag{3.23}$$

If we now use the normalisation condition and we require that the average of s be one to work with the normalised splitting, we fix $a_\beta = -\pi/2$ and $C' a_\beta = \pi/2$ and

$$p_{\text{Wigner}}(s) = \frac{\pi}{2} \, s \, e^{-\frac{\pi}{4} s^2} \; . \tag{3.24}$$

4 The Gaussian orthogonal ensemble

For a number of nuclear physics (and other applications) the relevant random matrix ensemble to use is the Gaussian orthogonal one in which the matrices are real and symmetric. The matrices have then $N(N+1)/2$ independent elements, M_{ij} with $i \le j$, and the probability distribution is

$$p(M) = \prod_{i \le j} p_{ij}(M_{ij}) \tag{4.25}$$

Since we argued that the choice of the p_{ij} is not important, as long as the variance is finite, one uses the same Gaussian pdf to draw each element in the (half of the) matrix. (In the nuclear physics context, the symmetry is a consequence of the invariance under space rotations, see the proof in [4] for example).

The ensemble is invariant under orthogonal transformations

$$M' = O^T M O \tag{4.26}$$

with O any real orthogonal matrix. Precisely, this means that the probability $p(M)dM$ that a matrix M will be in the volume element dM is invariant under orthogonal transformations:

$$p(M)dM = p(M')dM' \; . \tag{4.27}$$

The first numerical investigation of the distribution of successive eigenvalues associated with random matrices was carried out by Porter and Rozenzweig in the late 1950's [63]. They diagonalized a large number of matrices where the elements are generated randomly but constrained by a probability distribution. The analytical theory developed in parallel with their work. At the time it was clear that the spacing distribution was not influenced significantly by the chosen form of the probability distribution of the individual elements.

Remarkably, the $N \times N$ distributions had forms given almost exactly by the original Wigner 2×2 distribution.

The invariance of the measure with respect to the rotations imposes that it can only depend on N traces of powers of H,

$$\mathrm{Tr}\, M^j = \sum_{n=1}^{N} \lambda_n^j \qquad \text{with} \qquad j = 1, \dots, N \tag{4.28}$$

with $(\lambda_1, \dots, \lambda_N)$ the N eigenvalues of H. Moreover, one can prove that for matrices constructed with statistical independent elements, the traces of the first two powers are enough, $j = 1, 2$, and these can only occur in the exponential

$$p(M) = e^{a + b\,\mathrm{Tr}\, M + c\,\mathrm{Tr}\, M^2} \tag{4.29}$$

The issue is to deduce the joint probability distribution function of the eigenvalues $p(\lambda_1, \dots, \lambda_N)$ implied by the Gaussian probability of the matrix elements (further constrained to build a symmetric matrix).

The real symmetric matrix has $N \times (N+1)/2$ independent elements. Therefore, such a matrix can be parametrised with its N eigenvalues plus $N(N+1)/2 - N = N(N-1)/2$ other parameters that we call p_ℓ with $\ell = 1, \dots, N(N-1)/2$. From the change of variables form H_{ij} to λ_n, p_ℓ

$$\prod_{i \leq j} p(\{M_{ij}\}) dM_{ij} = \prod_n \prod_\ell p(\{\lambda_n, p_\ell\}) d\lambda_n dp_\ell \tag{4.30}$$

that implies

$$p(\{\lambda_n\}) = \int \prod_\ell dp_\ell \; p(\{\lambda_n, p_\ell\}) = \int \prod_\ell dp_\ell \; p(\{M_{ij}\}) \left| \frac{\partial(M_{11}, M_{12}, \dots, M_{NN})}{\partial(\lambda_1, \dots, \lambda_n, p_1, \dots, p_{N(N-1)/2})} \right| \tag{4.31}$$

The last factor is the Jacobian. The joint probability distribution of the eigenvalues is the integral over the parameters p_ℓ.

We now need to find the parameters p_ℓ. The matrix H can be diagonalised with an orthogonal transformation

$$M = O^T D O \qquad \text{with} \qquad O^T O = 1 \tag{4.32}$$

and D the diagonal matrix with elements equal to the eigenvalues of H (we will assume that the eigenvalues are non-degenerate, see [4] for the treatment of more general cases). O is a matrix with real elements whose columns are the normalised eigenvectors of H, that can be chosen to be orthogonal. By taking variations of these two expressions with respect to p_ℓ and manipulating in various ways the Jacobian one ends with [4]

$$p(\{\lambda_n\}) = \prod_{1<j<k<N} |\lambda_j - \lambda_k| \; e^{a + b \sum_n \lambda_n + c \sum_n \lambda_n^2} \tag{4.33}$$

With a simple linear change of variables from λ_n to x_n the linear term is eliminated and

$$p(\{x_n\}) = C \prod_{1<j<k<N} |x_j - x_k| \; e^{-\frac{1}{2}\sum_n x_n^2} \; . \tag{4.34}$$

4.1 The Coulomb gas

The joint probability density of the eigenvalues can be written as the Boltzmann factor for a gas of charged particles interacting via a two dimensional Coulomb force. The equilibrium density of this *Coulomb gas* is such as to make the potential energy a minimum, and this density is identified with the level density of the corresponding Gaussian ensembles. Consider a gas of N point charges with positions x_i free to move on the infinite straight line $-\infty < x_i < \infty$. Suppose that the potential energy of the gas is given by

$$V(\{x_i\}) = \frac{1}{2}\sum_{i=1}^{N} x_i^2 - \sum_{i<j} \ln|x_i - x_j| \tag{4.35}$$

The first term in H represents a harmonic potential which attracts each charge independently towards the origin $x = 0$; the second term represents an electrostatic repulsion between each pair of charges. The logarithmic function comes in if we assume the universe to be two-dimensional.

The potential energy is bounded from below

$$W \geq W_0 = -N(N-1)(1+\ln 2) + \sum_{j=1}^{N} j \ln j \tag{4.36}$$

and this minimum is attained when the positions of the charges coincide with the zeros of the Hermite polynomial $H_N(x)$.

Let this charged gas be in thermodynamical equilibrium at a temperature T, so that the probability density of the positions of the N charges is given by

$$P(\{x_i\}) = Z_N^{-1}(\beta) e^{-\beta H} \tag{4.37}$$

(One can add to the potential energy the kinetic energy. This, in any case, gives a trivial factor that depends on β and the mass of the particles. We can simply ignore it.) We note that this expression is the probability distribution of the eigenvalues provided that $\beta = 1$ in the GOE is interpreted as the inverse temperature here! The miracle now is that the integral over the positions of the point-like particles can be computed exactly for any N

$$Z_N(\beta) = \frac{(2\pi)^{N/2}}{\beta^{N/2} + \beta N(N-1)/4} [\Gamma(1+\beta/2)]^{-N} \prod_{j=1}^{N} \Gamma(1+\beta j/2) \; . \tag{4.38}$$

The density of eigenvalues can then be simply computed by performing the integral over $N = 1$ eigenvalues

$$\rho(x) = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_{N-1}\, P(x_1, \ldots, x_N) \; . \tag{4.39}$$

In this way, the eigenvalue density of the random matrices in the GOE can be derived for any finite N. In the limit $N \to \infty$ this distribution approaches the semi-circle law, depicted in Fig. 4.7 is obtained. This result is shown by taking a continuum limit in which the eigenvalue density becomes a continuous function ρ of the real variable x and the potential energy is written as

$$W = \frac{1}{2}\int_{-\infty}^{\infty} dx\, x^2\rho(x) - \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \ln|x-y|\rho(x)\rho(y)\,. \tag{4.40}$$

The level density is then obtained from the minimisation of this expression under the constraint

$$\int_{-\infty}^{\infty} dx\, \rho(x) = N\,. \tag{4.41}$$

After a number of steps detailed in [4] one finds the semi-circle law

$$\rho(x) = \begin{cases} 1/\pi\,\sqrt{2N - x^2} & |x| < (2N)^{-1/2} \\ 0 & |x| \geq (2N)^{-1/2} \end{cases} \tag{4.42}$$

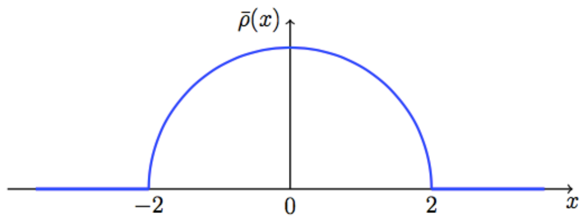

Figure 4.7: The semi-circle law eigenvalue distribution of matrices in the Gaussian orthogonal ensemble, with the horizontal axis normalised by $\sqrt{N}$ to make the non-trivial interval be of finite extent.

Interestingly enough, the semi-circle law for the eigenvalue density can be derived with the *replica trick* used to study disordered systems [26].

The calculation of the level spacing is long and a bit tedious. It can be found in [4] for this and the other ensembles.

Dyson proposed to define various expressions that relate to the energy-level series in complete analogy with the classical notions of entropy, specific heat, and the like. One can measure these observables from the experimental data and check in this way the theory.

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