Advanced Statistical Physics: Phase Transitions

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1 Phase transitions

Take a piece of material in contact with an external reservoir. The material will be characterised by certain global observables, energy, magnetisation, *etc.*. To characterise macroscopic systems it is convenient to consider densities of energy, magnetisation, *etc*, by diving the macroscopic value by the number of particles (or the volume) of the system. If the system is coupled to its surroundings, this external environment will be characterised by some parameters, like the temperature, magnetic field, pressure, *etc.* In principle, one is able to tune the latter and measure the former as a function of them. In isolated systems, the temperature can also be defined from the entropy-energy relation following the canonical prescription, and it can be used as one of the axis in the phase diagram.

Sharp changes in the behaviour of macroscopic systems at critical points (lines) in parameter space have been observed experimentally. These correspond to *phase transitions* [1, 2, 3, 4, 5, 6, 7], a non-trivial collective phenomenon arisng in the thermodynamic limit, $N \to \infty$ and $V \to \infty$. *Phase diagrams* as the one in Fig. 1.1 are used as a visual help to identify the global behaviour of a system according to the values that the *order parameters* (relevant observables) take in different regions of variation of the *control parameters* that give the axes to the phase diagram.

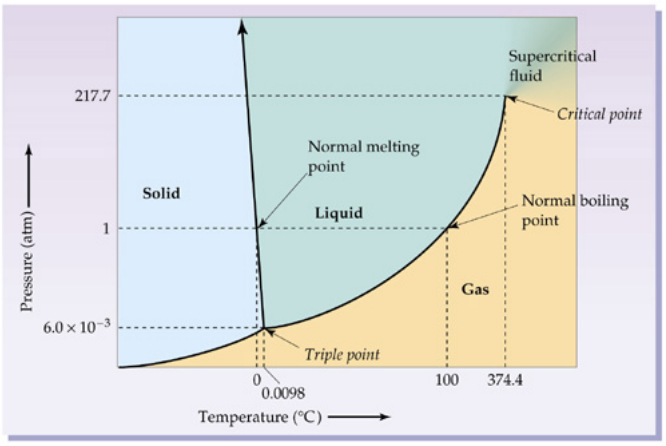

Figure 1.1: A quite generic phase diagram.

Macroscopic models of agents in interaction may have *static* and *dynamic phase transitions*. The former are the usual ones studied with statistical physics methods. For example, in the canonical ensemble, one finds the phase transitions by looking for *non-analyticities of the free-energy density* as a function of the control parameters, say just

$\beta = 1/(k_B T)$,

$$-\beta f(\beta) = N^{-1} \ln Z(\beta) \qquad \text{with} \qquad Z(\beta) = \sum_C e^{-\beta H(C)} \tag{1.1}$$

where C represents all the system configurations. One is interested in identifying the *order parameter* (in some cases this is easy, in others it is not), finding the *critical curves* of the control parameters in the phase diagram, studying the *critical phenomenon* that is to say the behaviour of the order parameter and other properties close to the phase transition, etc.

Dynamic phase transitions correspond to sharp changes in the dynamic evolution of a macroscopic system. We will not discuss them in these notes.

1.1 The standard models for magnetic systems

Let us analyse a magnetic system. The Hamiltonian describing all microscopic details is a rather complicated one. It depends on the electrons' magnetic moments giving rise to the macroscopic magnetization of the sample but also on the vibrations of the atomic crystal, the presence of structural defects, *etc.* If we call α a *microstate*, in the canonical ensemble its probability is $P_\alpha = e^{-\beta H_\alpha}/Z$ with Z the partition function, $Z = \sum_\alpha e^{-\beta H_\alpha}$. It is, however, impossible and not necessarily interesting to keep all details and work with all possible physical phenomena simultaneously. Imagine that we are only interested on the magnetic properties, characterized by the electronic magnetic moments.

The *Ising model* is a simplified mathematical representation of a magnetic system. It describes magnetic moments as *classical spins*, s_i, taking values ± 1, lying on the vertices of a cubic lattice in d dimensional space, and interacting via nearest-neighbor couplings, $J > 0$. The energy is then

$$H_J(\{s_i\}) = -\frac{J}{2} \sum_{\langle ij \rangle} s_i s_j - \sum_i h_i s_i \tag{1.2}$$

where h_i is a local external magnetic field. Most typically one works with a uniform field, $h_i = h$ for all sites. The justification for working with an Ising variable taking only two values is that in many magnetic systems the magnetic moment is forced to point along an *easy axis* selected by crystalline fields. We then need a model that focuses just on these.

There are two external parameters in H, the coupling strength J and the external field h. $J > 0$ favors the alignement of the spin in the same direction (ferromagnetism) while $J < 0$ favors the anti-alignement of the spins (antiferromagnetism). The magnetic field tends to align the spins in its direction.

The spins lie on a d dimensional lattice that can have different geometries. For instance, a cubic lattice is such that each vertex has *coordination number*, or number of neighbours, $z = 2d$. Triangular, honeycomb, *etc.* lattices are also familiar.

The Ising model is specially attractive for a number of reasons:
(i) It is probably the simplest example of modeling to which a student is confronted.
(ii) It can be solved in some cases: $d = 1$, $d = 2$, $d \to \infty$. The solutions have been the source of new and powerful techniques later applied to a variety of different problems in physics and interdisciplinary fields.
(iii) It has not been solved analytically in the most natural case, $d = 3$!
(iv) It has a phase transition, an interesting collective phenomenon, separating two phases that are well-understood and behave, at least qualitatively, as real magnets with a paramagnetic and a ferromagnetic phase.
(v) There is an *upper*, d_u, and a *lower*, d_l, *critical dimension*. Above d_u mean-field theory correctly describes the critical phenomenon. At and below d_l there is no finite T phase transition. Below d_u mean-field theory fails.
(vi) One can see at work generic tools to describe the critical phenomenon like *scaling* and the *renormalization group*.
(vii) Generalizations in which the interactions and/or the fields are random variables taken from a probability distribution are typical examples of problems with *quenched disorder*.
(viii) Generalizations in which spins are not just Ising variables but vectors with n components with a local constraint on their modulus are also interesting. Their energy is

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \vec{s}_i \vec{s}_j - \sum_i \vec{h}_i \vec{s}_i \tag{1.3}$$

with $n = 1$ (Ising), $n = 2$ (XY), $n = 3$ (Heisenberg), ... , $n \to \infty$ ($O(n)$) as particular cases. The *local constraint* on the length of the spin is

$$s_i^2 \equiv \sum_{a=1}^{n} (s_i^a)^2 = n \; . \tag{1.4}$$

Note that each component is now a continuous variable bounded in a finite interval, $-\sqrt{n} \leq s_i^a \leq \sqrt{n}$, that actually diverges in the $n \to \infty$ limit. When $n \to \infty$ it is sometimes necessary to redefine the coupling constants including factors of n that yield a sensible $n \to \infty$ limit of thermodynamic quantities.
(ix) One can add a dynamic rule to update the spins. We are then confronted to the *kinetic Ising model* and more generally to the new World of stochastic processes.
(x) *Dynamic phase transitions* occur in the properties of the system's evolution. We will not discuss them in these Lectures.
(xi) In the low temperature phase the progressive order is reached via *domain growth*, the simplest example of coarsening.
(xii) Last but not least, it has been a paradigmatic model extended to describe many problems going beyond physics like *neural networks*, *social ensembles*, *etc.*

Note the difference between the two parameters, N and n. N is the number of spins in the system. n is the number of components that each spin vector has. There is still one other dimension, the one of real space, that we call d.

1.2 Concepts

Let us now discuss some important concepts, *symmetries*, *order parameters*, *pinning fields*, *broken ergodicity* and *broken symmetry* [1, 2, 3, 4, 5, 6, 7], with the help of the concrete example of the Ising model. The discussion is however much more general.

1.2.1 Symmetries

Let us treat separately the case of continuous and discrete symmetries.

Continuous

In the absence of an applied magnetic field the Hamiltonian (1.3) remains invariant under the simultaneous *rotation* of all spins:

$$\begin{aligned} H[\vec{s}'] &= -\frac{J}{2}\sum_{\langle ij\rangle}\vec{s}'_i\vec{s}'_j = -\frac{J}{2}\sum_{\langle ij\rangle}R^{ab}s_i^b R^{ac}s_j^c \\ &= -\frac{J}{2}\sum_{\langle ij\rangle}R^{Tba}R^{ac}s_i^b s_j^c = -\frac{J}{2}\sum_{\langle ij\rangle}s_i^b s_j^b \end{aligned} \tag{1.5}$$

since R is an orthogonal transformation, such that $R^T R = I$. This symmetry is explicitly broken by the external field. (Summation over repeated a, b indices is assumed.)

Discrete

The Ising model with no applied field is invariant under the *reversal* of all spins, $s_i \to -s_i$, for all i, a discrete symmetry.

1.2.2 Order parameters

An order parameter is generically defined as a quantity – the average of an observable – that typically vanishes in one phase and is different from zero in another one (or other ones). One must notice though that the order parameter is not unique (any power of an order parameter is itself an order parameter) and that there can exist transitions without an order parameter as in the *topological* Kosterlitz-Thouless transition in the $2d$ XY model.

In the ferromagnetic Ising model the order parameter is the magnetisation density

$$m = \frac{1}{N}\sum_{i=1}^{N}\langle\, s_i\,\rangle \qquad \text{and} \qquad \langle\, s_i\,\rangle = \mathcal{Z}^{-1}\sum_C s_i\, e^{-\beta H(C)} \tag{1.6}$$

where N is the total number of spins and the angular brackets represent the thermal average.

1.2.3 Thermodynamic limit

The abrupt change in the order parameter at a particular value of the external parameters, say temperature and magnetic field (T, h), is associated to the divergence of some derivative of the free-energy (we use the canonical ensemble) with respect to one of these parameters. The partition function is a sum of positive terms. In a system with a finite number of degrees of freedom (as, for instance, in an Ising spin model where the sum has 2^N terms with N the number of spins) such a sum is an analytic function of the parameters. Thus, no derivative can diverge. One can then have a phase transition only in the *thermodynamic limit* in which the number of degrees of freedom diverges.

1.2.4 Pinning field

In the absence of a magnetic field for pair interactions the energy is an even function of the spins, $H_J(\{s_i\}) = H_J(\{-s_i\})$ and, consequently, the equilibrium magnetisation density computed as an average over *all* spin configurations with their canonical weight, $e^{-\beta H_J(C)}$, vanishes at all temperatures:

$$\langle s_i \rangle = 0 \quad \forall\, i \quad \text{if} \quad h_i = 0\,. \tag{1.7}$$

At high temperatures, $m = 0$ characterises completely the equilibrium properties of the system since there is a unique paramagnetic state with vanishing magnetisation density. At low temperatures instead if we perform an experiment we *do observe* a net magnetisation density. In practice, what happens is that when the experimenter takes the system through the transition he/she cannot avoid the application of tiny external fields – the experimental set-up, the Earth... – and there is always a small *pinning field* that actually selects one of the two possible equilibrium states, with positive or negative magnetisation density, allowed by symmetry. In the course of time, the experimentalist should see the full magnetisation density reverse, to ensure $m = 0$ in equilibrium. However, this is not seen in practice since astronomical time-scales would be needed. We shall see this phenomenon at work when solving mean-field models exactly.

To see $\langle s_i \rangle \neq 0$ one needs to compute

$$\lim_{h\to 0} \lim_{N\to\infty} \langle s_i \rangle_h = m \neq 0 \tag{1.8}$$

1.2.5 Broken ergodicity

Introducing dynamics into the problem,[1] *ergodicity breaking* can be stated as the fact that the temporal average over a long (but finite) time window

$$\overline{A}_t = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} dt' A(t') \tag{1.9}$$

[1] Note that Ising model does not have a natural dynamics associated to it. A dynamic rule can be attributed to the evolution of the spins.

is different from the statical one, with the sum running over all configurations with their associated Gibbs-Boltzmann weight:

$$\overline{A}_t \neq \langle A \rangle \, . \tag{1.10}$$

In practice the temporal average is done in a long but finite interval $\tau < \infty$. During this time, the system is positively or negatively magnetized depending on whether it is in "one or the other degenerate equilibrium states" (see Fig. 1.2). Thus, the temporal average of the orientation of the spins, for instance, yields a non-vanishing result $\overline{A}_t = m \neq 0$. If, instead, one computes the statistical average summing over *all* configurations of the spins, the result is zero, as one can see using just symmetry arguments, explained in Sec. 1.2.6. The reason for the discrepancy is that with the time average we are actually summing over half of the available configurations of the system. If time τ is not as large as a function of N, the trajectory does not have enough time to visit all configurations in phase space. One can reconcile the two results by, in the statistical average, summing only over the configurations with positive (or negative) magnetization density and recovering in this way a non-vanishing result. We shall see this at work in a concrete calculation below.

Note that ergodicity breaking is a statement about the dynamics of a system.

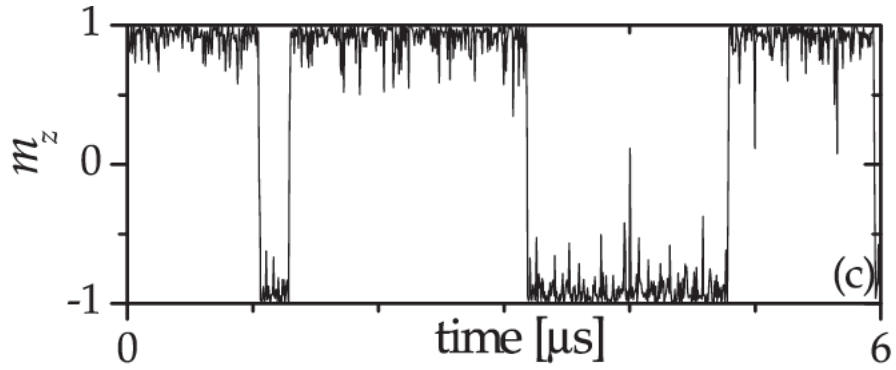

Figure 1.2: Time dependence of the global magnetization.

1.2.6 Spontaneous broken symmetry

In the absence of an external field the Hamiltonian is symmetric with respect to the simultaneous reversal of all spins, $s_i \to -s_i$ for all i. The phase transition corresponds to a *spontaneous symmetry breaking* between the states of positive and negative magnetization. One can determine the one that is chosen when going through T_c either by applying a small *pinning field* that is taken to zero only after the thermodynamic limit, or by imposing adequate *boundary conditions* like, for instance, all spins pointing up on the borders of

the sample. Once a system sets into one of the equilibrium states this is completely stable in the $N \to \infty$ limit. The mathematical statement of spontaneous symmetry breaking is then

$$\lim_{h\to 0^+} \lim_{N\to\infty} \langle s_i \rangle = - \lim_{h\to 0^-} \lim_{N\to\infty} \langle s_i \rangle \neq 0 \,. \tag{1.11}$$

Ergodicity breaking necessarily accompanies spontaneous symmetry breaking but the reverse is not true; an example is provided by systems with quenched disorder. Indeed, spontaneous symmetry breaking generates disjoint ergodic regions in phase space, related by the broken symmetry, but one cannot prove that these are the only ergodic components in total generality. Mean-field spin-glass models provide a counterexample of this implication.

1.2.7 Landau scheme

Without getting into the details of the Landau description of phase transitions (that you will certain study in the Statistical Field Theory lectures) we just summarize here, in Figs. 1.3 taken from [2] the two scenarii corresponding to second order (the panels in the first row) and first order phase transitions (the next six panels). The figures show the evolution of the free-energy density as a function of the order parameter η when temperature (called T in the first three panels and t in the next six ones) is modified.

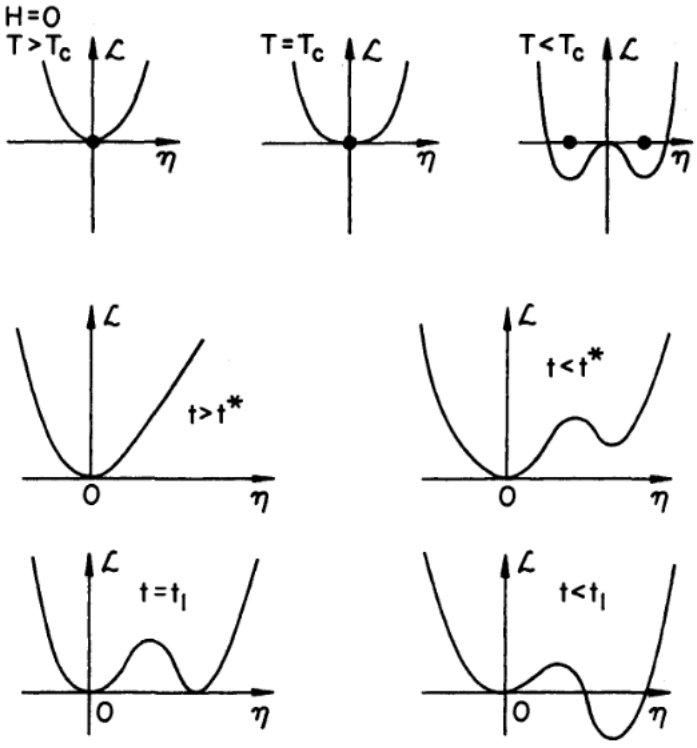

Figure 1.3: Second order and First order phase transitions. Figures taken from [2].

The saddle-point equation typically takes the form $x =$ a sigmoid function. The difference between second order and first order solutions in the way in which the sigmoid function changes when the control parameter is modified. Figure 1.4 shows two sketches of this evolution for second order (labelled $p = 2$) and first order (labelled $p = 3$) transitions.

In a second order phase transition the non-vanishing solutions split from the vanishing one in a continuous way. A possible strategy to find the critical parameters is, then, to look for the values at which the slope of the sigmoid function close to zero equals one.

In a first order phase transition the sigmoid function touches the diagonal axis at a non-vanishing value when the local minimum at $x \neq 0$ first appears. Further changing the parameters this point splits in two and the sigmoid function crosses the diagonal at three points, say $x = 0$, x_1 (a maximum of the free-energy density) and x_2 (the non-zero minimum of the free-energy function). Other two crossings are symmetrically placed on $x < 0$ values if the model is invariant under $x \mapsto -x$.

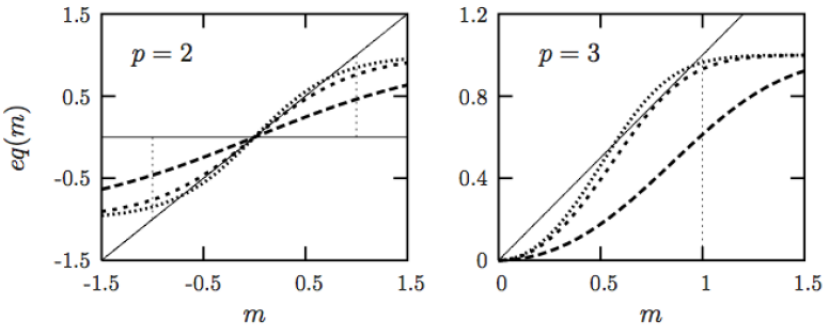

Figure 1.4: Sketch of the graphical solution of the mean-field equation for the order parameter.

1.2.8 Energy *vs* entropy

Let us first use a thermodynamic argument to describe the high and low temperature phases of a magnetic system.

The free energy of a system is given by $F = U - k_B T S$ where U is the internal energy, $U = \langle H \rangle$, and S is the entropy. The equilibrium state may depend on temperature and it is such that it minimises its free-energy F. A competition between the energetic contribution and the entropic one may then lead to a change in phase at a definite temperature, *i.e.* a different group of micro-configurations, constituting a state, with different macroscopic properties dominate the thermodynamics at one side and another of the transition.

At zero temperature the free-energy is identical to the internal energy U. In a system with ferromagnetic couplings between magnetic moments, the magnetic interaction is such that the energy is minimised when neighbouring moments are parallel. Thus the preferred configuration is such that all moments are parallel, the system is fully ordered and $U = -\#$ pairs.

Switching on temperature thermal agitation provokes the reorientation of the moments and, consequently, misalignments. Let us then investigate the opposite, infinite temperature case, in which the entropic term dominates and the chosen configurations are such that entropy is maximised. This is achieved by the magnetic moments pointing in random independent directions. For example, for a model with N Ising spins, the entropy at infinite temperature is $S \sim k_B N \ln 2$.

Decreasing temperature magnetic disorder becomes less favourable. The existence or not of a finite temperature phase transitions depends on whether long-range order, as the one observed in the low-temperature phase, can remain stable with respect to *fluctuations*, or the reversal of some moments, induced by temperature. Up to this point, the discussion has been general and independent of the dimension d.

The competition argument made more precise allows one to conclude that there is no finite temperature phase transition in $d = 1$ while it suggests there is one in $d > 1$. Take a one dimensional ferromagnetic Ising model with closed boundary conditions (the case of open boundary conditions can be treated in a similar way),

$$H_J[\{s_i\}] = -J \sum_{i=1}^{N} s_i s_{i+1} \ , \tag{1.12}$$

and $s_{N+1} = s_1$. At zero temperature it is ordered and its internal energy is just

$$U_o = -JN \tag{1.13}$$

with N the number of links and spins. Since there are two degenerate ordered configurations (all spins up and all spins down) the entropy is

$$S_o = k_B \ln 2 \tag{1.14}$$

The internal energy is extensive while the entropy is just a finite number. At temperature T the free-energy of the completely ordered state is then

$$F_o = U_o - k_B T S_o = -JN - k_B T \ln 2 \ . \tag{1.15}$$

This is the *ground state* at finite temperature or global configuration that minimises the free-energy of the system.

Adding a *domain* of the opposite order in the system, *i.e.* reversing n spins, two bonds are unsatisfied and the internal energy becomes

$$U_2 = -J(N-2) + 2J = -J(N-4) \ , \tag{1.16}$$

for any n. Since one can place the misaligned spins anywhere in the lattice, there are N equivalent configurations with this internal energy. The entropy of this state is then

$$S_2 = k_B \ln(2N) \ . \tag{1.17}$$

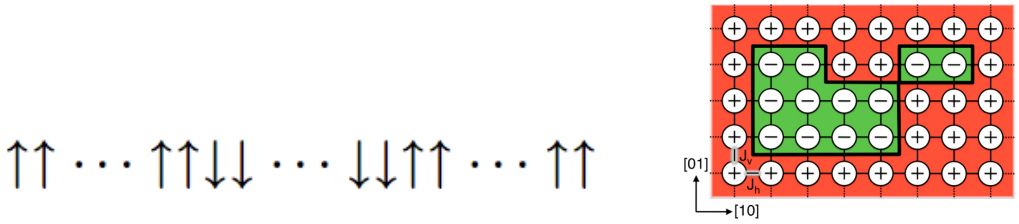

Figure 1.5: Left, a domain wall in a one dimensional Ising system and right, two bidimensional domains in a planar (artificial) Ising system.

The factor of 2 inside the logarithm arises due to the fact that we consider a reversed domain in each one of the two ordered states. At temperature T the free-energy of a state with one reversed spin and two *domain walls* is

$$F_2 = U_2 - k_B T S_2 = -J(N-4) - k_B T \ln(2N) \; . \tag{1.18}$$

The variation in free-energy between the ordered state and the one with one domain is

$$\Delta F = F_2 - F_o = 4J - k_B T \ln N \; . \tag{1.19}$$

Thus, even if the internal energy increases due to the presence of the domain wall, the increase in entropy is such that the free-energy of the state with a droplet in it is much lower, and therefore the state much more favourable, at any finite temperature T. We conclude that spin flips are favourable and order is destroyed at any finite temperature. The ferromagnetic Ising chain does not have a finite temperature phase transition.

A similar argument in $d > 1$ suggests that one can have, as indeed happens, a finite temperature transition in these cases (see, *e.g.* [2]).

1.2.9 Field theories

A field theory for the magnetic problem can be rather simply derived by coarse-graining the spins over a coarse-graining length ℓ. This simply amounts to computing the averaged spin on a box of linear size ℓ. In the limit $\ell \gg a$ where a is the lattice spacing many spins contribute to the sum. For instance, an Ising bimodal variable is thus transformed into a continuous real variable taking values in $[-1, 1]$. Studying the problem at long distances with respect to ℓ (or else taking a continuum spatial limit) the problem transforms into a *field theory*. This is the route followed by Landau.

Field theories are the natural tool to describe particle physics and cosmology. For example, the Big Bang leaves a radiation-dominated universe at very high temperature close to the Planck scale. As the initial fireball expands, temperature falls precipitating a sequence of phase transitions. The exact number and nature of these transitions is not known. It is often considered that they are at the origin of the structures (galaxies, clusters, *etc.*) seen in the universe at present, the original seeds being due to density fluctuations left behind after the phase transition.

The similarity between the treatment of condensed matter problems and high energy physics becomes apparent once both are expressed in terms of field theories. It is however often simpler to understand important concepts like spontaneous symmetry breaking in the language of statistical mechanics problems.

1.3 Models with continuous symmetry

The energy of spin models with continuous variables, such as the XY, Heisenberg or generic $O(n)$ models introduced in (1.3) and (1.4) in the absence of an applied field ($\vec{h} = \vec{0}$), is invariant under the simultaneous rotation of all the spin variables:

$$s_i^a \mapsto R^{ab} s_i^b \, . \tag{1.20}$$

(R^{ab} are the n^2 elements of a rotation matrix in an n-dimensional space. As all rotation matrices in real space it has real elements and it is orthogonal, that is to say, $R^T = R^{-1}$ with $\det R = \pm 1$.) This is a *continuous global symmetry* to be confronted to the *discrete global* spin reversal invariance, $s_i \mapsto -s_i$, of the Ising case. In group theoretical terms, the continuous symmetry is $O(n)$ and the discrete one is Z_2.

The spontaneous magnetization at low temperatures can point in any of the infinite equivalent directions constrained to satisfy (1.4). This gives rise to an infinite degeneracy of ground states that are translational invariant (in real space). These equilibrium states are controlled by a continuous variable, determining the direction on the n-dimensional hypersphere of radius 1.

1.3.1 The d-dimensional XY model: spin-waves

Let us consider one such equilibrium state and call it $\vec{s}_i^{\,\mathrm{eq}}$. It is clear that if one slightly modifies the angle of the $\vec{s}$ vector on neighbouring space points, the energy cost of such a perturbation would vanish in the limit of vanishing angle. More precisely, these configurations are called *spin-waves* and they differ from the uniformly ordered state by an arbitrarily small amount.

In the particular case of the XY model, see Fig. 1.6, the local spins are constrained to rotate on the *plane*; therefore, each spin has only two components ($n = 2$) and it can be parametrized as

$$\vec{s}_i = (s_i^1, s_i^2) = |\vec{s}_i|(\cos\phi_i, \sin\phi_i) = (\cos\phi_i, \sin\phi_i) \tag{1.21}$$

where $0 \leq \phi_i \leq 2\pi$ is the angle with respect to the x axis of the easy plane on each d-dimensional lattice site i. The modulus of each vector spin is fixed to one. The Hamiltonian (1.3) then becomes

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \cos\phi_{ij} \tag{1.22}$$

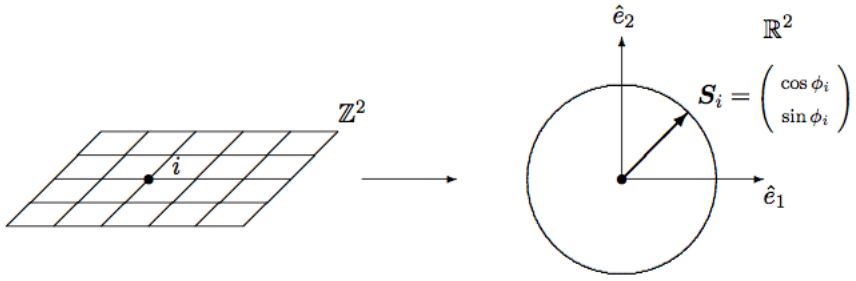

Figure 1.6: A sketch of the $2d$ XY model definition. On the left the square lattice in $2d$, on the right the $n = 2$ spin vector.

where $\phi_{ij} = \phi_i - \phi_j$ is the angle between the spins at neighbouring sites i and j. Equation (1.22) remains invariant under the global translation of all angles, $\phi_i \to \phi_i + \phi_0$ by the same amount, that corresponds to the global rotational invariance. The zero-temperature ground state is the fully aligned state $\phi_i = \phi$ for all i, with ϕ in $[0, 2\pi]$. The ground state energy $E_0 = -JNz/2$ with z is the coordination number of the lattice and N the total number of spins in the system. If one assumes that at low enough T the angles between contiguous spins can only be small, $|\phi_i - \phi_j| \ll 2\pi$, the cosine in the Hamiltonian can be expanded to second order and

$$H \simeq E_0 + \frac{J}{4} \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2 \;=\; E_0 + \frac{J}{4} \sum_{\vec{r},\vec{a}} [\phi(\vec{r} + \vec{a}) - \phi(\vec{r})]^2 \; . \tag{1.23}$$

If $\phi(\vec{r})$ is a slowly varying function of $\vec{r}$ one can approximate the finite difference by a derivative, *e.g.* along the x axis $\phi(\vec{r} + \vec{a}) = \phi(\vec{r} + a\hat{e}_x) - \phi(\vec{r}) \simeq a\partial_x \phi(\vec{r})$ the sum over lattice sites by an integral $\sum_{\vec{r}} \cdots \simeq a^{-d} \int d^d r \ldots$, and write

$$H \simeq E_0 + \frac{J}{4a^{d-2}} \int d^d r \; [\vec{\nabla} \phi(\vec{r})]^2 \; . \tag{1.24}$$

a is the lattice spacing. We ended up with a quadratic form that, if we relax the constraint $\phi \in [0, 2\pi]$, acts on a real unbounded field

$$-\infty < \phi < \infty \; . \tag{1.25}$$

This is also called the *elastic representation* of the Hamiltonian. We also note that if we use a Fourier representation $\phi_{\vec{k}} = \int d^d r \; e^{i\vec{k}\cdot\vec{r}} \, \phi(\vec{r})$, the Hamiltonian is one of independent harmonic oscillators

$$H \simeq E_0 + \frac{J}{4a^{d-2}} \frac{1}{V} \sum_{\vec{k}} k^2 \phi^2(\vec{k}) \; . \tag{1.26}$$

From the harmonic Hamiltonian, assuming the smooth character of the field $\phi(\vec{r})$ one finds

$$\nabla^2\phi(\vec{r}) = 0 \; . \tag{1.27}$$

This equation admits the trivial solution $\phi(\vec{r}) = \text{cst}$. Note that this equation is identical to the *Laplace equation* for the electrostatic potential in the absence of any charge density.

First of all one may want to compute the average magnetisation $\vec{m} = \langle \vec{s}(\vec{r}) \rangle = \lim_{\vec{h}\to\vec{0}} \langle \vec{s}(\vec{r}) \rangle_{\vec{h}}$ where $\vec{h}$ is a pinning field. Mermin's exact calculation [9] (that we will not present here, see [7] for a description of the proof) leads to $\vec{m} = \vec{0}$ at all temperatures, excluding usual magnetic order.

The interest is in computing the *spin-spin correlation function*

$$G(r) \equiv \langle \vec{s}(\vec{r})\vec{s}(\vec{0}) \rangle = \text{Re}\langle e^{i[\phi(\vec{r})-\phi(\vec{0})]} \rangle = e^{-\frac{1}{2}\langle [\phi(\vec{r})-\phi(\vec{0})]^2 \rangle} \equiv e^{-\frac{1}{2}g(r)} \; , \tag{1.28}$$

where the second identity holds for Gaussian fields[2]. $G(r)$ here is a space-dependent correlation function and its Fourier transform is called the *structure factor*. One should analyse whether at long distances it converges to a finite value (*long-range order*) or zero (no long-range order). We shall not give the details of this calculation which can be found in many textbooks (and your field-theory lectures, I presume) and just give the results:

$$\frac{Ja^{2-d}}{k_B T} \; g(r) \simeq \begin{cases} \Omega_d/(d-2)\; (\pi/L)^{d-2} & d > 2 \; , \\ (2\pi)^{-1} \; \ln(r/L) & d = 2 \; , \\ r/2 & d = 1 \; , \end{cases}$$

that imply

$$G(r) \simeq \begin{cases} e^{-\text{const } k_B T} & d > 2 & \text{long-range order} \; , \\ (r/L)^{-\eta(k_B T/J)} & d = 2 & \text{quasi-long-range order} \; , \\ \exp[-k_B T/(2Ja)\; r] & d = 1 & \text{short-range order} \end{cases}$$

(hint to prove it, use the Fourier transform representation.) The behaviour is special in $d = 2$. Interestingly enough, we find that the $2d$ XY model does not support long-range order but its correlation function decays algebraically at all temperatures. This is the kind of decay found *at* a critical point, $G(r) \simeq r^{d-2+\eta}$, so the system behaves as at criticality at all temperatures. This does not seem feasible physically and, indeed, we shall see that other excitations, not taken into account by the continuous expansion above, are responsible for a phase transition of a different kind, a so-called *topological phase transition*. After these have been taken care of, the low-T phase remains well described by the spin-wave approximation but the high-T one is dominated by the proliferation of topological defects.

The exponent $\eta(k_B T/J)$ continuously depends on temperature, $\eta(k_B T/J) = k_B T/(2\pi J)$. This is a signature of the *criticality of the low-T phase*. The criticality is also accompanied

[2]Gaussian identity: $\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{z^2}{2\sigma^2}} \, e^{iz} = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{1}{2}\left(\frac{z}{\sigma}-i\sigma\right)^2} e^{-\frac{\sigma^2}{2}} = e^{-\frac{\sigma^2}{2}}$.

by other special features, such as, for example, the non-trivial fluctuations of the "failed" order parameter $m = N^{-1}\langle|\sum_{i=1}^{N}\vec{s}_i|\rangle$ for finite system size [10]. Indeed, the thermally averaged value of the order parameter m has abnormally large finite size corrections. Within a spin wave calculation one finds $m = (1/(2N))^{k_BT/(8\pi J)}$ with the expected vanishing value in the thermodynamic limit but rather large values at low temperature and finite sizes. Monte Carlo simulations demonstrate that the distribution function, $P(y)$ with $y = N^{-1}|\sum_{i=1}^{N}\vec{s}_i|$ is a universal asymmetric form with interesting characteristics.

1.3.2 The $2d$ XY model: high temperature expansions

A first quantitative hint on the fact that there must be a phase transition in the $2d$ XY model came from the study of the high temperature expansion [11, 12]. The method is very similar to the one used to study Ising spin systems. With the aim of developing a small β Taylor expansion, the partition function is written as

$$Z = \int \prod_i d\theta_i \; e^{\frac{\beta J}{2}\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)} = \int \prod_i d\theta_i \prod_{\langle ij\rangle} e^{\frac{\beta J}{2}\cos(\theta_i-\theta_j)} \tag{1.29}$$

Exploiting the periodicity of the exponential of the cos, one can use several tricks to derive

$$G(r) = e^{-r/\xi} \qquad \text{with} \qquad \xi = a/\ln(4k_BT/J) \tag{1.30}$$

an exponential decay of the correlation function. This calculation strongly suggests that there must be a phase transition between the high temperature disordered phase and a low temperature phase, the latter with, possibly, quasi long-range order predicted by the spin-wave approximation.

1.3.3 The $2d$ XY model: vortices and the Kosterlitz-Thouless transition

The failure of the spin-wave approximation at high temperatures is rooted in that this approximation only allows for small and smooth deviations (gradient expansion) about the ferromagnetically ordered state. In particular, it excludes configurations in which the angular field is singular at some isolated point(s). In other words, only *single-valued* functions ϕ satisfying,

$$\sum_{\vec{r},\vec{r}'}^{C}[\phi(\vec{r}) - \phi(\vec{r}')] \mapsto \int_C d\vec{r}' \cdot \vec{\nabla}\phi(\vec{r}') = 0 \tag{1.31}$$

for any closed path C are admitted in the spin-wave expansion. However, in the $2d$ XY model, only the spin $\vec{s}_i$ should be single-valued and the original Hamiltonian (1.22) defined on the lattice has a *discrete symmetry*

$$\phi_i - \phi_j \to \phi_i - \phi_j \pm 2\pi \tag{1.32}$$

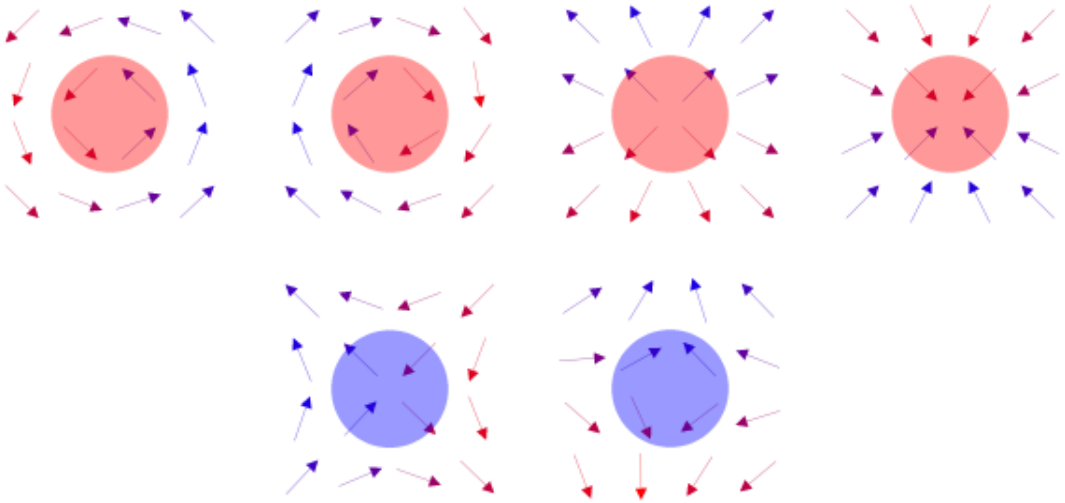

Figure 1.7: Four examples of vortices with charge $q = 1$ (first line) and two examples of anti-vortices with $q = -1$ (second line). Figures borrowed from [16].

that is lost in the continuous approximation (1.24). This symmetry permits the existence of *vortices*, a particular kind of *topological defects*. These excitations are the ones that kill the simple spin-wave prediction of there being quasi long-range order at all temperatures, as explained by Kosterlitz & Thouless in the series of papers [13, 14, 15]. Kosterlitz & Thouless (together with Haldane) were retributed the Nobel Prize in 2016 for having exhibited a new class of phase and phase transitions, called *topological phases*.

> *Topological defects* are configurations, in this case spin configurations, that are local minima of the potential energy and that *cannot* be smoothly transformed into the ground state, in this case the configuration in which all the spins are aligned, by a continuous transformation of variables, in this case a continuous rotation of all spins.

On the lattice a vortex configuration is such that

$$\sum_{ij \in C} [\phi_i - \phi_j] = 2\pi q \tag{1.33}$$

with q an integer ensuring that the spin be single-valued on each site of the lattice. The center of the vortex is located on a site of the *dual lattice*.

In a continuous description of the lattice problem, this means that there is no transformation of the kind

$$\vec{s}(\vec{r}) \mapsto R(\vec{r})\vec{s}(\vec{r}) \tag{1.34}$$

with a continuous rotation matrix $R(\vec{r})$ that transforms the configuration with a topological defect into one of the ground state (and continuously transformable into a spin-wave state).

In the $2d$ XY model, the topological defects are vortices. Concretely, vortex configurations are local minima of the Hamiltonian

$$\frac{\delta H}{\delta\phi(\vec{r})} = 0 \qquad \text{and} \qquad \frac{\delta^2 H}{\delta\phi(\vec{r})\delta\phi(\vec{r}')} \quad \text{positive definite} \tag{1.35}$$

where the second condition ensures their stability.

The solution $\phi(\vec{r}) = \text{cst}$ is not the only field configuration that represents the continuous limit of the original discrete problem. Vortex configurations, $\phi(\vec{r})$, in which the field has a *singularity* at the location of a point-like *charge*, are also solutions. A vortex configuration can be written as

$$\phi(\vec{r}) = q\varphi(\vec{r}) + \phi_0 \tag{1.36}$$

with q the *integer charge* and $\varphi(\vec{r})$ the polar angle (angle with the horizontal x axis) of the space point $\vec{r}$ and ϕ_0 an additive constant. As a example, let us take $q = 1$ and $\phi_0 = 0$. One can easily construct the spin configuration associated with this $\phi(\vec{r})$, that is $\vec{s}(\vec{r}) = (\cos\phi(\vec{r}), \sin\phi(\vec{r}))$. The arrows point as in the third panel in the first line in Fig. 1.7. Another choice is to use $q = 1$ and $\phi_0 = \pi/2$, leading to a configuration in which the spins turn anti-clockwise as in the left Fig. 1.10. Finally, one can use $q = 1$ and $\phi_0 = \pi$ to construct a configuration in which all spins point inwards, as in the last snapshot in the first line in Fig. 1.7.

All the configurations with the same q can be continuously transformed into one another. In the cases listed in the previous paragraph, $q = 1$, and

$$\vec{s}(\vec{r}, t) = (\cos(\varphi + t), \sin(\varphi + t)) \tag{1.37}$$

with t a real parameter, taking the values $t = 0, \pi/2$ and π in these particular cases. However, there is no parameter t that makes this configuration be the one of a constant field.

The divergence of the configuration $\phi(\vec{r}) = q\varphi(\vec{r}) + \phi_0$ is

$$\begin{aligned}\vec{\nabla}\phi(\vec{r}) &= q\vec{\nabla}\varphi(\vec{r}) = q\vec{\nabla}\arctan\left(\frac{y}{x}\right) = -q\frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}}\hat{e}_x + q\frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}}\hat{e}_y \\ &= -q\,\frac{y}{x^2+y^2}\,\hat{e}_x + q\,\frac{x}{x^2+y^2}\,\hat{e}_y = -q\,\frac{r\sin\varphi}{r^2}\,\hat{e}_x + q\,\frac{r\cos\varphi}{r^2}\,\hat{e}_y\end{aligned}$$

and

$$\boxed{\vec{\nabla}\phi(\vec{r}) = \frac{q}{r}\,\hat{e}_\varphi} \tag{1.38}$$

where we used $\hat{e}_\varphi = \cos\varphi\,\hat{e}_y - \sin\varphi\,\hat{e}_x$. One clearly sees the divergence for $r \to 0$. The problem is *spherically symmetric* in the sense that the modulus of the gradient of the field only depends on the modulus of r, $|\vec{\nabla}\phi(\vec{r})| = f(r)$.

Let us see whether the Laplacian of the angle vanishes for all $r \neq 0$:

$$\vec{\nabla}\cdot\vec{\nabla}\phi(\vec{r}) = q\,\frac{2xy}{(x^2+y^2)^2} - q\,\frac{2xy}{(x^2+y^2)^2} = 0\,.$$

At the origin one has to be more careful because of the divergence of the gradient.

Taking a circle with radius R and centred at the centre of the vortex, the circulation of the field ϕ around C yields

$$\oint_C d\phi(\vec{r}) = \oint_C d\vec{l} \cdot \vec{\nabla}\phi(\vec{r}) = \int_0^{2\pi} R\, d\varphi\, \hat{e}_\varphi \cdot \frac{q}{R}\, \hat{e}_\varphi = 2\pi q \; . \tag{1.39}$$

Actually, in a single vortex configuration the angle winds around the topological defect for any contour C around the centre of the vortex[3]

$$\oint_C d\phi(\vec{r}) = \oint_C d\vec{l} \cdot \vec{\nabla}\phi(\vec{r}) = 2\pi q \; . \tag{1.40}$$

(Note that the spin has to point in the same direction after coming back to the starting point of the circulation, this condition implies that q must be an integer.) The integral yields this non-vanishing result for all paths C that encircle the centre of the vortex and vanishes on paths that do not. The position of the vortex corresponds to a singularity in the field that is constructed with the *coarse-graining* procedure. The discrete nature of the charge makes it impossible to find a continuous deformation which returns the state to the uniformly ordered configuration in which the charge is zero. (One justifies the continuous treatment of the spin rotation by taking a curve around the vortex core with a sufficiently large "radius" so that the variations in angle will be small and the lattice structure can be ignored. The continuous approximation fails close to the core of the vortex.) A vortex creates a distortion in the phase field $\phi(\vec{r})$ that persists infinitely far from the centre of the vortex.

The electromagnetic analogy, that is explained in detail in the book by Chaikin & Lubensky [19], is such that

$$\begin{array}{rrcl} \text{magnetic induction} & \vec{B} & \leftrightarrow & \vec{\nabla}\phi \\ \text{electric current density} & \vec{J} & \leftrightarrow & \vec{\mathcal{M}} = \vec{\nabla} \times \vec{\nabla}\phi \\ \text{vector potential} & \vec{\nabla} \times \vec{A} & \leftrightarrow & \vec{\nabla}\phi \end{array} \tag{1.41}$$

The current density is singular at the location of the centre of the vortices as

$$\vec{\mathcal{M}}(\vec{r}) = 2\pi \sum_i q_i \delta(\vec{r} - \vec{r}_i)\; \hat{e}_z = 2\pi \rho(\vec{r})\; \hat{e}_z \tag{1.42}$$

where $\vec{r}$ lives on the two dimensional plane and $\hat{e}_z$ is perpendicular to it. $\rho(\vec{r})$ is the charge density constituted by point-like charges located at positions $\vec{r}_i$.

Several singular configurations are shown in Fig. 1.7, with vortices ($q = 1$) in the first row and antivortices ($q = -1$) in the second row (figures borrowed from [16]). A simple

[3]Recall Gauss' divergence theorem $\int dV\; \vec{\nabla} \cdot \vec{F} = \int dS\; \hat{n} \cdot \vec{F}$, where the volume integral on the left transforms into the surface integral on the right. Applied to a volume in two dimensions and $\vec{F} = \vec{\nabla}\phi$, one goes from eq. (**??**) to eq. (1.40) for a single vortex with charge q.

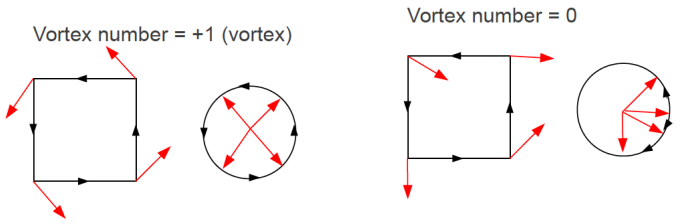

Figure 1.8: A graphical way to visualize the charge of a vortex. One places on the circle an arrow corresponding the the "firs" (arbitrary choice) spin. One takes the next spin on the plaquette, conventionally turning in anti-clockwise order, and places a second arrow on the circle. One repeats the procedure until the last spin on the plaquette. The points on the circle are numbered according to the order of the spins on the plaquette, $1, \ldots, 4$ in these examples. If the points make one turn on the circle the charge is $q = 1$. If it has made an anti-turn the charge is $q = -1$. If they make more than one turn the charge is higher than 0.

visualisation of the winding angle is sketched in Fig. 1.8. Vortices with higher charge are also possible (though as they have a higher energetic cost they are less common), see Fig. 1.9. A vortex and a nearby anti-vortex configuration are shown in Fig. 1.10 and some constant spin lines around vortex-antivortex pairs are shown in Fig. 1.11. The latter appear bounded in the low temperature phase, see Fig. 1.12.

An angular configuration with M vortices with charge q_i situated at the points $\vec{r}_i$ is

$$\phi(\vec{r}) = \sum_i^M q_i \arctan\left(\frac{(\vec{r}-\vec{r}_i)_y}{(\vec{r}-\vec{r}_i)_x}\right) \tag{1.43}$$

where the sub-scripts $_x$ and $_y$ indicate the horizontal and vertical components.

Let us evaluate the *energy* of a single vortex configuration. We have already argued that the vortex configuration satisfies

$$\vec{\nabla}\phi(\vec{r}) = \frac{q}{r}\,\hat{e}_\varphi \tag{1.44}$$

where, without loss of generality, we set the origin of coordinates at the center of the vortex, θ is the angle of the position $\vec{r}$ with respect to the x axis, and q is the charge of the vortex. Using the expression (1.24) where $\vec{\nabla}\phi(\vec{r})$ is replaced by (1.44),

$$E_{1\ vortex} = \frac{J}{2}\int d^2r\ [\vec{\nabla}\phi(\vec{r})]^2 = \frac{J}{2}q^2\int_0^{2\pi} d\varphi \int_a^L dr\ r\ \frac{1}{r^2} = \pi J q^2 \ln\frac{L}{a} \tag{1.45}$$

with L the linear dimension of the system. The energy of a single vortex

- increases quadratically with its charge
- diverges logarithmically in the infinite size limit

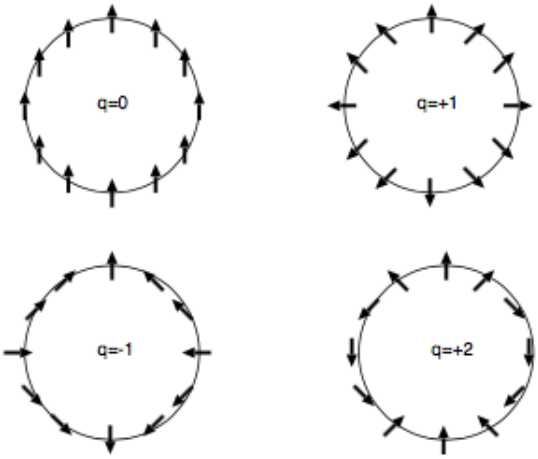

Figure 1.9: Four vortices with charges $q = 0, 1, -1, 2$.

and one might conclude that these configurations cannot exist in equilibrium at any temperature. However, as already discussed in the Ising chain, at finite T one needs to estimate the *free-energy* difference between configurations with and without a vortex to decide for their existence or not. The entropy of a single vortex is $S = k_B \ln \mathcal{N} = k_B \ln(L/a)^2$ since in a $2d$ lattice the centre of the vortex can be located on $(L/a)^2$ different sites. Then

$$\Delta F = F_{1\ vortex} - F = (\pi J q^2 - 2k_B T) \ln(L/a) \ . \tag{1.46}$$

This quantity changes sign at $k_B T = \pi J q^2/2$ therefore *there cannot be isolated vortices in equilibrium below* $k_B T_{KT} = \pi J/2$ but they can at higher temperature. Indeed, at $T > T_{KT}$, isolated vortices proliferate (favoured by the entropic contribution), destroy the quasi long-range order and make correlations decay exponentially on a length-scale given by the typical spacing between vortices

$$G(r) \simeq e^{-r/\xi(T)} \qquad\qquad \xi(T) \simeq e^{b|T-T_{KT}|^{-1/2}} \tag{1.47}$$

close to T_{KT}. This very fast divergence of the correlation length, $\nu \to \infty$, can be rigorously proven with an RG analysis [15] that we shall not present here.

The estimate of T_{KT} just given represents only a bound for the stability of the system towards the condensation of topological defects. Pairs (dipoles) of defects may appear at larger couplings or lower temperatures.

Although the energy of a single vortex diverges as $\ln L$, the energy of a *bound pair* of vortex-antivortex does not diverge, since, the total vorticity of the pair vanishes, see the Fig. 1.11 taken from [6]. Below T_{KT} vortices exist only in bound pairs with opposite vorticity held together by a logarithmic confining potential

$$E_{pair}(\vec{r}_1, \vec{r}_2) = -\pi J q_1 q_2 \ln(|\vec{r}_1 - \vec{r}_2|/a) \ . \tag{1.48}$$

This expression follows uniquely from the fact that at distances much larger than the pair's size there is no net vorticity, so the energy of the pair must be finite, and as the pair's size diverges E_{pair} should yield the sum of the energies for an isolated vortex and an isolated antivortex. (A more detailed calculation uses an integral over a contour in the $2d$ plane that excludes the centers of the vortices. In particular, this approach allows one to show that a sum over the energies of the single vortices appears multiplied by $\sum_i q_i$ and this divergence is eliminated if the total vorticity is zero, i.e. $\sum_i q_i = 0$.) Such pairs can thus be thermally excited, and the low temperature phase will host a gas of such pairs. The insight by Kosterlitz and Thouless was that at a certain temperature T_{KT} the pairs will break up into individual vortices. It is this vortex pair unbinding transition that will take the system to a high temperature phase with exponentially decaying correlations.

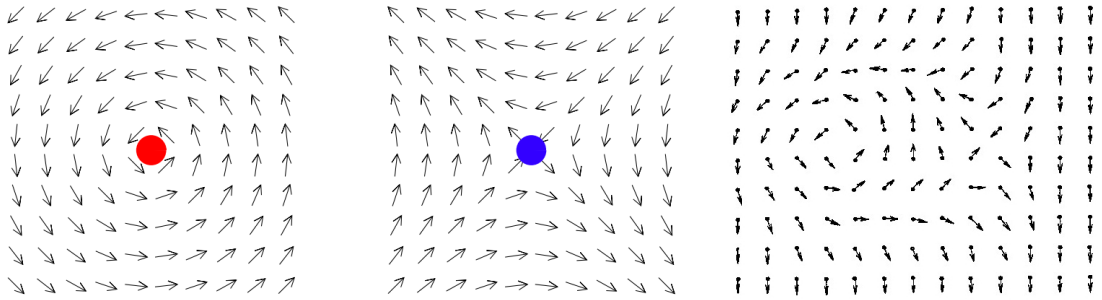

Figure 1.10: A vortex and a near-by anti-vortex configuration as they may appear bounded in the low temperature phase.

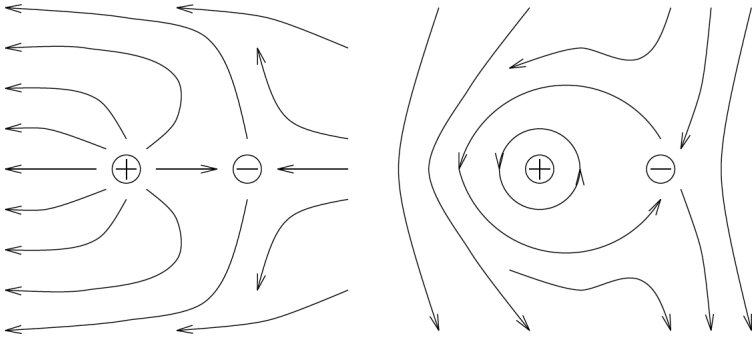

Figure 1.11: Lines of spin direction close to a vortex-antivortex pair. As one observes the spin configurations far from the vortex cores, the lines of constant spin are smooth.

The (single) vortices and anti-vortices act as if they were two point particles with

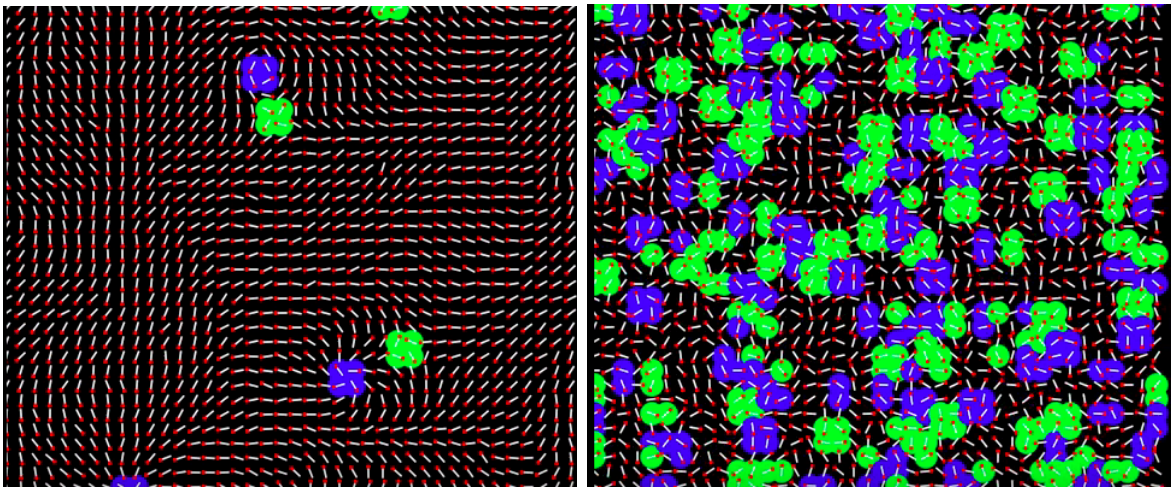

Figure 1.12: At low T there are few vortices and they are bound in pairs. At high T there are many more vortices, they are free and can separate apart. Image taken from [17].

charges $q = +1$ and $q = -1$ interacting with a $1/r$ force. Since this corresponds to the Coulomb interaction in two dimensions, the physics of the topological defects is just like the physics of a two-dimensional *neutral Coulomb gas*. Note that the energy increases if one tries to unbind – separate – the vortices in the pair. The vortices remain paired and do not change much the behaviour in the low temperature phase. The correlation still decays as a power-law and there is no spontaneous symmetry breaking in this phase since the order parameter vanishes – in agreement with the Mermin-Wagner theorem that we discuss below. In terms of the electrostatic analogy, the high temperature phase is a plasma. A detailed description of the vortex influence on the equilibrium properties of the $2d$ XY goes beyond the scope of these Lectures. A detailed description can be found in several book, in particular in [7].

This argument shows that two qualitatively different equilibrium states exist at high and low T but it does not characterize the transition. The naive order parameter vanishes on both sides of the transition but there is still a topological order, with the spin-spin correlation decaying exponentially on one side (high T) and as a power law on the other (low T) of the transition. In contrast to usual continuous phase transitions, the KT-transition does not break any symmetry.

1.3.4 The $2d$ XY model: Nobel Prize and applications

From the Nobel Lecture: *In 1972 J. Michael Kosterlitz and David J. Thouless identified a completely new type of phase transition in two-dimensional systems where topological*

defects play a crucial role. Their theory applied to certain kinds of magnets and to superconducting and superfluid films, and has also been very important for understanding the quantum theory of one-dimensional systems at very low temperatures.

Other two dimensional systems, notably those of particles in interaction that would like to form solids at sufficiently low temperature and high densities, also fall into the scheme of the Kosterlitz-Thouless phase transitions. Indeed, in 1935, Peierls argued that thermal motion of *long-wave length phonons* will destroy the long-range order or a two dimensional solid in the sense that the mean square deviation of an atom from its equilibrium position increases logarithmically with the size of the system. He also proposed a model, just atoms sitting on a lattice in $2d$ and linked together by Hookean springs, that has quasi long-range order at all temperatures [18]. Quasi long range order means in this context that he mean square deviation of an atom from its equilibrium position increases logarithmically with the size of the system. It was later understood that the mechanism for distabilising this critical phase is through the unbinding of topological defects that are of a different kind from the ones we studied here. (For more details, see, for example the slides that I included in my web page.)

For similar reasons, the expectation value of the *superfluid order parameter* in a two dimensional Bose fluid is zero. In 1978, Bishop and Reppy studied the superfluid transition of a thin two dimensional helium film absorbed on an oscillating substrate. The observation results on superfluid mass and dissipation supported the Kosterlitz-Thouless picture of the phase transition in a two dimensional superfluid. The jump in the superfluid density at the transition given by Kosterlitz and Thouless is in good agreement with estimates from experiment.

1.3.5 The Mermin-Wagner theorem

What happens in $d = 2$ and below? Indeed, the logarithmic behaviour of the angle correlation function in the XY model or the transverse correlation in the generic $O(n)$ model, see below, are signatures of the fact that this is a special dimension.

In 1968, using a mathematical inequality, Mermin showed that the magnetisation density m is strictly zero at all $T > 0$ in the $2d$ XY model. This proof is part of what is nowadays called the Mermin-Wagner theorem.

The *Mermin-Wagner theorem* states that for any system with short-range interactions there is a *lower critical dimension* below which no spontaneous broken symmetry can exist at finite temperature. In other words, fluctuations are so large that any ordering that breaks a continuous symmetry is destroyed by thermal fluctuations. $d_c = 1$ for discrete symmetries and $d_c = 2$ for systems with continuous symmetries. The absence of long-range order in the $2d$ XY case, for example, is demonstrated by the fact that the finite temperature correlation decays to zero at long distances – albeit as a power law – and thus there is no net magnetisation in the system.

In a continuous spin model the cost of an interface is proportional to its surface divided by its thickness (note that spins can smoothly rotate from site to site to create a thick

interface). The thickness of the interface depends on the details of the model, temperature, *etc.* This means that interfaces are much easier to create in continuous spin models than in discrete ones.

The Mermin-Wagner theorem is known as Coleman-Weinberg theorem/result in field theory.

1.3.6 O(n) model: Ginzburg-Landau field theory and Goldstone modes

We lift here the constraint on the modulus of the vector spins and we let it fluctuate. It is simple to derive a continuum limit of the lattice model in analogy with the Landau approach. One first coarse-grains the two-component spin to construct a n-component field

$$\vec{\psi}(\vec{r}) = \ell^{-d} \sum_{i \in V_{\vec{r}}} \vec{s}_i \ . \tag{1.49}$$

Let us first focus on the *d dimensional $O(2)$ model*, where the field has just two components. One proposes a Landau ψ^4 action for the field $\vec{\psi}$,

$$F[\vec{\psi}] = \int d^d r \left[\frac{1}{2} [\vec{\nabla} \vec{\psi}(\vec{r})]^2 + \frac{T - T_c}{T_c} \psi^2(\vec{r}) + \frac{\lambda}{4!} \psi^4(\vec{r}) + \vec{h} \vec{\psi}(\vec{r}) \right] \tag{1.50}$$

and parametrises the field by its modulus and angle,

$$\vec{\psi}(\vec{r}) = |\phi_0(\vec{r})| (\cos \phi(\vec{r}), \sin \phi(\vec{r})) \qquad (\text{or } \ \vec{\psi}(\vec{r}) = |\phi_0(\vec{r})| e^{i\phi(\vec{r})}) \ . \tag{1.51}$$

to rewrite the Landau free-energy of a generic configuration in the absence of the external field $\vec{h}$ as

$$F[\phi_0, \phi] = \int d^d r \left[\frac{1}{2} (\vec{\nabla} \phi_0(\vec{r}))^2 + \frac{T - T_c}{T_c} \phi_0^2(\vec{r}) + \frac{\lambda}{4!} \phi_0^4(\vec{r}) \right] \tag{1.52}$$

$$+ \frac{\phi_0^2}{2} \int d^d r \ [\vec{\nabla} \phi(\vec{r})]^2 \ . \tag{1.53}$$

The first term is just similar to the Landau free-energy of a *massive scalar field* configuration in the Ising model. The second-term quantifies the free-energy of the spin-wave configurations (in higher dimensions topological defects also exist, for example, in $d = 3$ this model has vortex lines with linear singularities). The local angle is simply a *massless scalar field* in d dimensional space. The correlation functions of the ϕ field behave as

$$\langle \, \phi(\vec{r}) \phi(\vec{r}') \, \rangle \sim (2 - d)^{-1} |\vec{r} - \vec{r}'|^{2-d} \tag{1.54}$$

in the large $|\vec{r} - \vec{r}'|$ limit for $d = 1, 2$. The behaviour is logarithmic in $d = 2$ (the $2d$ XY model). The correlation reaches a constant in $d > 2$.

Let us now focus on the generic *d dimensional $O(n)$ model*. The free-energy à la Landau is the one in Eq. (1.50)

$$F[\vec{\psi}] = \int d^d r \left[\frac{1}{2} (\vec{\nabla} \vec{\psi}(\vec{r}))^2 + \frac{T - T_c}{T_c} \psi^2(\vec{r}) + \frac{\lambda}{4!} \psi^4(\vec{r}) + \vec{h} \vec{\psi}(\vec{r}) \right] \tag{1.55}$$

Figure 9.2. A 'Mexican hat' potential.

Figure 1.13: A Mexican hat potential, figure taken from [8].

where $\psi^2 \equiv \sum_{a=1}^{N} \psi_a^2$ is the result of a sum over n components. The potential $V(\psi^2)$ has the Mexican hat form sketched in Fig. 1.13 (credit to A. M. Tsvelik), with extrema at

$$\vec{\psi} = \vec{0} \qquad \text{or} \qquad \psi^2 = -\frac{4!}{2\lambda}\frac{T - T_c}{T_c} \tag{1.56}$$

Clearly, the latter exists only if $T < T_c$ and we focus on this range of temperatures. It is clear that the condition on ψ^2 admits an infinite number of solutions, in other words, there is a *ground state manifold*, corresponding to the circular bottom of the valley in the Mexican hat potential. The pinning field $\vec{h}$ can then be used to force the system to choose one among all these degenerate directions in the n dimensional space, in which the field "condenses". Let us suppose that this is the nth direction that we therefore call *longitudinal*. The rotation symmetry in the remaining *transverse* $n-1$ directions remains unbroken and the symmetry is therefore spontaneously broken to $O(n-1)$. The expected values of such a configuration is then

$$\langle \psi_a(\vec{r}) \rangle = \psi \delta_{an} \tag{1.57}$$

while the fluctuations are

$$\begin{aligned} \psi_n(\vec{r}) &= \langle \psi_n(\vec{r}) \rangle + \delta\psi_n(\vec{r}) \\ \psi_{a\neq n}(\vec{r}) &= \delta\psi_{a\neq n} \end{aligned} \tag{1.58}$$

(think of the case $n = 3$, choosing the n direction to be the z vertical one and the rotations around this axis). Replacing these forms in the Landau free-energy one finds that the longitudinal mode is massive while the transverse ones are massless (just decoupled Gaussian fields).

The correlation functions, $C_{ab}(\vec{r}) = \langle \psi_a(\vec{r})\psi_b(\vec{0}) \rangle$, can be written as

$$C_{ab}(\vec{r}) = \delta_{ab}\left[C_L(r)\delta_{an} + C_T(r)(1 - \delta_{an})\right] \ . \tag{1.59}$$

We recall that a and b label the components in the n-dimensional space. C_L is the *longitudinal* correlation (parallel to an infinitesimal applied field that selects the ordering

direction, $\vec{h} = h\hat{e}_n$) and C_T is the *transverse* (orthogonal to the applied field) one. A simple calculation shows that the longitudinal component behaves just as the correlation in the Ising model. It is a massive scalar field. The transverse directions, instead, are massless: there is no restoring force to the tilt of the full system. These components behave just as the angle in the XY model, $C_T(\vec{r}) \sim r^{2-d}$ (the power law decay becomes a logarithm in $d = 2$). These are called *Goldstone modes* or *soft modes*.

1.3.7 The Higgs mechanism

A particular feature of models with continuous symmetry breaking in *gauge theories* is that gauge fields acquire a mass through the process of spontaneous symmetry breaking. Take the classical Abelian field theory

$$\mathcal{L}[A_\mu, \phi] = \int d^d r \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) + V(\phi) \right] \tag{1.60}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + ieA_\mu$ and ϕ a complex field. The potential is

$$V(\phi) = \mu(\phi^*\phi) + \lambda(\phi^*\phi)^2 \tag{1.61}$$

with $\mu < 0$ and $\lambda > 0$. The ϕ configuration that renders V minimum is such that $\phi_0^*\phi_0 = -2\mu/\lambda$. Without loss of generality one can choose ϕ_0 to be real through a uniform rotation over all space. It is easy to verify that replacing ϕ by $(\phi_0 + \delta\phi) + i\phi_2$ where ϕ_2 is an imaginary part (playing the role of the transverse components in the analysis of the $O(n)$ model) one finds that the quadratic Lagrangian does not have a ϕ^2 term (massless field) but instead a quadratic term in A appears. The gauge field acquired a mass (there is also a $A_\mu \partial^\mu \phi_2$ term that can be eliminated with a change of variables).

This phenomenon has been discovered in the study of superconductors by P. W. Anderson. Indeed, one can find a short account of the historic development in Wikipedia: *The mechanism was proposed in 1962 by Philip W. Anderson, who discussed its consequences for particle physics but did not work out an explicit relativistic model. The relativistic model was developed in 1964 by three independent groups Ð Robert Brout and François Englert, Peter Higgs and Gerald Guralnik, Carl Richard Hagen, and Tom Kibble. Slightly later, in 1965, but independently from the other publications the mechanism was also proposed by Alexander Migdal and Alexander Polyakov at that time Soviet undergraduate students. However, the paper was delayed by the Editorial Office of JETP, and was published only in 1966.* The Nobel Prize was given to F. Englert and P. Higgs in 2013 *"for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of the predicted fundamental particle, by the ATLAS and CMS experiments at CERN's Large Hadron Collider"*.

A Polar coordinate system

The polar coordinate system is such that

$$\begin{aligned} \hat{e}_r &= \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi &= -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{aligned} \tag{A.62}$$

and

$$\hat{e}_\varphi = \hat{e}_z \times \hat{e}_r \ . \tag{A.63}$$

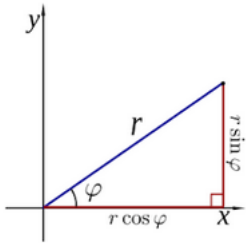

Figure 1.14: Polar coordinates notation convention.

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