Advanced Statistical Physics: Quenched disordered systems

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November 13, 2017

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1 Random fields, random interactions

No material is perfectly homogeneous: impurities of different kinds are distributed randomly throughout the samples. In ultra-cold atom systems, so much studied nowadays, disorder can be realized, for example, using speckle laser light.

A natural effect of disorder should be to lower the critical temperature. Much attention has been payed to the effect of *weak disorder* on phase transitions, that is to say, situations in which the nature of the ordered and disordered phases is not modified by the impurities but the critical phenomenon is. On the one hand, the critical exponents of second order phase transitions might be modified by disorder, on the other hand, disorder may smooth out the discontinuities of first order phase transitions rendering them of second order. *Strong disorder* instead changes the nature of the low-temperature phase and before discussing the critical phenomenon one needs to understand how to characterize the new ordered 'glassy' phase.

In this Section we shall discuss several types of *quenched disorder* and models that account for it. We shall also overview some of the theoretical methods used to deal with the static properties of models with quenched disorder, namely, scaling arguments and the droplet theory, mean-field equations, and the replica method.

1.1 Quenched and annealed disorder

First, one has to distinguish between *quenched or frozen-in* and *annealed* disorder. Imagine that one mixes some random impurities in a melt and then very slowly cools it down in such a way that the impurities and the host remain in thermal equilibrium. If one wants to study the statistical properties of the full system one then has to compute the full partition function in which one sums over all configurations of original components and impurities. This is called annealed disorder. In the opposite case in which upon cooling the host and impurities do not equilibrate but the impurities remain blocked in random fixed positions, one talks about quenched disorder. Basically, the relaxation time associated with the diffusion of the impurities in the sample is so long that these remain trapped:

$$\tau_o \sim 10^{-12} - 10^{-15} \text{ sec} \ll t_{\text{obs}} \sim 10^4 \text{ sec} \ll t_{\text{diff}} \ , \tag{1.1}$$

where τ_o is the microscopic time associated to the typical scale needed to reverse a spin.

The former case is easier to treat analytically but is less physically relevant. The latter is the one that leads to new phenomena and ideas that we shall discuss next.

Quenched disorder is static. Instead, in annealed disorder the impurities are in thermal equilibrium in the experimental time-scales, and they can simply be included in the statistical mechanic description of the problem.

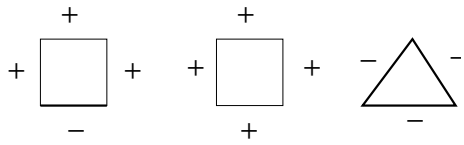

Figure 1.1: A frustrated (left) and an unfrustrated (center) square plaquette. A frustrated triangular plaquette (right).

1.2 Properties

1.2.1 Lack of homogeneity

It is clear that the presence of quench disorder, in the form of random interactions, fields, dilution, *etc.* breaks spatial homogeneity and renders single samples heterogenous. Homogeneity is recovered though, if one performs an average of all possible realizations of disorder, each weighted with its own probability.

1.2.2 Frustration

Depending on the value of the distance r_{ij} the numerator in Eq. (1.10) can be positive or negative implying that both ferromagnetic and antiferromagnetic interactions exist. This leads to *frustration*, which means that in any configuration some two-body interactions remain unsatisfied. In other words, there is no spin configuration that minimizes all terms in the Hamiltonian. An example with four sites and four links is shown in Fig. 1.1-left, where we took three positive exchanges and one negative one all, for simplicity, with the same absolute value, J. Four configurations minimize the energy, $E_f = -2J$, but none of them satisfies the lower link. One can easily check that any closed loop such that the product of the interactions takes a negative sign is frustrated. Frustration naturally leads to a *higher energy* and a *larger degeneracy* of the number of ground states. This is again easy to grasp by comparing the number of ground states of the frustrated plaquette in Fig. 1.1-left to its unfrustrated counterpart shown on the central panel. Indeed, the energy and degeneracy of the ground state of the unfrustrated plaquette are $E_u = -4J$ and $n_u = 2$, respectively.

Frustration may also be due to pure geometrical constraints. The canonical example is an antiferromagnet on a triangular lattice in which each plaquette is frustrated, see Fig. 1.1-right.

In short, frustration arises when the geometry of the lattice and/or the nature of the interactions make impossible to simultaneously minimize the energy of all pair couplings between the spins. Any loop of connected spins is said to be frustrated if the product of the signs of connecting bonds is negative. In general, energy and entropy of the ground states increase due to frustration.

1.2.3 Self-averageness

If each sample is characterized by its own realization of the exchanges, should one expect a totally different behavior from sample to sample? Fortunately, many generic static and dynamic properties of spin-glasses (and other systems with quenched disorder) do not depend on the specific realization of the random couplings and are *self-averaging*. This means that the typical value is equal to the average over the disorder:

$$A_J^{\rm typ} = [\, A_J \,] \tag{1.2}$$

in the thermodynamic limit. Henceforth, we use square brackets to indicate the average over the random couplings. More precisely, in self-averaging quantities sample-to-sample fluctuations with respect to the mean value are expected to be $\mathrm{O}(N^{-a})$ with $a > 0$. Roughly, observables that involve summing over the entire volume of the system are expected to be self-averaging. In particular, the free-energy density of models with short-ranged interactions is expected to be self-averaging in this limit.

An example: the disordered Ising chain

The meaning of this property can be grasped from the solution of the random bond Ising chain defined by the energy function $H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1}$ with spin variables $s_i = \pm$, for $i = 1, \ldots, N$ and random bonds J_i independently taken from a probability distribution $P(J_i)$. For simplicity, we consider periodic boundary conditions. The disorder-dependent partition function reads

$$Z_J = \sum_{\{s_i = \pm 1\}} e^{\beta \sum_i J_i s_i s_{i+1}} \tag{1.3}$$

and this can be readily computed introducing the change of variables $\sigma_i \equiv s_i s_{i+1}$. (Note that these new variables are not independent, since they are constrained to satisfy $\prod_i \eta_i = 1$. This constraint is irrelevant in the thermodynamic limit.) One finds

$$Z_J = \prod_i 2\cosh(\beta J_i) \qquad \Rightarrow \qquad -\beta F_J = \sum_i \ln\cosh(\beta J_i) + N \ln 2 \,. \tag{1.4}$$

The partition function is a *product* of *i.i.d.* random variables and it is itself a random variable with a log-normal distribution. The free-energy density instead is a *sum* of *i.i.d.* random variables and, using the central limit theorem, in the large N limit becomes a Gaussian random variable narrowly peaked at its maximum. The typical value, given by the maximum of the Gaussian distribution, coincides with the average, $\lim_{N\to\infty}(f_J^{\rm typ} - [\, f_J \,]) = 0$ with $f_J = F_J/N$.

General argument

A simple argument justifies the self-averageness of the free-energy density in generic finite dimensional systems with short-range interactions. Let us divide a, say, cubic system of volume $V = L^d$ in n subsystems, say also cubes, of volume $v = \ell^d$ with $V = nv$.

If the interactions are short-ranged, the total free-energy is the sum of two terms, a contribution from the bulk of the subsystems and a contribution from the interfaces between the subsystems: $-\beta F_J = \ln Z_J = \ln \sum_{\rm conf} e^{-\beta H_J({\rm conf})} = \ln \sum_{\rm conf} e^{-\beta H_J({\rm bulk})-\beta {\rm H_J(surf)}} \approx \ln \sum_{\rm bulk} e^{-\beta H_J({\rm bulk})} + \ln \sum_{\rm surf} e^{-\beta H_J({\rm surf})} = -\beta F_J^{\rm bulk} - \beta F_J^{\rm surf}$ (we neglected the contributions from the interaction between surface and bulk). If the interaction extends over a short distance σ and the linear size of the boxes is $\ell \gg \sigma$, the surface energy is negligible with respect to the bulk one and $-\beta F_J \approx \ln \sum_{\rm bulk} e^{-\beta H_J({\rm bulk})}$. In the thermodynamic limit, the disorder dependent free-energy is then a sum of $n = (L/\ell)^d$ random numbers, each one being the disorder dependent free-energy of the bulk of each subsystem: $-\beta F_J \approx \sum_{k=1}^{n} \ln \sum_{{\rm bulk}_k} e^{-\beta H_J({\rm bulk}_k)}$. In the limit of a very large number of subsystems ($L \gg \ell$ or $n \gg 1$) the central limit theorem implies that the total free-energy is Gaussian distributed with the maximum reached at a value $F_J^{\rm typ}$ that coincides with the average over all realizations of the randomness $[\, F_J \,]$. Morever, the dispersion about the typical value vanishes in the large n limit, $\sigma_{F_J}/[\, F_J \,] \propto \sqrt{n}/n = n^{-1/2} \to 0$. Similarly, $\sigma_{f_J}/[f_J] \sim O(n^{-1/2})$ where $f_J = F_J/N$ is the intensive free-energy. In a sufficiently large system the typical F_J is then very close to the averaged $[\, F_J \,]$ and one can compute the latter to understand the static properties of typical systems.

Lack of self-averageness in the correlation functions

Once one has $[F_J]$, one derives all disordered average thermal averages by taking derivatives of the disordered averaged free-energy with respect to sources introduced in the partition function. For example,

$$[\, \langle\, s_i \,\rangle\,] = - \left. \frac{\partial [\, F_J \,]}{\partial h_i} \right|_{h_i=0} , \tag{1.5}$$

$$[\, \langle\, s_i s_j \,\rangle - \langle\, s_i \,\rangle \langle\, s_j \,\rangle\,] = -T \left. \frac{\partial^2 [\, F_J \,]}{\partial h_i h_j} \right|_{h_i=0} , \tag{1.6}$$

with $H_J \to H_J - \sum_i h_i s_i$. Connected correlation functions, though, *are not* self-averaging quantities. This can be seen, again, studying the random bond Ising chain. Take $i < j$. One can easily check that

$$\langle\, s_i s_j \,\rangle_J - \langle\, s_i \,\rangle_J \langle\, s_j \,\rangle_J = Z_J^{-1} \frac{\partial}{\partial \beta J_{j-1}} \dots \frac{\partial}{\partial \beta J_i} Z_J = \tanh(\beta J_i) \dots \tanh(\beta J_j) \, , \tag{1.7}$$

where we used $\langle\, s_i \,\rangle = 0$ (valid for a distribution of random bonds with zero mean) and the dots indicate all sites on the chain between the ending points i and j, *i.e.* $i+1 \le k \le j-1$. The last expression is a product of random variables and it is not equal to its average (1.6) – not even in the large separation limit $|\vec{r}_i - \vec{r}_j| \to \infty$.

1.2.4 Annealed disorder

The thermodynamics of a system with annealed disorder is obtained by averaging the partition function over the impurity degrees of freedom,

$$Z = [Z_J] \tag{1.8}$$

since one needs to do the partition sum over the disorder degrees of freedom as well.

1.3 Models

1.3.1 Bethe lattices and random graphs

The *Bethe lattice* is a tree, in which each site has z neighbours and each branch gives rise to $z-1$ new branches. Two important properties of these lattices are:
- there are no closed loops.
- the number of sites on the border is of the same order of magnitude as the total number of sites on the lattice.
- All sites on the lattice are equivalent, there is no notion of a central site.

Exercise 1.1 Show that the total number of sites on the Bethe lattice with $z = 3$ and g generations (or the distance from the site designed as the central one) is $n_{\rm tot} = 3\ 2^g - 1$ and the number of sites on the border is $n_{\rm border} = 3\ 2^{g-1}$. The surface to volume ratio tends to 1/2.

Exercise 1.2 Take a hypercubic lattice in d dimensions and estimate the surface to volume ratio. Show that this ratio tends to a finite value only if $d \to \infty$.

A *random graph* is obtained by starting with a set of n isolated vertices and adding successive edges between them at random. A popular ensemble is the one denoted $G(n,p)$, in which every possible edge occurs independently with probability $0 < p < 1$. Random graphs with fixed connectivity are also commonly used.

Random graphs are used in social sciences modeling (nodes representing individuals and edges the friendship relationship), technology (interconnections of routers in the Internet, pages of the WWW, or production centers in an electrical network), biology (interactions of genes in a regulatory network) [40, 41]. Disordered systems are usually defined on random graphs, especially the ones motivated by combinatorial optimisation.

1.3.2 Dilute spin models

Lattice models with site or link dilution are

$$H_J^{\text{site dil}} = -J \sum_{\langle ij \rangle} s_i s_j \epsilon_i \epsilon_j \ , \qquad H_J^{\text{link dil}} = -J \sum_{\langle ij \rangle} s_i s_j \epsilon_{ij} \ , \tag{1.9}$$

with $P(\epsilon_i = 1, 0) = p, 1-p$ in the first case and $P(\epsilon_{ij} = 1, 0) = p, 1-p$ in the second. These models are intimately related to Percolation theory. Physically, dilution is realised by vacancies or impurity atoms in a crystal.

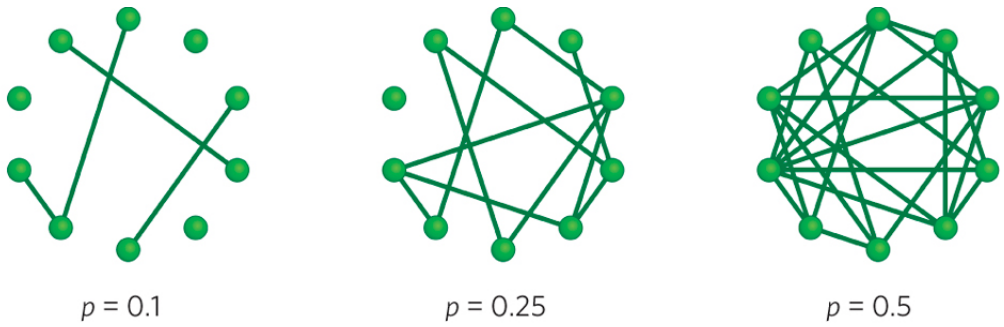

Figure 1.2: Random graphs with $N = 10$ and different probabilities p.

1.3.3 Spin-glass models

Spin-glasses are alloys in which magnetic impurities substitute the original atoms in positions randomly selected during the chemical preparation of the sample [43, 45, 46]. The interactions between the impurities are of RKKY type:

$$V_{\text{rkky}} = -J \, \frac{\cos(2k_F r_{ij})}{r_{ij}^3} \, s_i s_j \tag{1.10}$$

with $r_{ij} = |\vec{r}_i - \vec{r}_j|$ the distance between them and s_i a spin variable that represents their magnetic moment. Clearly, the initial location of the impurities varies from sample to sample. The time-scale for diffusion of the magnetic impurities is much longer than the time-scale for spin flips. Thus, for all practical purposes the positions $\vec{r}_i$ can be associated to quenched random variables distributed according to a uniform probability distribution that in turn implies a probability distribution of the exchanges. This is called *quenched disorder*.

In early 70s *Edwards and Anderson* proposed a rather simple model that should capture the main features of spin-glasses. The interactions (1.10) decay with a cubic power of the distance and hence they are relatively short-ranged. This suggests to put the spins on a regular cubic lattice model and to trade the randomness in the positions into random nearest neighbour exchanges taken from a Gaussian probability distribution:

$$H_J^{\text{ea}} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad \text{with} \qquad P(J_{ij}) = (2\pi\sigma^2)^{-\frac{1}{2}} \, e^{-\frac{J_{ij}^2}{2\sigma^2}} \, . \tag{1.11}$$

The precise form of the probability distribution of the exchanges is supposed not to be important, though some authors claim that there might be non-universality with respect to it.

Another natural choice is to use bimodal exchanges

$$P(J_{ij}) = p\delta(J_{ij} - J_0) + (1-p)\delta(J_{ij} + J_0) \tag{1.12}$$

with the possibility of a bias towards positive or negative interactions depending on the parameter p. A tendency to non-zero average J_{ij} can also be introduced in the Gaussian pdf.

A natural extension of the EA model in which all spins interact has been proposed by Sherrington and Kirkpatrick

$$H_J^{\rm SK} = -\sum_{i\neq j} J_{ij} s_i s_j - \sum_i h_i s_i \tag{1.13}$$

and it is called the *SK model*. The interaction strengths J_{ij} are taken from a Gaussian pdf and they scale with N in such a way that the thermodynamic limit is non-trivial:

$$P(J_{ij}) = (2\pi\sigma_N^2)^{-\frac{1}{2}}\, e^{-\frac{J_{ij}^2}{2\sigma_N^2}} \qquad\qquad \sigma_N^2 = \sigma^2 N\ . \tag{1.14}$$

The first two-moments of the exchange distribution are $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2/(2N)$. This is a case for which a mean-field theory is expected to be exact.

1.3.4 Glass models

A further extension of the EA model is called the *p spin model*

$$H_{J^{\rm p-spin}} = -\sum_{i_1<\cdots<i_p} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} - \sum_i h_i s_i \tag{1.15}$$

with $p \geq 3$. The sum can also be written as $\sum_{i_1<i_2<\cdots<i_p} = 1/p! \sum_{i_1\neq i_2\neq i_p}$. The exchanges are now taken from a Gaussian probability distribution

$$P(J_{ij}) = (2\pi\sigma_N^2)^{-\frac{1}{2}}\, e^{-\frac{J_{ij}^2}{2\sigma_N^2}} \qquad\qquad \sigma_N^2 = J^2 p!/(2N^{p-1})\ . \tag{1.16}$$

with $[J_{i_1\ldots i_p}] = 0$ and $[J_{i_1\ldots i_p}^2] = \frac{J^2 p!}{2N^{p-1}}$. Indeed, an extensive free-energy is achieved by scaling $J_{i_1\ldots i_p}$ with $N^{-(p-1)/2}$. This scaling can be justified as follows. The 'local field' $h_i = 1/(p-1)! \sum_{ii_2\neq i_p} J_{ii_2\ldots i_p} m_{i_2}\ldots m_{i_p}$ should be of order one. At low temperatures the m_i's take plus and minus signs. In particular, we estimate the order of magnitude of this term by working at $T = 0$ and taking $m_i = \pm 1$ with probability $\frac{1}{2}$. In order to keep the discussion simple, let us take $p = 2$. In this case, if the strengths J_{ij}, are of order one, h_i is a sum of N *i.i.d.* random variables, with zero mean and unit variance[1], and h_i has zero mean and variance equal to N. Therefore, one can argue that h_i is of order $\sqrt{N}$. To make it finite we then chose J_{ij} to be of order $1/\sqrt{N}$ or, in other words, we impose $[\,J_{ij}^2\,] = J^2/(2N)$. The generalization to $p \geq 2$ is straightforward.

We classify this model in the "glass" class since it has been shown that its behaviour mimics the one of so-called fragile glasses.

[1]The calculation goes as follow: $\langle\, F_i\,\rangle = \sum_j J_{ij}\langle\, m_j\,\rangle = 0$ and $\langle\, F_i^2\,\rangle = \sum_{jk} J_{ij}J_{ik}\langle\, m_j m_k\,\rangle = \sum_j J_{ij}^2$

1.3.5 Vector spins

Extensions to *vector spins* with two (XY), three (Heisenberg) or N components also exist. In the former cases can be relevant to describe real samples. One usually keeps the modulus of the spins fixed to be 1 in these cases.

But there is another way to extend the spin variables and it is to use a *spherical constraint*,

$$-\infty \leq s_i \leq \infty \qquad \sum_{i=1} s_i^2 = N \; . \tag{1.17}$$

In this case, the spins s_i are the components of an N-dimensional vector, constrained to be an N-dimensional sphere.

1.3.6 Optimization problems

Cases that find an application in **computer science** [7] are defined on random graphs with fixed or fluctuating finite connectivity. In the latter case one places the spins on the vertices of a graph with links between couples or groups of p spins chosen with a probability c. These are *dilute spin-glasses* on graphs (instead of lattices).

Optimisation problems can usually be stated in a form that requires the minimisation of a cost (energy) function over a large set of variables. Typically these **cost functions** have a very large number of local minima – an exponential function of the number of variables – separated by barriers that scale with N and finding the truly absolute minimum is hardly non-trivial. Many interesting optimisation problems have the great advantage of being defined on random graphs and are then mean-field in nature. The mean-field machinery that we will discuss at length is then applicable to these problems with minor (or not so minor) modifications due to the finite connectivity of the networks.

Let us illustrate this kind of problems with two examples. The **graph partitioning** problem consists in, given a graph $G(N, E)$ with N vertices and E edges, to partition it into smaller components with given properties. In its simplest realisation the uniform graph partitioning problem is how to partition, in the optimal way, a graph with N vertices and E links between them in two (or k) groups of equal size $N/2$ (or N/k) and the minimal the number of edges between them. Many other variations are possible. This problem is encountered, for example, in computer design where one wishes to partition the circuits of a computer between two chips. More recent applications include the identification of clustering and detection of cliques in social, pathological and biological networks.

Another example, that we will map to a spin model, is **k-satisfiability (k-SAT)**. The problem is to determine whether the variables of a given Boolean formula can be assigned in such a way to make the formula evaluate to 'TRUE'. Equally important is to determine whether no such assignments exist, which would imply that the function expressed by the formula is identically 'FALSE' for all possible variable assignments. In this latter case, we would say that the function is unsatisfiable; otherwise it is satisfiable.

We illustrate this problem with a concrete example. Let us use the convention x for the requirement x = TRUE and $\overline{x}$ for the requirement x = FALSE. For example, the formula $C_1 : x_1$ OR $\overline{x}_2$ made by a single clause C_1 is satisfiable because one can find the values x_1 = TRUE (and x_2 free) or x_2 = FALSE (and x_1 free), which make $C_1 : x_1$ OR x_2 TRUE. This formula is so simple that 3 out of 4 possible configurations of the two variables solve it. This example belongs to the $k = 2$ class of satisfiability problems since the clause is made by two literals (involving different variables) only. It has $M = 1$ clauses and $N = 2$ variables.

Harder to decide formulæ are made of M clauses involving k literals required to take the true value (x) or the false value ($\overline{x}$) each, these taken from a pool of N variables. An example in 3-SAT is

$$\mathrm{F} = \begin{cases} C_1 \ : \ x_1 \ \mathrm{OR} \ \overline{x}_2 \ \mathrm{OR} \ x_3 \\ C_2 \ : \ \overline{x}_5 \ \mathrm{OR} \ \overline{x}_7 \ \mathrm{OR} \ x_9 \\ C_3 \ : \ x_1 \ \mathrm{OR} \ \overline{x}_4 \ \mathrm{OR} \ x_7 \\ C_4 \ : \ x_2 \ \mathrm{OR} \ \overline{x}_5 \ \mathrm{OR} \ x_8 \end{cases} \tag{1.18}$$

All clauses have to be satisfied simultaneously so the formula has to be read

$$\mathrm{F}\text{: } C_1 \ \mathrm{AND} \ C_2 \ \mathrm{AND} \ C_3 \ \mathrm{AND} \ C_4 \ . \tag{1.19}$$

It is not hard to believe that when $\alpha \equiv M/N \gg 1$ the problems typically become unsolvable while many solutions exist for $\alpha \ll 1$. One could expect to find a sharp threshold between a region of parameters $\alpha < \alpha_c$ where the formula is satisfiable and another region of parameters $\alpha \geq \alpha_c$ where it is not.

In **random k-SAT** an instance of the problem, i.e. a formula, is chosen at random with the following procedure: first one takes k variables out of the N available ones. Second one decides to require x_i or $\overline{x}_i$ for each of them with probability one half. Third one creates a clause taking the OR of these k literals. Forth one returns the variables to the pool and the outlined three steps are repeated M times. The M resulting clauses form the final formula.

The Boolean character of the variables in the k-SAT problem suggests to transform them into Ising spins, i.e. x_i evaluated to TRUE (FALSE) will correspond to $s_i = 1$ (-1) . The requirement that a formula be evaluated TRUE by an assignment of variables (i.e. a configuration of spins) will correspond to the ground state of an adequately chosen energy function. In the simplest setting, each clause will contribute zero (when satisfied) or one (when unsatisfied) to this cost function. There are several equivalent ways to reach this goal. For instance C_1 above can be represented by a term $(1 - s_1)(1 + s_2)(1 - s_3)/8$. The fact that the variables are linked together through the clauses suggests to define k-uplet interactions between them. We then choose the interaction matrix to be

$$J_{ai} = \begin{cases} 0 & \text{if neither } x_i \text{ nor } \overline{x}_i \in C_a \\ 1 & \text{if } x_i \in C_a \\ -1 & \text{if } \overline{x}_i \in C_a \end{cases} \tag{1.20}$$

and the energy function as

$$H_J[\{s_i\}] = \sum_{a=1}^{M} \delta(\sum_{i=1}^{N} J_{ai} s_i, -k) \tag{1.21}$$

where $\delta(x, y)$ is a Kronecker-delta that equals one when the arguments are identical and zero otherwise. This cost function is easy to understand. The Kronecker delta contributes one to the sum only if all terms in the sum $\sum_{i=1}^{N} J_{ai} s_i$ are equal to -1. This can happen when $J_{ai} = 1$ and $s_i = -1$ or when $J_{ai} = -1$ and $s_i = 1$. In both cases the condition on the variable x_i is not satisfied. Since this is required from all the variables in the clause, the clause itself and hence the formula are not satisfied.

The energy (1.21) can be rewritten in a way that resembles strongly physical spin models,

$$H_J[\{s_i\}] = \frac{M}{2^K} + \sum_{R=1}^{K} (-1)^R \sum_{i_1 < \cdots < i_R} J_{i_1 \ldots i_R} s_{i_1} \ldots s_{i_R} \tag{1.22}$$

and

$$J_{i_1 \ldots i_R} = \frac{1}{2^K} \sum_{a=1}^{M} J_{a i_1} \ldots J_{a i_R} \ . \tag{1.23}$$

These problems are "solved" numerically, with algorithms that do not necessarily respect physical rules. Thus, one can use non-local moves in which several variables are updated at once – as in cluster algorithms of the Swendsen-Wang type used to beat critical slowing down close to phase transitions – or one can introduce a temperature to go beyond cost-function barriers and use dynamic local moves that do not, however, satisfy a detail balance. The problem is that with hard instances of the optimization problem none of these strategies is successful. Indeed, one can expect that glassy aspects, such as the proliferation of metastable states separated by barriers that grow very fast with the number of variables, can hinder the resolutions of these problems in polynomial time, that is to say a time that scales with the system size as N^{ζ}, for *any* algorithm. These are then **hard combinatorial problems**.

1.3.7 Random bond ferromagnets

Let us now discuss some, *a priori* simpler cases. An example is the Mattis random magnet with generic nergy (1.15) in which the interaction strengths are given by

$$J_{i_1 \ldots i_p} = \xi_{i_1} \ldots \xi_{i_p} \qquad \text{with} \qquad \xi_j = \pm \quad \text{with} \quad \text{prob} = 1/2 \tag{1.24}$$

for any p and any kind of graph. In this case a simple *gauge transformation*, $\eta_i \equiv \xi_i s_i$, allows one to transform the disordered model in a ferromagnet, showing that there was no true frustration in the system.

Random bond ferromagnets (RBFMs) are systems in which the strengths of the interactions are not all identical but their sign is always positive. One can imagine such a exchange as the sum of two terms:

$$J_{ij} = J + \delta J_{ij} \ , \qquad \text{with } J > 0 \quad \text{and} \quad \delta J_{ij} \qquad \text{small and random} \ . \tag{1.25}$$

There is no frustration in these systems either.

As long as all J_{ij} remain positive, this kind of disorder should not change the two bulk phases with a paramagnetic-ferromagnetic second-order phase transition. Moreover the up-down spin symmetry is not broken by the disorder. The disorder just changes the local tendency towards ferromagnetism that can be interpreted as a change in the *local critical temperature*. Consequently, this type of disorder is often called random-T_c disorder, and it admits a Ginzburg-Landau kind of description, with a random distance from criticality, $\delta u(\mathbf{x})$,

$$F[m(\vec{r})] = \int d^d r \left\{ -h\, m(\vec{r}) + [r + \delta r(\mathbf{x})]\, m^2(\vec{r}) + (\nabla\, m(\vec{r}))^2 + u\, m^4(\vec{r}) + \ldots \right\} \ .. \tag{1.26}$$

The disorder couples to the m^2 term in the free-energy functional. In quantum field theory, this term is called the mass term and, therefore, random-T_c disorder is also called *random-mass* disorder. (In addition to random exchange couplings, random-mass disorder can also be realized by random dilution of the spins.)

1.3.8 Random field ferromagnets

Link randomness is not the only type of disorder encountered experimentally. Random fields, that couple linearly to the magnetic moments, are also quite common; the classical model is the *ferromagnetic random field Ising model* (RFIM):

$$H_J^{\text{rfim}} = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i s_i h_i \qquad \text{with} \qquad P(h_i) = (2\pi\sigma^2)^{-\frac{1}{2}}\ e^{-\frac{h_i^2}{2\sigma^2}} \ . \tag{1.27}$$

The *dilute antiferromagnet in a uniform magnetic field* is believed to behave similarly to the ferromagnetic random field Ising model. Experimental realizations of the former are common and measurements have been performed in samples like $Rb_2Co_{0.7}Mg_{0.3}F_4$.

Note that the up-down Ising symmetry is not preserved in models in the RFIMm and any spin model such that the disorder couples to the local order parameter.

In the Ginzburg-Landau description this model reads

$$F[m(\vec{r})] = \int d^d r \left\{ -h(\mathbf{x})\, m(\vec{r}) + r\, m^2(\vec{r}) + (\nabla\, m(\vec{r}))^2 + u\, m^4(\vec{r}) + \ldots \right\} \tag{1.28}$$

where $h(\vec{r})$ is the local random variable that breaks the up-down spin symmetry. Whether or not the symmetry is broken globally depends on the probability distribution of the

random fields. A particularly interesting situation arises if the distribution is even in h such that the up-down symmetry is globally preserved in the statistical sense.

Random-field disorder is generally stronger than random-mass disorder.

The random fields give rise to many metastable states that modify the equilibrium and non-equilibrium behaviour of the RFIM. In one dimension the RFIM does not order at all, in $d = 2$ there is strong evidence that the model is disordered even at zero temperature, in $d = 3$ it there is a finite temperature transition towards a ferromagnetic state. Whether there is a glassy phase near zero temperture and close to the critical point is still and open problem.

The RFIM at zero temperature has been proposed to yield a generic description of material cracking through a series of avalaches. In this problem one cracking domain triggers others, of which size, depends on the quenched disorder in the samples. In a random magnetic system this phenomenon corresponds to the variation of the magnetization in discrete steps as the external field is adiabatically increased (the time scale for an avalanche to take place is much shorter than the time-scale to modify the field) and it is accessed using Barkhausen noise experiments. Disorder is responsible for the jerky motion of the domain walls. The distribution of sizes and duration of the avalanches is found to decay with a power law tail and cut-off at a given size. The value of the cut-off size depends on the strength of the random field and it moves to infinity at the critical point.

1.3.9 Random manifolds

Once again, disorder is not only present in magnetic systems. An example that has received much attention is the so-called *random manifold*. This is a d dimensional *directed elastic manifold* moving in an embedding $N + d$ dimensional space under the effect of a quenched random potential. The simplest case with $d = 0$ corresponds to a particle moving in an embedding space with N dimensions. If, for instance $N = 1$, the particle moves on a line, if $N = 2$ it moves on a plane and so on and so forth. If $d = 1$ one has a line that can represent a domain wall, a polymer, a vortex line, *etc.* The fact that the line is directed means it has a preferred direction, in particular, it does not have overhangs. If the line moves in a plane, the embedding space has $(N = 1) + (d = 1)$ dimensions. One usually describes the system with an N-dimensional coordinate, $\vec{\phi}$, that locates in the transverse space each point on the manifold, represented by the internal d-dimensional coordinate $\vec{r}$,

The elastic energy is $H_{\rm elas} = \gamma \int d^d x \, \sqrt{1 + (\nabla \phi(\vec{r}))^2}$ with γ the deformation cost of a unit surface. Assuming the deformation is small one can linearise this expression and get, upto an additive constant, $H_{\rm elas} = \frac{\gamma}{2} \int d^d r \, (\nabla \phi(\vec{r}))^2$.

Disorder is introduced in the form of a random potential energy $V(\vec{\phi}(\vec{r}), \vec{r})$ characterised by its pdf.

The random manifold model is then

$$H_V(\vec{\phi}) = \int d^d r \left[\frac{\gamma}{2} \left(\nabla \phi(\vec{r}) \right)^2 + V(\vec{\phi}(\vec{r}), \vec{r}) \right] . \tag{1.29}$$

If the random potential is the result of a large number of impurities, the central limit theorem implies that its probability density is Gaussian. Just by shifting the energy scale one can set its average to zero, $[\, V \,] = 0$. As for its correlations, one typically assumes, for simplicity, that they exist in the transverse direction only:

$$[\, V(\vec{\phi}(\vec{r}), \vec{r}) V(\vec{\phi}'(\vec{r}'), \vec{r}') \,] = \delta^d(\vec{r} - \vec{r}') \mathcal{V}(\vec{\phi}, \vec{\phi}') \, . \tag{1.30}$$

If one further assumes that there is a statistical isotropy and translational invariance of the correlations, $\mathcal{V}(\vec{\phi}, \vec{\phi}') = W/\Delta^2 \, \mathcal{V}(|\vec{\phi} - \vec{\phi}'|/\Delta)$ with Δ a correlation length and $(W \Delta^{d-2})^{1/2}$ the strength of the disorder. The disorder can now be of two types: short-ranged if $\mathcal{V}$ falls to zero at infinity sufficiently rapidly and long-range if it either grows with distance or has a slow decay to zero. An example involving both cases is given by the power law $\mathcal{V}(z) = (\theta + z)^{-\gamma}$ where θ is a short distance cut-off and γ controls the range of the correlations with $\gamma > 1$ being short-ranged and $\gamma < 1$ being long-ranged.

This model also describes directed domain walls in random systems. One can derive it in the long length-scales limit by taking the continuum limit of the pure Ising part (that leads to the elastic term) and the random part (that leads to the second disordered potential). In the pure Ising model the second term is a constant that can be set to zero while the first one implies that the ground state is a perfectly flat wall, as expected. In cases with quenched disorder, the long-ranged and short-ranged random potentials mimic cases in which the interfaces are attracted by pinning centres ('random field' type) or the phases are attracted by disorder ('random bond' type), respectively. For instance, random bond disorder is typically described by a Gaussian pdf with zero mean and delta-correlated $[V(\vec{\phi}(\vec{r}), \vec{r}), V(\vec{\phi}'(\vec{r}'), \vec{r}')] = W \Delta^{d-2} \, \delta^d(\vec{r} - \vec{r}') \delta(\vec{\phi} - \vec{\phi}')$.

1.4 Properties of finite dimensional disordered systems

Once various kinds of quenched disorder introduced, a number of questions on their effect on the equilibrium and dynamic properties arise. Concerning the former:

- Are the equilibrium phases qualitatively changed by the random interactions?
- Is the phase transition still sharp, or is it smeared because different parts of the system undergo the transition independently?
- If there is still a phase transition, does its order (first order *vs.* continuous) change?
- If the phase transition remains continuous, does the critical behavior, *i.e.*, the values of the critical exponents, change?

Now, for the latter:

- Is the dynamic behaviour of the system modified by the quenched randomness?

In the following we explain a series of classical results in this field: the Harris criterium, the proof of non-analyticity of the free-energy of quenched disordered systems below their critical temperature given by Griffiths, the analysis of droplets and their domain wall stiffness, and the derivation of some exact results using the gauge invariance.

We first focus on impurities or defects that lead to spatial variations with respect to the tendency to order but do not induce new types of order, that is to say, no changes are induced in the two phases at the two sides of the transition. Only later we consider the spin-glass case.

1.4.1 The Harris criterium

The first question to ask is how does the average disorder strength behave under coarse-graining or, equivalently, how is it seen at long distances. This is the question answered by the Harris argument.

The Harris' criterion [51] states that if the specific-heat of a pure system

$$C^{\text{pure}}(T) \simeq |T - T_c^{\text{pure}}|^{-\alpha} \tag{1.31}$$

presents a power-like divergence with

$$\alpha_{\text{pure}} > 0 \, , \tag{1.32}$$

the disorder may induce a new universality class. Otherwise, if $\alpha_{\text{pure}} < 0$, disorder is irrelevant in a renormalisation group sense and the critical behaviour of the model remains unchanged. The criterium does not decide in the marginal case $\alpha_{\text{pure}} = 0$ case. Note that the Harris criterium is a necessary condition for a change in critical behaviour but not a sufficient one.

The hyper-scaling relation $2 - d\nu_{\text{pure}} = \alpha_{\text{pure}}$ allows to rewrite the Harris criterium as

$$\text{critical behaviour} = \begin{cases} \text{unchanged if } \nu_{\text{pure}} > 2/d \\ \text{changed} \quad \text{if } \nu_{\text{pure}} < 2/d \end{cases} \tag{1.33}$$

where ν_{pure} is the correlation length exponent

$$\langle s_0 s_{\vec{r}} \rangle \simeq e^{-r/\xi_{\text{pure}}} \qquad \text{and} \qquad \xi \simeq |T - T_c^{\text{pure}}|^{-\nu_{\text{pure}}} \, , \tag{1.34}$$

of the pure system.

The proof of the Harris result is rather simple and illustrates a way of reasoning that is extremely useful [51, 50]. Take the full system with frozen-in disorder at a temperature T slightly above its critical temperature T_c^{dis}. Divide it into equal pieces with linear size ξ_{dis}, the correlation length at the working temperature. By construction, the spins within

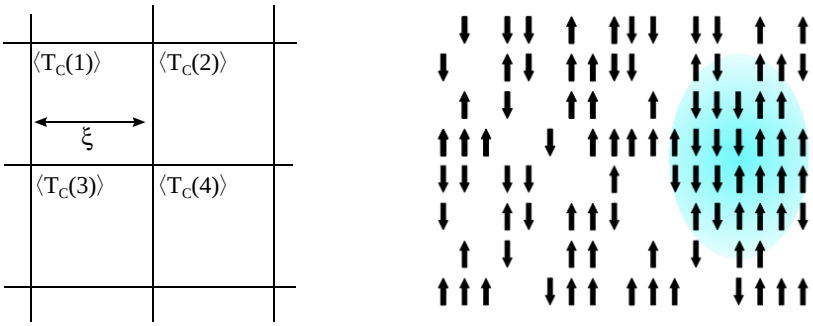

Figure 1.3: Left: scheme of the Harris construction. The disordered system is divided into cells with linear length $\xi_{\rm dis}$, its correlation length. Right: a typical configuration of the dilute Ising ferromagnet. Figures taken from [50].

each of these blocks behave as a super-spin since they are effectively parallel. Because of disorder, each block k has its own local critical temperature $T_c^{(k)}$ determined by the interactions (or dilution) within the block. Harris proposes to compare the fluctuations in the local critical temperatures $\Delta T_c^{(k)} \equiv T_c^{(k)} - T_c^{\rm dis}$ with respect to the global critical one $T_c^{\rm dis}$, with the distance from the critical point $\Delta T \equiv T - T_c^{\rm dis} > 0$, taken to be positive:

- If $\Delta T_c^{(k)} < \Delta T$ for all k, all blocks have critical temperature below the working one, $T_c^{(k)} < T$, and the system is 'uniform' with respect to the phase transition.
- If $\Delta T_c^{(k)} > \Delta T$ for some k, some blocks are in the disordered (paramagnetic) phase and some are in the ordered (ferromagnetic) phase, making a uniform transition impossible. The inhomogeneity in the system may then be important.

Require now $\Delta T_c^{(k)} < \Delta T$ for all k to have an unmodified critical behaviour. Use also that an unmodified critical behaviour implies $\xi_{\rm dis} = \xi_{\rm pure} = \xi$ and, consequently, $\nu_{\rm dis} = \nu_{\rm pure}$.

As this should be the case for all k we call $\Delta T_c^{(\rm loc)}$ the generic one. $\Delta T_c^{(\rm loc)}$ can be estimated using the central limit theorem. Indeed, as each local $T_c^{(k)}$ is determined by an average of a large number of random variables in the block (*e.g.*, the random J_{ij} in the Hamiltonian), its variations decay as the square root of the block volume, $\Delta T_c^{(\rm loc)} \simeq \xi^{-d/2}$. On the other hand, $\Delta T \simeq \xi^{-1/\nu_{\rm pure}}$. Therefore,

$$\Delta T_c^{(k)} < \Delta T_c \qquad \Rightarrow \qquad d\nu_{\rm pure} > 2 \; . \tag{1.35}$$

The interpretation of this inequality is the following. If the Harris criterion $d\nu_{\rm pure} > 2$ is fulfilled, the ratio $\Delta T_c^{(\rm loc)}/\Delta T$ goes to zero as the critical point is approached. The system looks less and less disordered on larger length scales, the effective disorder strength vanishes right at criticality, and the disordered system features the same critical behaviour as the clean one. An example of a transition that fullfills the Harris criterion is the

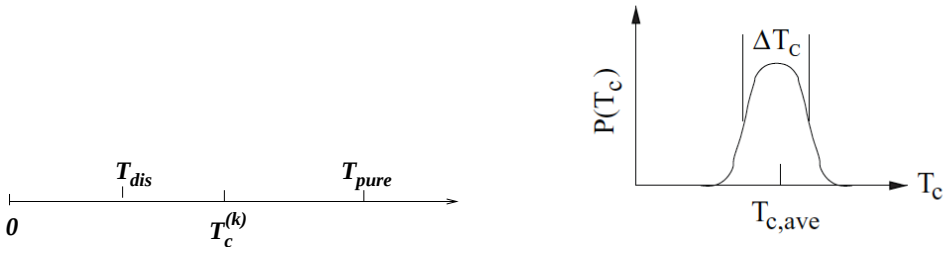

Figure 1.4: The characteristic temperatures. $T_{\rm pure}$ and $T_{\rm dis}$ are the critical temperatures of the pure and disordered systems, respectively. $T_c^{(k)}$ is the critical temperature of the local region with linear size $\xi_{\rm dis}$ labelled k, see the sketch in Fig. 1.3-left. The distance from the disordered critical point is measured by $\Delta T_c^{(k)} = T_c^{(k)} - T_{\rm dis}$ for the critical temperature of block k and $\Delta T = T - T_{\rm dis}$ for the working temperature T. Right: the probability distribution function of the local critical temperatures $T_c^{(k)}$. The width depends on $\xi_{\rm dis}$ and clearly decreases with increasing ξ as the local temperatures fluctuate less and less.

ferromagnetic transition in a three-dimensional classical Heisenberg model. Its clean correlation length exponent is $\nu_{\rm pure} \approx 0.69 > 2/d = 2/3$.

In contrast, if $d\nu_{\rm pure} < 2$, the ratio $\Delta T_c^{(\rm loc)}/\Delta T$ increases upon approaching the phase transition. The blocks differ more and more on larger length scales. Eventually, some blocks are on one side of the transition while other blocks are on the other side. This makes a uniform sharp phase transition impossible. The clean critical behavior is unstable and the phase transition can be erased or it can remain continuous but with different critical behaviour. More precisely, the disordered system can be in a new universality class featuring a correlation length exponent that fullfills the inequality $d\nu_{\rm dis} > 2$. Many phase transitions in classical disordered systems follow this scenario, for example the three-dimensional classical Ising model. Its clean correlation length exponent is $\nu_{\rm pure} \approx 0.63$ which violates the Harris criterion. In the presence of random-mass disorder, the critical behavior changes and $\nu_{\rm dis} \approx 0.68$. (Note, however, that the difference between these exponents is tiny!)

In the marginal case $d\nu_{\rm pure} = 2$, more sophisticated methods are required to decide the stability of the clean critical point.

Chayes et al. [53] turned this argument around to show rigorously that for all the continuous phase transitions in presence of disorder, the correlation-length critical exponent of the disordered system, $\nu_{\rm dis}$ verifies $\nu_{\rm dis} \geq 2/d$, independently of whether or not the critical behaviour is the same as in the uniform system and even when the system does not have a uniform analogue.

Finally, note that the Harris criterion $d\nu_{\rm pure} > 2$ applies to uncorrelated or short-range correlated disorder. If the disorder displays long-range correlations in space, the inequality needs to be modified because the central-limit theorem estimate of $\Delta T_c^{(\rm loc)}$ changes.

Long-range correlated disorder is especially important in quantum phase transitions.

The reason is the fact that the statistical properties of quantum systems are studied in an imaginary time formulation that makes a d-dimensional quantum problem equivalent to a $d+1$ dimensional classical one. Along Along this additional spatial direction, quenched randomness is long-range correlated.

1.4.2 The Griffiths phase

The critical temperature of a spin system is usually estimated from the high temperature expansion and the evaluation of its radius of convergence (see App. A.1). However, Griffiths showed that the temperature at which the free-energy of models with quenched disorder starts being non-analytical falls above the critical temperature where the order parameter detaches from zero [49]. The argument applies to models with second order phase transitions.

Griffiths explained his argument using the dilute ferromagnetic Ising model. First, he argued that the critical temperature of the disordered model should decrease for increasing p, the probability of *empty* sites. This is 'intuitively obvious' since no spontaneous magnetisation can occur at a finite temperature if the probability of occupied sites is less than the critical percolation probability at which an 'infinite cluster' first appears. See Fig. 1.5 where the phase diagram of the dilute Ising ferromagnet is shown.

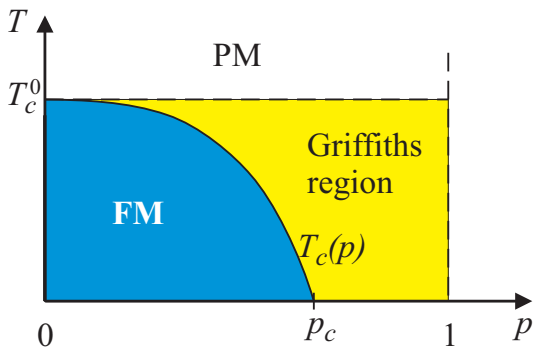

Figure 1.5: The phase diagram of the dilute ferromagnetic Ising model. p is the probability of empty sites in this figure, taken from [50]. With increasing dilution the ordered phase is eventually suppressed.

In the following paragraph we sketch Griffiths' argument and we use his notation in which p is the probability of occupying a site. For any concentration $p < 1$ the magnetisation m is not an analytic function of h at $h = 0$ at any temperature below T_c^{pure}, the critical temperature of the regular Ising model $p = 1$. As he explains, this fact is most easily explained for $p < p_c$. The magnetisation m per lattice site in the

thermodynamic limit has the form

$$m = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle = \sum_c P(c)m(c) \tag{1.36}$$

where $P(c)$ is the probability that a particular site on the lattice belongs to a cluster c that is necessarily finite for $p < p_c$, and $m(c)$ is the magnetisation density of the cluster c, that is to say $m(c) = N^{-1}(c)\sum_{i\in c}\langle s_i\rangle$ with $N(c)$ the number of sites in the cluster.

Griffiths uses the Yang-Lee theorem, see App. A, to express $m(c)$ as

$$m(c) = 1 + \frac{2z}{N(c)}\sum_{i\in c}\frac{1}{\xi_i - z} \qquad \text{with} \qquad z = e^{-2\beta h} \tag{1.37}$$

and ξ_i, $i = 1, \ldots, N(c)$, complex numbers with $|\xi_i| = 1$. The total magnetisation density is then of the same form

$$m = 1 + zf(z) \qquad f(z) = \sum_i \eta_i(\xi_i - z)^{-1} \tag{1.38}$$

with $\eta_i = 2P(c)/N(c)$. He then argues that this form is analytic for $z < 1$ but non-analytic at $z = 1$ that corresponds to $h = 0$.

A more intuitive understand of what is going on in the temperature region above the critical temperature of the disordered model, $T_c^{\rm dis}$, and below the critical temperature the pure one, $T_c^{\rm pure}$, can be reached as follows [50]. The effects of quenched disorder show up already in the paramagnetic phase of finite dimensional systems. Below the critical point of the pure case (no disorder) finite regions of the system can order due to fluctuations in the couplings or, in a dilute ferromagnetic model, they can be regions where all sites are occupied, as shown in Fig. 1.3. As such rare regions are finite-size pieces of the clean system, their spins align parallel to each other below the clean critical temperature $T_c^{\rm pure}$. Because they are of finite size, these regions cannot undergo a true phase transition by themselves, but for temperatures between the actual transition temperature $T_c^{\rm dis}$ and $T_c^{\rm pure}$ they act as large superspins.

Note that using the ideas of percolation theory, one can estimate the scaling of $P(c)$ with its size. Recall the one dimensional case. Take a segment of length $L + 2$ on the lattice. A cluster of size L will occupy the internal sites with empty borders with probability $p^L(1-p)^2$. This is because one needs L contiguous sites to be occupied and its boundary sites be empty. In larger dimensions, this probability will be approximately $p^{L^d}(1-p)^{L^{d-1}}$ with the first factor linked to the filled volume and the second to the empty surface. In the large L limit one can make a harsh approximation and use $\simeq \exp\{\ln[p^{L^d}(1-p)^{L^{d-1}}]\} = \exp[\ln p^{L^d} + \ln(1-p)^{L^{d-1}}] \simeq \exp[-{\rm c}(p)L^d]$ with ${\rm c}(p) = \ln 1/p$.

The sum in eq. (1.36) is made of two contributions. On the one hand, there are the large clusters that are basically frozen at the working temperature. On the other,

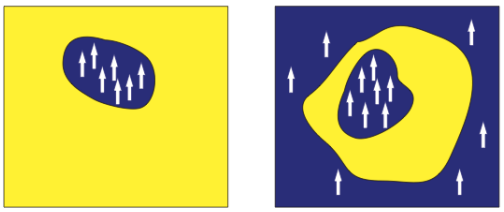

Figure 1.6: Rare regions in a random ferromagnet, figure taken from [50]. On the left, a ferromagnetically ordered region in the paramagnetic bulk ($T > T_c^{\rm dis}$). On the right, a paramagnetic band in a system that is ordered ferromagnetically in a patchwork way ($T < T_c^{\rm dis}$).

there are the free spins that belong to small clusters and are easy to flip at the working temperature. Let us focus on the former. Their magnetic moment is proportional to their volume $m(c) \simeq \mu L^d$. The energy gain due to their alignment with the field is $\Delta E(c) = -hm(c) = -h\mu L^d$ where h is a small uniform field applied to the system, say to measure its susceptibility.

The separation of the clusters in the two groups is then controlled by $\Delta E(c)$: the small clusters with $|\Delta E(c)| < k_B T$ can be flipped by thermal fluctuations, and the large clusters with $|\Delta E(c)| > k_B T$ and are frozen.

Then effect of the frozen clusters for which $|\Delta E(c)| > k_B T$ is then

$$m(T,h) \approx \sum_{|\Delta E(c)|>k_B T} P(c)m(c) \approx \int_{L_c}^{\infty} dL\; e^{-{\rm c}(p)L^d}\; \mu L^d \tag{1.39}$$

and $L_c^d \approx k_B T/(\mu h)$. This integral can be computed by the saddle-point method, see App. **??**, and it is dominated by the lower border. The result is

$$m(T,h) \approx e^{-{\rm c}(p)L_c^d} = e^{-{\rm c}(p)k_B T/(\mu h)} \tag{1.40}$$

and this contribution has an essential singularity in the $h \to 0$ limit.

It is important to note that the clusters that contribute to this integral are *rare regions* since they occur with probability $P(c) \simeq e^{-{\rm c}(p)L^d}$ that is exponentially small in their volume. Still they are the cause of the non-analytic behaviour of $m(h)$.

The magnetic susceptibility χ can be analyzed similarly. Each locally ordered rare region makes a Curie contribution $m^2(c)/k_B T$ to χ. The total rare region susceptibility can therefore be estimated as

$$\chi(T,h) \sim \int_{L_c}^{\infty} dL\; e^{-{\rm c}(p)L^d} \mu^2\, L^{2d}/(k_B T) \approx e^{-{\rm c}(p)k_B T/(\mu h)}\;. \tag{1.41}$$

This equation shows that the susceptibility of an individual rare region does not increase fast enough to overcome the exponential decay of the rare region probability with increasing size L. Consequently, large rare regions only make an exponentially small contribution to the susceptibility.

Rare regions also exist on the ordered side of the transition $T < T_c$. One has to consider locally ordered islands inside holes that can fluctuate between up and down because they are only very weakly coupled to the bulk ferromagnet outside the hole, see Fig. 1.6. This conceptual difference entails a different probability for the rare events as one needs to find a large enough vacancy-rich region around a locally ordered island.

There are therefore slight differences in the resulting Griffiths singularities on the two sides of the transition. In the site-diluted Ising model, the ferromagnetic Griffiths phase comprises all of the ferromagnetic phase for $p > 0$. The phase diagram of the dilute ferromagnetic Ising model is sketched in Fig. 1.5.

1.4.3 Scenario for the phase transitions

The argument put forward by Harris is based on the effect of disorder on average over the local critical temperatures. The intuitive explanation of the Griffiths phase shows the importance of rare regions on the behaviour of global observables such as the magnetisation or the susceptibility, The analysis of the effect of randomness on the phase transitions should then be refined to take into account the effect of *rare regions* (tails in the distributions). Different classes of rare regions can be identified according to their dimension $d_{\rm rr}$. This leaves place for three possibilities for the effect of (still weak in the sense of not having frustration) disorder on the phase transition.

- The rare regions have dimension $d_{\rm rr}$ smaller than the lower critical dimension of the pure problem, $d_{\rm rr} < d_L$; therefore the critical behaviour is not modified with respect to the one of the clean problem.

- When the rare regions have dimension equal to the lower-critical one, $d_{\rm rr} = d_L$, the critical point is still of second order with conventional power law scaling but with different exponents that vary in the Griffiths phase. At the critical point the Harris criterium is satisfied $d\nu_{\rm dis} > 2$.

- Infinite randomness strength, appearing mostly in problems with correlated disorder, lead to a complete change in the critical properties, with unconventional activated scaling. This occurs when $d_{\rm rr} > d_L$.

In the derivation of this scenario the rare regions are supposed to act independently, with no interactions among them. This picture is therefore limited to systems with short-range interactions.

1.4.4 Domain-wall stiffness and droplets

Let us now just discuss one simple argument that is at the basis of what is needed to

derive the results of the *droplet theory* for spin-glasses without entering into the complications of the calculations.

At very high temperature the configurations are disordered and one does not see large patches of ordered spins.

Close but above the critical temperature T_c finite patches of the system are ordered (in all possible low-temperature equilibrium states) but none of these include a finite fraction of the spins in the sample and the magnetization density vanishes. However, these patches are enough to generate non-trivial thermodynamic properties very close to T_c and the richness of critical phenomena.

At criticality one observes ordered domains of the two equilibrium states at all length scales – with *fractal* properties.

Below the critical temperature thermal fluctuations induce the spin reversal with respect to the order selected by the spontaneous symmetry breaking. It is clear that the structure of *droplets*, meaning patches in which the spins point in the opposite direction to the one of the background ordered state, plays an important role in the thermodynamic behaviour at low temperatures.

M. Fisher and others developed a droplet phenomenological theory for critical phenomena in clean systems. Later D. S. Fisher and D. Huse extended these arguments to describe the effects of quenched disorder in spin-glasses and other random systems; this is the so-called *droplet model*.

Domain-wall stiffness

Ordered phases resist spatial variations of their order parameter. This property is called *stiffness* or *rigidity* and it is absent in high-temperature disordered phases.

More precisely, in an ordered phase the *free-energy cost* for changing one part of the system with respect to another part far away is proportional to $k_B T$ and usually diverges as a power law of the system size. In a disordered phase the information about the reversed part propagates only a finite distance (of the order of the correlation length, see below) and the stiffness vanishes.

Concretely, the free-energy cost of installing a *domain-wall* in a system, gives a measure of the stiffness of a phase. The domain wall can be imposed by special boundary conditions. Compare then the free-energy of an Ising model with linear length L, in its ordered phase, with periodic and anti-periodic boundary conditions on one Cartesian direction and periodic boundary conditions on the $d-1$ other directions of a d-dimensional hypercube. The $\pm$ boundary conditions forces an interface between the regions with positive and negative magnetisations. At $T = 0$, the minimum energy interface is a $d-1$ flat hyper-plane and the energy cost is

$$\Delta E(L) \simeq \sigma L^{\theta} \qquad \text{with} \qquad \theta = d - 1 \tag{1.42}$$

and $\sigma = 2J$ the interfacial energy per unit area or the interfacial tension of the domain wall.

Droplets - generalisation of the Peierls argument

In an ordered system at finite temperature domain walls, surrounding *droplet fluctuations*, or domains with reversed spins with respect to the bulk order, are naturally generated by thermal fluctuations. The study of droplet fluctuations is useful to establish whether an ordered phase can exist at low (but finite) temperatures. One then studies the free-energy cost for creating large droplets with thermal fluctuations that may destabilise the ordered phase, in the way usually done in the simple Ising chain (the Peierls argument).

Indeed, temperature generates fluctuations of different size and the question is whether these are favourable or not. These are the *droplet excitations* made by simply connected regions (domains) with spins reversed with respect to the ordered state. Because of the surface tension, the minimal energy droplets with linear size or radius L will be compact spherical-like objects with volume L^d and surface L^{d-1}. The surface determines their energy and, at finite temperature, an entropic contribution has to be taken into account as well. Simplifying, one argues that the free-energy cost is of the order of L^θ, that is L^{d-1} in the ferromagnetic case but can be different in disordered systems.

Summarising, in system with symmetry breaking the free-energy cost of an excitation of linear size L is expected to scale as

$$\Delta F(L) \simeq \sigma(T) L^\theta \ . \tag{1.43}$$

The sign of θ determines whether thermal fluctuations destroy the ordered phase or not. For $\theta > 0$ large excitations are costly and very unlikely to occur; the order phase is expected to be stable. For $\theta < 0$ instead large scale excitations cost little energy and one can expect that the gain in entropy due to the large choice in the position of these excitations will render the free-energy variation negative. A proliferation of droplets and droplets within droplets is expected and the ordered phase will be destroyed by thermal fluctuations. The case $\theta = 0$ is marginal and its analysis needs the use of other methods.

As the phase transitions is approached from below the surface tension $\sigma(T)$ should vanish. Moreover, one expects that the stiffness should be independent of length close to T_c and therefore, $\theta_c = 0$.

Above the transition the stiffness should decay exponentially

$$\Delta F(L) \simeq e^{-L/\xi} \tag{1.44}$$

with ξ the equilibrium correlation length.

1.4.5 Stability of ordered phases

A ferromagnet under a magnetic field

Let us study the stability properties of an equilibrium ferromagnetic phase under an applied external field that tends to destabilize it. If we set $T = 0$ the free-energy is just

the energy. In the ferromagnetic case the free-energy cost of a spherical droplet of radius R of the equilibrium phase parallel to the applied field embedded in the dominant one (see Fig. 1.7-left) is

$$\Delta F(R) = -2\Omega_d R^d h m_{\rm eq} + \Omega_{d-1} R^{d-1} \sigma_0 \tag{1.45}$$

where σ_0 is the interfacial free-energy density (the energy cost of the domain wall) and Ω_d is the volume of a d-dimensional unit sphere. We assume here that the droplet has a regular surface and volume such that they are proportional to R^{d-1} and R^d, respectively. The excess free-energy reaches a maximum

$$\Delta F_c = \frac{\Omega_d}{d} \frac{\Omega_{d-1}^d}{\Omega_d^d} \left(\frac{d-1}{2dhm_{\rm eq}} \right)^{d-1} \sigma_0^d \tag{1.46}$$

at the critical radius

$$R_c = \frac{(d-1)\Omega_{d-1}\sigma_0}{2d\Omega_d h m_{\rm eq}} , \tag{1.47}$$

see Fig. 1.7-right ($h > 0$ and $m_{\rm eq} > 0$ here, the signs have already been taken into account). The free-energy difference vanishes at

$$\Delta F(R_0) = 0 \qquad \Rightarrow \qquad R_0 = \frac{\Omega_{d-1}\sigma_0}{2\Omega_d h m_{\rm eq}} . \tag{1.48}$$

Several features are to be stressed:

- The barrier vanishes in $d = 1$; indeed, the free-energy is a linear function of R in this case.
- Both R_c and R_0 have the same dependence on $hm_{\rm eq}$: they monotonically decrease with increasing $hm_{\rm eq}$ vanishing for $hm_{\rm eq} \to \infty$ and diverging for $hm_{\rm eq} \to 0$.
- In dynamic terms that we shall discuss later, the passage above the barrier is done *via* thermal activation; as soon as the system has reached the height of the barrier it rolls on the right side of 'potential' ΔF and the favorable phase nucleates.
- As long as the critical size R_c is not reached the droplet is not favorable and the system remains positively magnetized.

The Imry-Ma argument for the random field Ising model

Take a ferromagnetic Ising model in a random field, defined in eq. (1.27). In zero applied field and low enough temperature, if $d > 1$ there is a phase transition between a ferromagnetic and a paramagnetic phase at a critical value of the variance of the random fields, $\sigma_h^2 = [h_i^2] \propto h^2$, that sets the scale of the values that these random fields can take. Under the effect of a random field with very strong typical strength, the spins align with the local external fields that point in both directions and the system is paramagnetic. It

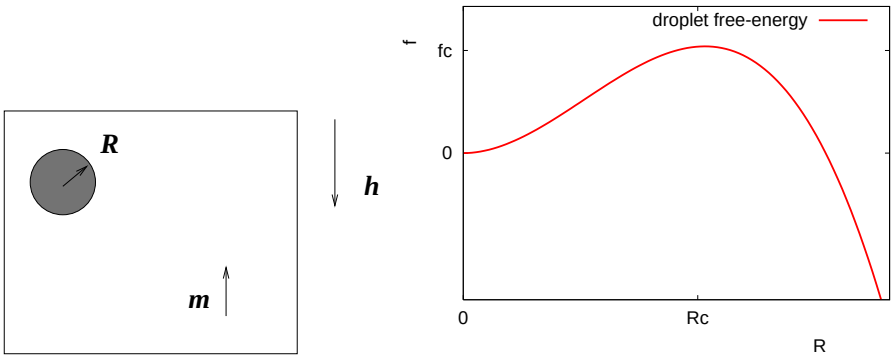

Figure 1.7: Left: the droplet. Right: the free-energy density $f(R)$ of a spherical droplet with radius R.

is, however, non-trivial to determine the effect of a relatively weak random field on the ferromagnetic phase at sufficiently low temperature. The long-range ferromagnetic order could be preserved or else the field could be enough to break up the system into large but finite domains of the two ferromagnetic phases.

A qualitative argument to decide whether the ferromagnetic phase survives or not in presence of the external random field is due to Imry and Ma [54]. Let us fix $T = 0$ and switch on a random field. If a compact domain $\mathcal{D}$ of the opposite order (say down) is created within the bulk of the ordered state (say up) the system pays an energy due to the unsatisfied links lying on the boundary that is

$$\Delta E_{\text{border}} \sim 2JR^{d-1} \tag{1.49}$$

where R is the radius of the domain and $d-1$ is the dimension of the border of a domain embedded in d a dimensional volume, assuming the interface is not fractal. By creating a domain boundary the system can also gain a magnetic energy in the interior of the domain due to the external field:

$$\Delta E_{\text{random field}} \sim -hR^{d/2} \tag{1.50}$$

since there are $N \propto R^d$ spins inside the domain of linear scale R (assuming now that the bulk of the domain is not fractal) and, using the central limit theorem, $-h\sum_{j\in\mathcal{D}} s_i \sim -h\sqrt{N} \propto -hR^{d/2}$. $h \approx \sigma_h$ is the width of the random field distribution.

Dimension lower than two. In $d = 1$ the energy difference is a monotonically decreasing function of R thus suggesting that the creation of droplets is very favourable and there is no barrier to cross to do it. Indeed, for any $d < 2$, the random field energy increases faster with R than the domain wall energy. Even for weak random fields, there will be a critical R beyond which forming domains that align with the local random field becomes favourable. Consequently, *the uniform ferromagnetic state is unstable* against

domain formation for arbitrary random field strength. In other words, in dimensions $d < 2$ random-field disorder prevents spontaneous symmetry breaking.

Dimension larger than two. The functional form of the total energy variation $\Delta E = \Delta E_{\text{border}} + \Delta E_{\text{random field}}$ as a function of R is characterised by $\Delta E \to 0$ for $R \to 0$ and $\Delta E \to \infty$ for $R \to \infty$. The function has a minimum at

$$R_c \sim \left(\frac{hd}{4J(d-1)} \right)^{2/(d-2)} \tag{1.51}$$

and crosses zero at R_0 to approach ∞ at $R \to \infty$. The comparison between these two energy scales yields

$$2JR_0^{d-1} \sim hR_0^{d/2} \qquad \Rightarrow \qquad R_0 \sim \left(\frac{h}{2J} \right)^{\frac{2}{d-2}} \tag{1.52}$$

This equation clearly shows a change in $d = 2$, with

$$\lim_{h/J \to 0} R_0(h/J) = \begin{cases} 0 & \text{if} \quad d > 2 \,, \\ \infty & \text{if} \quad d < 2 \,. \end{cases} \tag{1.53}$$

Therefore, in $d > 2$ the energy difference first decreases from $\Delta E(R = 0) = 0$ to reach a negative minimum at R_c, and then increases back to pass through zero at R_0 and diverge at infinity. This indicates that *the creation of domains at zero temperature is not favourable in $d > 2$*. Just domains of finite length, up to R_0 can be created. Note that R_0 increases with h/J in $d > 2$. Therefore, a higher field tends to generate larger droplets and thus disorder more the sample.

The marginal case $d = 2$ is more subtle and more powerful techniques are needed to decide.

With this argument one cannot show the existence of a phase transition at h_c nor the nature of it. The argument is such that it suggests that order can be supported by the system at zero temperature and small fields in $d > 2$.

Again, we stress that these results hold for short-range correlated disorder.

There are rigorous proofs that random fields destroy long-range order (and thus prevent spontaneous symmetry breaking) in all dimensions $d \leq 2$ for discrete (Ising) symmetry and in dimensions $d \leq 4$ for continuous (Heisenberg) symmetry. The existence of a phase transition from a FM to a PM state at zero temperature in $3d$ was shown in [56].

An elastic line in a random potential

The interfacial tension, σ, will tend to make an interface, forced into a system as flat as possible. However, this will be resisted by thermal fluctuations and, in a system with random impurities, by quenched disorder.

Let us take an interface model of the type defined in eq. (1.29) with $N = 1$. If one assumes that the interface makes an excursion of longitudinal length L and transverse

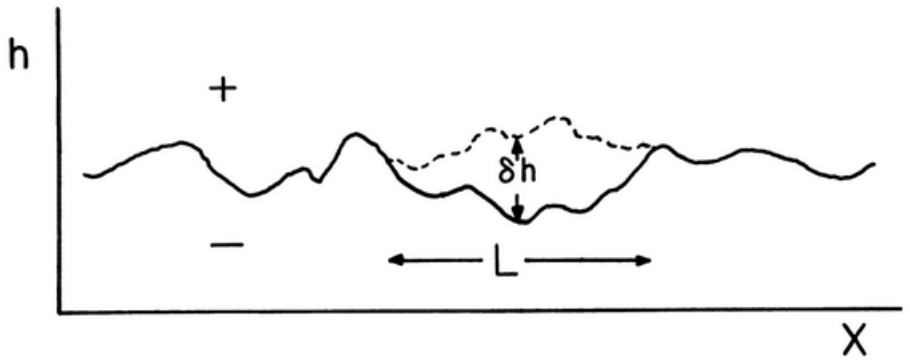

Figure 1.8: Illustration of an interface modeled as a directed manifold. In the example, the domain wall separates a region with positive magnetisation (above) from one with negative magnetisation (below). The line represents a lowest energy configuration that deviates from a flat one due to the quenched randomness. An excitation on a length-scale L is shown with a dashed line. The relative displacement is $\delta h \equiv \delta\phi \simeq L^{\alpha}$ and the excitation energy $\Delta E(L) \simeq L^{\theta}$. Figure taken from [57].

length ϕ the elastic energy cost is

$$E_{\rm elast} = \frac{c}{2}\int d^d x\ (\nabla\phi(\vec{x}))^2 \qquad \Rightarrow \qquad \Delta E_{\rm elast} \sim cL^d(L^{-1}\phi)^2 = cL^{d-2}\phi^2 \tag{1.54}$$

Ignore for the moment the random potential. Thermal fluctuations cause fluctuations of the kind shown in Fig. 1.8. The interfaces *roughens*, that is to say, it deviates from being flat. Its mean-square displacement between two point $\vec{x}$ and $\vec{y}$, or its *width* on a scale L satisfies

$$\langle[\phi(\vec{x}) - \phi(\vec{y})]^2\rangle \simeq T\ |\vec{x} - \vec{y}|^{2\zeta_T} \tag{1.55}$$

with ζ_T the *roughness exponent*.

The elastic energy cost of an excitation of length L is then

$$\Delta E_{\rm elast}(L) \simeq cL^{d-2}\phi^2(L) \simeq cTL^{d-2}L^{2\zeta_T} \tag{1.56}$$

and this is of order one if

$$\zeta_T = \frac{2-d}{2}\ . \tag{1.57}$$

In the presence of quenched randomness, the deformation energy cost competes with gains in energy obtained from finding more optimal regions of the random potential. Naively, the energy gain due to the randomness is

$$\int d^d x\ V \simeq [W^2 L^d]^{1/2} \simeq WL^{d/2} \tag{1.58}$$

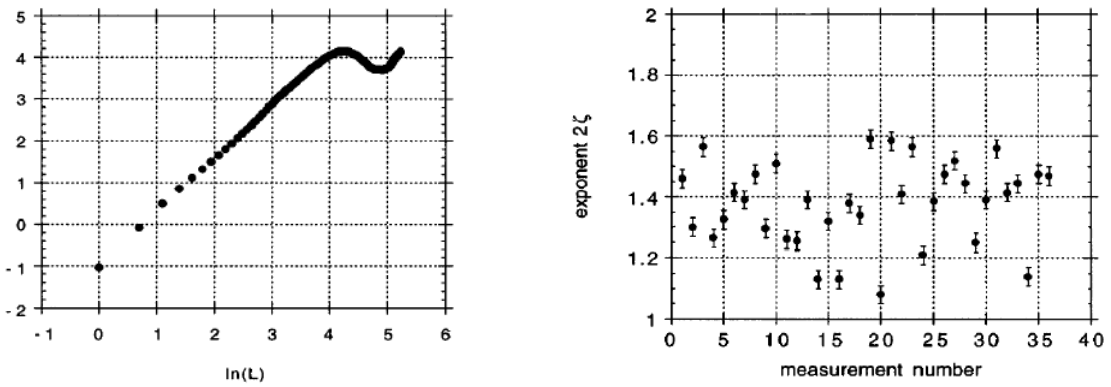

Figure 1.9: The interface width and the roughness exponent in a magnetic domain wall in a thin film. The value measured $\zeta_D \simeq 0.6$ is compatible with the Flory value 2/3 expected for a one dimensional domain wall in a two dimensional space ($N = 1$ and $d = 1$ in the calculations discussed in the text.) [58].

and the balance with the elastic cost, assumed to be the same as with no disorder, yields

$$cTL^{d-2}L^{2\zeta_D} \simeq WL^{d/2} \qquad \Rightarrow \qquad \zeta_D = \frac{4-d}{2} \tag{1.59}$$

This result turns out to be an upper bound of the exponent value [57]. It is called the *Flory exponent* for the roughness of the surface. One then concludes that for $d > 4$ disorder is irrelevant and the interface is flat ($\phi \to 0$ when $L \to \infty$). Since the linearization of the elastic energy [see the discussion leading to eq. (1.29)] holds only if $\phi/L \ll 1$, the result (1.59) may hold only for $d > 1$ where $\alpha < 1$.

Destruction of first order phase transitions under randomness

A first order phase transition is characterized by macroscopic phase coexistence at the transition point. For example, at the liquid-gas phase transition of a fluid, a macroscopic liquid phase coexists with a macroscopic vapour phase. Random-mass disorder locally favors one phase over the other. The question is whether the macroscopic phases survives in the presence of disorder or the system forms domains (droplets) that follow the local value of the random-mass.

Consider a single domain or droplet (of linear size L) of one phase embedded in the other phase. The free energy cost due to forming the surface is

$$\Delta F_{\rm surf} \sim \sigma L^{d-1} \tag{1.60}$$

where σ is the surface energy between the two phases. The energy gain from the random-mass disorder can be estimated via the central limit theorem, resulting in a typical magnitude of

$$|\Delta F_{\rm dis}| \sim W^{1/2} L^{d/2} \tag{1.61}$$

where W is the variance of the random-mass disorder.

The macroscopic phases are stable if $|\Delta F_{\rm dis}| < \Delta F_{\rm surf}$, but this is impossible in dimensions $d \leq 2$ no matter how weak the disorder is. In dimensions $d > 2$, phase coexistence is possible for weak disorder but will be destabilized for sufficiently strong disorder.

We thus conclude that random-mass disorder destroys first-order phase transitions in dimensions $d \leq 2$. In many examples, the first-order transition is replaced by ('rounded to') a continuous one, but more complicated scenarios cannot be excluded.

The 3d Edwards-Anderson model in a uniform magnetic field

A very similar reasoning is used to argue that there cannot be spin-glass order in an Edwards-Anderson model in an external field [71, 72]. The only difference is that the domain wall energy is here assumed to be proportional to L^y with an *a priori* unknown d-dependent exponent y that is related to the geometry of the domains.

Comments

These arguments are easy to implement when one knows the equilibrium states (or one assumes what they are). They cannot be used in models in which the energy is not a slowly varying function of the domain wall position.

1.4.6 Consequences of the gauge invariace

H. Nishimori used the gauge transformation explained in Sec. **??** to derive a series of exact results for averaged observables of finite dimensional disordered systems [46].

The idea follows the steps by which one easily proves, for example, that the averaged local magnetization of a ferromagnetic Ising model vanishes, that is to say, one applies a transformation of variables within the partition sum and evaluates the consequences over the averaged observables. For example,

$$\langle s_i \rangle = \sum_{\{s_j\}} s_i \; e^{\beta J \sum_{ij} s_i s_j} = \langle s_i \rangle = \sum_{\{s_j\}} (-s_i) \; e^{\beta J \sum_{ij} s_i s_j} = -\langle s_i \rangle \; . \tag{1.62}$$

This immediately implies $\langle s_i \rangle = 0$ and, more generally, the fact that the average of any odd function under $\{s_i\} \to \{-s_i\}$ vanishes exactly.

In the case of disordered systems, one is interested in observables that are averaged over the random variables weighted with their probability distribution. The gauge transformation that leaves the Hamiltonian unchanged involves a change of spins accompanied by a transformation of the exchanges:

$$\overline{s}_i = \eta_i s_i \qquad \overline{J}_{ij} = \eta_i \eta_j J_{ij} \tag{1.63}$$

with $\eta_i = \pm 1$. The latter affects their probability distribution as this one, in general, is not gauge invariant. For instance, the bimodal pdf $P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J)$

can be rewritten as

$$P(J_{ij}) = \frac{e^{K_p J_{ij}/J}}{2\cosh K_p} \qquad \text{with} \qquad e^{2K_p} = \frac{p}{1-p}\,, \tag{1.64}$$

as one can simply check. $\tau_{ij} \equiv J_{ij}/J$ are just the signs of the J_{ij}. Under the gauge transformation $P(J_{ij})$ transforms as

$$\overline{P}(\overline{J}_{ij})d\overline{J}_{ij} = P(J_{ij})dJ_{ij} \qquad \Rightarrow \qquad \overline{P}(\overline{J}_{ij}) = P(J_{ij}(\overline{J}_{ij}))\,\frac{dJ_{ij}}{d\overline{J}_{ij}} \tag{1.65}$$

that implies

$$\overline{P}(\overline{J}_{ij}) = \frac{e^{K_p \overline{J}_{ij}/(\eta_i\eta_j J)}}{2\cosh K_p}\frac{1}{\eta_i\eta_j} \qquad \Rightarrow \qquad \overline{P}(\overline{J}_{ij}) = \eta_i\eta_j\,\frac{e^{K_p \overline{J}_{ij}\eta_i\eta_j/J}}{2\cosh K_p} \tag{1.66}$$

For instance, applying the gauge transformation to the internal energy of an Ising spin-glass model with bimodal disorder, after a series of straightforward transformations one finds

$$[\langle H_J\rangle]_J = -N_B J \tanh K_p \tag{1.67}$$

with N_B the number of bonds in the lattice, under the condition $\beta J = K_p$. This relation holds for any lattice. The constraint $\beta J = K_p$ relates the inverse temperature $J/(k_B T)$ and the probability $p = (\tanh K_p + 1)/2$. The curve $\beta J = K_p$ connects the points $(p = 1, T = 0)$ and $(p = 1/2, T \to \infty)$ in the (p, T) phase diagram and it is called the *Nishimori line*.

The proof of the relation above goes as follows. The full pdf of the interactions is

$$P(\{J_{ij}\}) = \prod_{\langle ij\rangle} P(J_{ij}) \tag{1.68}$$

and the average of any disorder dependent quantity is expressed as

$$[A_J] = \sum_{\{J_{ij}=\pm J\}} \prod_{\langle ij\rangle} P(J_{ij}) A_J \tag{1.69}$$

The disorder average Hamiltonian reads

$$[\langle H_J\rangle]_J = \sum_{\{J_{ij}\}} \frac{e^{K_p \sum_{\langle ij\rangle} J_{ij}/J}}{(2\cosh K_p)^{N_B}}\,\frac{\sum_{\{s_i\}}(-\sum_{ij} J_{ij}s_i s_j)\; e^{\beta\sum_{\langle ij\rangle} J_{ij}s_i s_j}}{\sum_{\{s_i\}} e^{\beta\sum_{\langle ij\rangle} J_{ij}s_i s_j}} \tag{1.70}$$

with N_B the number of bonds in the graph or lattice. Performing the gauge transformation

$$[\langle H_J\rangle]_J = \sum_{\{J_{ij}\}} \frac{e^{K_p \sum_{\langle ij\rangle} J_{ij}\eta_i\eta_j/J}}{(2\cosh K_p)^{N_B}}\,\frac{\sum_{\{s_i\}}(-\sum_{ij} J_{ij}s_i s_j)\; e^{\beta\sum_{\langle ij\rangle} J_{ij}s_i s_j}}{\sum_{\{s_i\}} e^{\beta\sum_{\langle ij\rangle} J_{ij}s_i s_j}} \tag{1.71}$$

where gauge invariance of the Hamiltonian has been used and the spins and interactions have been renamed J_{ij} and s_i. As this is independent of the choice of the parameters $\{\eta_i\}$ used in the transformation, one can sum over all possible 2^N choices and divide by this number keeping the result unchanged:

$$[\langle H_J\rangle]_J = \frac{1}{2^N}\sum_{\{J_{ij}\}} \frac{\sum_{\{\eta_i\}} e^{K_p \sum_{\langle ij\rangle} J_{ij}\eta_i\eta_j/J}}{(2\cosh K_p)^{N_B}} \; \frac{\sum_{\{s_i\}}(-\sum_{ij} J_{ij}s_is_j)\; e^{\beta\sum_{\langle ij\rangle} J_{ij}s_is_j}}{\sum_{\{s_i\}} e^{\beta\sum_{\langle ij\rangle} J_{ij}s_is_j}} \tag{1.72}$$

If β is chosen to be $\beta = K_p/J$ the sum over the spins in the denominator (the partition sum in the normalisation) cancels out the sum over the parameters η_i introduced via the gauge transformation. The sum over J_{ij} and the remaining sum over the spin configurations can be rewritten

$$[\langle H_J\rangle]_J = \frac{1}{2^N}\frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial\beta}\right) \sum_{\{s_i\}} \prod_{\langle ij\rangle} \sum_{\{J_{ij}=\pm J\}} e^{\beta J_{ij}s_is_j} \; . \tag{1.73}$$

Changing now variables in the sum over $J_{ij} = \pm J$ to $\tau_{ij} = J_{ij}s_is_j = \pm J$,

$$\begin{aligned}[\langle H_J\rangle]_J &= \frac{1}{2^N}\frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial\beta}\right) \sum_{\{s_i\}} \prod_{\langle ij\rangle} \sum_{\tau_{ij}=\pm J} e^{\beta\tau_{ij}} \\ &= \frac{1}{2^N}\frac{1}{(2\cosh K_p)^{N_B}} \left(-\frac{\partial}{\partial\beta}\right) 2^N (2\cosh K_p)^{N_B} \; ,\end{aligned} \tag{1.74}$$

where the sum over the spin configurations yields the 2^N factor and the sum over the independent τ_{ij} configurations yields the last factor. Finally, taking the derivative with respect to β:

$$[\langle H_J\rangle]_J = -N_B J \tanh K_p \tag{1.75}$$

with $K_p = \beta J$, defining the Nishimori line in the phase diagram.

For Gaussian distributed quenched randomness there also exists a Nishimori line and the averaged internal energy can also be computed exactly on this line.

Many other relations of this kind exist and are explained in [46].

A Classical results in statistical physics

A.1 High temperature expansion

The partition function of the Ising ferromagnet reads

$$Z = \sum_{s_i=\pm 1} e^{\beta J \sum_{\langle ij \rangle} s_i s_j} = \sum_{s_i=\pm 1} \prod_{\langle ij \rangle} e^{\beta J s_i s_j} \tag{A.76}$$

Using the identity $e^{\beta J s_i s_j} = a(1 + b s_i s_j)$ with $a = \cosh(\beta J)$ and $b = \tanh(\beta J)$ and the fact that b is order β, an expansion if powers of b can be established. The average of products of the spins s's that remains can be non-zero only if each spin appears an even number of times s. The expansion can then be represented as graphs on the lattice, a representation that makes the enumeration of terms easier.

A.2 Lee-Yang theorem

The LeeYang theorem states that if partition functions of models with ferromagnetic interactions are considered as functions of an external field, then all zeros are purely imaginary (or on the unit circle after a change of variable) [96].

A.3 Critical behaviour

Second order phase transitions are characterised by the diverge of the correlation length. In normal conditions, far from the critical point, the correlation function of the fluctuations of an observable decay as an exponential of the distance between the measuring points:

$$C(\vec{r}) \equiv \langle [O(\vec{r} + \vec{r}') - \langle (O(\vec{r} + \vec{r}') \rangle][O(\vec{r}') - \langle (O(\vec{r}') \rangle)\rangle] \rangle \simeq e^{-r/\xi} \, . \tag{A.77}$$

ξ is the correlation length that diverges at the critical point as

$$\xi \simeq |T - T_c|^{-\nu} \tag{A.78}$$

with ν a critical exponent. A power-law singularities in the length scales leads to power-law singularities in observable quantities. We summarise in Table 1 all the critical exponents associated to various quantities in a second order phase transition. The values of the critical exponents generally do not depend on the microscopic details but only on the space dimensionality and the symmetries of the system under consideration.

The collection of all these power laws characterizes the critical point and is usually called the critical behavior.

	exponent	definition	conditions
Specific heat	α	$c \propto \lvert u\rvert^{-\alpha}$	$u \to 0,\ h = 0$
Order parameter	β	$m \propto (-u)^{\beta}$	$u \to 0-,\ \ h = 0$
Susceptibility	γ	$\chi \propto \lvert u\rvert^{-\gamma}$	$u \to 0,\ h = 0$
Critical isotherm	δ	$h \propto \lvert m\rvert^{\delta}\mathrm{sign}(m)$	$h \to 0,\ u = 0$
Correlation length	ν	$\xi \propto \lvert r\rvert^{-\nu}$	$r \to 0,\ h = 0$
Correlation function	η	$G(\vec{r}) \propto \lvert\vec{r}\rvert^{-d+2-\eta}$	$r = 0,\ h = 0$

Table 1: Definitions of the commonly used critical exponents. m is the order parameter, *e.g.* the magnetization, h is an external conjugate field, *e.g.* a magnetic field, u denotes the distance from the critical point, *e.g.* $|T - T_c|$, and d is the space dimensionality.

Whether fluctuations influence the critical behavior depends on the space dimensionality d. In general, fluctuations become less important with increasing dimensionality.

In sufficiently low dimensions, *i.e.* below the lower critical dimension d_l, fluctuations are so strong that they completely destroy the ordered phase at all (nonzero) temperatures and there is no phase transition. Between d_l and the upper critical dimension d_u, order at low temperatures is possible, there is a phase transition, and the critical exponents are influenced by fluctuations (and depend on d). Finally, for $d > d_u$, fluctuations are unimportant for the critical behavior, and this is well described by mean-field theory. The exponents become independent of d and take their mean-field values. For example, for Ising ferromagnets, $d_l = 1$ and $d_u = 4$, for Heisenberg ferromagnets $d_l = 2$ and $d_u = 4$.

References

[1] H. E. Stanley, *Introduction to phase transitions and critical phenomena* (Oxford University Press, New York, 1971).

[2] N. Goldenfeld, *Lectures on phase transitions and the renormalization group* (Addison-Wesley, 1992).

[3] J. Cardy, *Scaling and renormalization in Statistical Physics*, Cambridge Lecture notes in physics (Cambridge University Press, 1996).

[4] D. J. Amit, *Field theory, the renormalization group and critical phenomena* (World Scientific, Singapore, 1984).

[5] G. Parisi, *Statistical Field Theory*, Frontiers in Physics, Lecture Notes Series (Addison-Wesley, 1988).

[6] P. Clifford and A. Sudbury, *Model for spatial conflict*, Biometrika 60, 581 (1973).

[7] M. Mézard and A. Montanari, *Information, Physics, and Computation*, (Oxford University Press, 2009).

[8] R. Aster, B. Borchers, and C. Thurber, *Parameter Estimation and Inverse Problems*, 2nd ed. (Elsevier, 2012).

[9] G. Toulouse, *Theory of frustration effects in spin-glasses*, Comm. Phys. **2**, 115 (1977).

[10] J. F. Sadoc and R. Mosseri, *Geometrical Frustration*, (Cambridge Univ. Press, 1999), reedited 2007.

[11] R. Moessner and A. P. Ramírez, *Geometrical Frustration*, Physics Today 2006.

[12] *Frustrated spin systems*, H. T. Diep ed. (World Scientific, 2004).

[13] R. M. F. Houtappel, Physica **16**, 425 (1950).

[14] G. H. Wannier, *Antiferromagnetism. The tringular Ising net*, Phys. Rev. **79**, 357 (1950).

[15] B. D. Metcalf and C. P. Yang, *Degeneracy of antiferromagnetic Ising lattices at critical magnetic field and zero temperature*, Phys. Rev. B **18**, 2304 (1978).

[16] C. Rottman and M. Wortis, *Statistical mechanics of equilibrium crystal shapes - interfacial phase diagrams and phase transitions* , Phys. Rep. **103**, 59 (1984).

[17] L. Pauling, *The structure and entropy of ice and other crystals with some randomness of atomic arrangements*, J. Chem. Phys. **57**, 2680 (1935).

[18] E. H. Lieb, *Exact solution of problem of entropy of 2-dimensional ice*, Phys. Rev. Lett. **18**, 692 (1967), *Exact solution of F model of an antiferroelectric*, Phys. Rev. Lett. **18**, 1046 (1967), *Exact solution of 2-dimensional Slater KDP model of a ferroelectric*, Phys. Rev. Lett. **19**, 108 (1967).

[19] R. J. Baxter *Exactly Solved Models in Statistical Mechanics* (Dover, New York, 1982).

[20] W. F. Giauque and J. W. Stout, *The entropy of water and the third law of thermodynamics - The heat capacity of ice from 15 to 273 degrees K* J. Am. Chem. Soc. **58**, 1144 (1936). A. P. Ramírez, A. Hayashi, R. J. Cava, R. S. Siddharthan and B. S. Shastry, *Zero-point entropy in 'spin ice'*, Nature **399**, 333 (1999).

[21] J. Villain, *Spin glass with non-random interactions*, J. Phys. C: C: Solid State Phys. **10**, 1717 (1977).

[22] M. E. Fisher, *Statistical mechanics of dimers on a place lattice*, Phys. Rev. **124**, 1664 (1961).

[23] M. Grousson, G. Tarjus, and P. Viot, *Phase diagram of an Ising model with long-range frustrating interactions: A theoretical analysis*, Phys. Rev. E **62**, 7781 (2000).

[24] M. Antonietti and C. Göltner, Angew. Chem., Int. Ed. Engl. **36**, 910 (1997)

[25] D. R. Nelson, *Order, frustration and defects in liquids and glasses*, Phys. Rev. B **28**, 5515 (1983).

[26] D. R. Nelson, *The structure and statistical mechanics of glass*, Lecture Notes in Physics **216**, 13 (1985).

[27] G. Tarjus, S. A. Kivelson, Z. Nussinov and P. Viot, *The frustration-based approach of supercooled liquids and the glass transition: a review and critical assessment*, ?J. Phys.: Condens. Matter **17**, R1143 (2005).

[28] J. Villain, R. Bidaux, J.-P. Carton, and R. Conte, *Order as an effect of disorder*, J. Physique **41**, 1263 (1980).

[29] E. F. Shender and P. W. C. Holdsworth, *Order by disorder and frustration in magnetic systems*, in *Fluctuations and order*, M. Millonas ed. (Springer, 1996).

[30] H. A. Kramers and G. H. Wannier, *Statistics of the two-dimensional ferromagnet Part I*, Phys. Rev. **60**, 252 (1941). *Statistics of the two-dimensional ferromagnet Part II*, Phys. Rev. **60**, 263 (1941).

[31] L. Onsager, *Crystal statistics I A two-dimensional model with an order-disorder transition* , Phys. Rev. **65**, 117 (1944).

[32] A. L. Efros, *Physics and geometry of disorder: percolation theory*, (Mir Publishers, Mockba, 1986)

[33] D. Stauffer and A. Aharony, *Introduction To Percolation Theory*, (Taylor and Francis, London,1994)

[34] *Percolation theory*, unpublished (2002).

[35] A. A. Saberi, Phys. Rep. **578**, 1 (2015).

[36] T. T. Kantzas, K. D. O'Neil and O. A. Semenikhin, Electrochim. Acta **53**, 1225 (2007).

[37] M. E. J. Newman, R. M. Ziff, *A fast Monte Carlo algorithm for site or bond percolation*, Phys. Rev. E **64**, 016706 (2001).

[38] P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Japan **26** (Suppl.), l l (1969).

[39] C. M. Fortuin and P. W. Kasteleyn, Physica **57** 536 (1972).

[40] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).

[41] A. Barrat, M. Barthelemy, and A. Vespignani, *Dynamical processes on complex networks* (Cambridge University Press, Cambridge, 2008).

[42] J. A. Mydosh, *Spin Glasses* (Taylor & Francis, 1985).

[43] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin glasses and beyond; an intriduction to the replica method and its applications*, (World Scientific, Singapore, 1986).

[44] K. Binder and A. P. Young, *Spin glasses: Experimental facts, theoretical concepts, and open question*, Rev. Mod. Phys. **58** 801 (1986).

[45] K. H. Fischer and J. Hertz, *Spin glasses* (Cambridge Univ. Press, 1991).

[46] H. Nishimori, *Statistical phyiscs of spin glasses and information processing*, (Oxford Univ. Press, 2001).

[47] P. Nordblad, *Spin Glasses: Model systems for non-equilibrium dynamics*, arXiv:cond-mat/0309602.

[48] E. Vincent, J. Hammann, M. Ocio, J.-P. Bouchaud and L. F. Cugliandolo, *Slow dynamics and aging in spin glasses*, Complex Behaviour of Glassy Systems, 184-219 (1997).

[49] R. B. Griffiths, *Non-analytic behavior above the critical point in a random Ising ferromagnet*, Phys. Rev. Lett. **23**, 17 (1969).

[50] T. Vojta, *Phases and phase transitions in disordered quantum systems*, arXiv:1301.7746

[51] A. B. Harris, *Effects of random defects on critical behaviour of Ising models*, J. Phys. C **7**, 1671 (1974).

[52] H. A. Brooks, *The Harris criterion lives on* J. Phys.: Condens. Matter **28**, 421006 (2016).

[53] J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer, *Correlation length bounds for disordered Ising ferromagnets*, Comm. Math. Phys. **120**, 501 (1989).

[54] Y. Imry and S-k Ma, *Random-Field Instability of the Ordered State of Continuous Symmetry*, Phys. Rev. Lett. **35**, 1399 (1975).

[55] M. Aizenman and Wehr, Phys. Rev. Lett. **62**, 2503 (1989).

[56] J. Bricmont and A. Kupiainen, *Phase transition in the 3d random field Ising model* Comm. Math. Phys. **116**, 539 (1988).

[57] D. S. Fisher, *Low temperature phases, ordering and dynamics in random media*, in *Phase transitions and relaxation in system with competing energy scales*, T. Riste and D. S. Sherrington eds. (Kruwer Academic Series, 1993)

[58] S. Lemerle, J. Ferré, C. Chappert, V. Mathet, T. Giamarchi, and P. Le Doussal, *Domain Wall Creep in an Ising Ultrathin Magnetic Film*, Phys. Rev. Lett. **80**, 849 (1998).

[59] A. I. Larkin, Sov. Phys. JETP **31**, 784 (1970).

[60] T. H. Berlin and M. Kac, *The spherical model of a ferromagnet*, Phys. Rev. **86**, 821 (1952).

[61] J. M. Kosterlitz, D. J. Thouless, and R. C. Jones, *Spherical Model of a Spin-Glass*, Phys. Rev. Lett. **36**, 1217 (1976).

[62] M. L. Mehta, *Random matrices* (Academic Press, 2004).

[63] S. F. Edwards and R. C. Jones, *The eigenvalue of a large symmetric random matrix*, J. Phys. A: Math. Gen. **9**, 1595 (1976).

[64] G. Semerjian and L. F. Cugliandolo, J. Phys. A **35**, 4837 (2002). G. J. Rodgers and A. J. Bray, Phys. Rev. B **37**, 3557 (1988). A. D. Mirlin and Y. V. Fyodorov, J. Phys. A **24**, 2273 (1991).

[65] L. F. Cugliandolo and D. S. Dean, *Full dynamical solution for a spherical spin-glass model*, J. Phys. A **28**, 4213 (1995).

[66] F. Corberi, E. Lippiello, and M. Zannetti, *Slow relaxation in the large-N model for phase ordering*, Phys. Rev. E **65**, 046136 (2002).

[67] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large N limit: a review*, Phys. Rep. **385**, 69 (2003).

[68] *The Large N Expansion in Quantum Field Theory and Statistical Physics From Spin Systems to 2-Dimensional Gravity*, edited by E Brézin and S. R. Wadia (World Scientific, Singapore, 1993).

[69] L. Lévy, *Nonlinear dynamic susceptibilities at the spin-glass transition of Ag:Mn*, Phys. Rev. Lett. **57**, 3288 (1986).

[70] L. P. Lévy amd A. T. Ogielski, *Nonlinear Dynamic Susceptibilities at the Spin-Glass Transition of Ag:Mn*, Phys. Rev. Lett. **57**, 3288 (1987).

[71] A. J. Bray and M. A. Moore, *Metastable states in spin glasses*, J. Phys. C: Solid State Phys. **13**, L469 (1980), *Chaotic nature of the spin-glass phase*, Phys. Rev. Lett. **58**, 57 (1987).

[72] D. S. Fisher and D. A. Huse, *Equilibrium behavior of the spin-glass ordered phase*, Phys. Rev. B **38**, 386 (1988).

[73] J-M Luck, *Systèmes désordonnés unidimensionnels*, ALEA-Saclay (1992).

[74] S. F. Edwards and P. W. Anderson, *Theory of spin glasses*, J. Phys. F: Metal Phys. **5**, 965 (1975).

[75] D. S. Sherrington and S. Kirkpatrick, *Solvable model of a spin-glass*, Phys. Rev. Lett. **35**, 1792 (1975).

[76] D. J. Thouless, P. W. Anderson and R. G. Palmer, *Solution of 'Solvable model of a spin glass'* Phil. Mag. **35**, 593 (1977).

[77] G. Biroli, *Dynamical TAP approach to mean field glassy systems*, J. Phys. A: Math. Gen. **32**, 8365 (1999).

[78] C. Göbl and N. Tjandra, *Application of Solution NMR Spectroscopy to Study Protein Dynamics*, Entropy **14**, 581 (2012).

[79] W. Janke, B. A. Berg and A. Billoire, *Multi-overlap simulations of spin glasses*, NIC Symposium 2001, Proceedings, Horst Rollnik, Dietrich Wolf (Editors), John von Neumann Institute for Computing, Jülich, NIC Series, Vol. 9, ISBN 3-00-009055-X, pp. 301-314, 2002.

[80] A. P. Young, *Direct determination of the probability distribution for the spin-glass order parameter*, Phys. Rev. Lett. **51**, 1206 (1983).

[81] M. Hasenbusch, Andrea Pelissetto, and Ettore Vicari, *The critical behavior of 3D Ising spin glass models: universality and scaling corrections*, J. Stat. Mech PL02001 (2008).

[82] H. Bouchiat, *Determination of the critical exponents in the Ag Mn spin glass*, J. Physique **47**, 71 (1986).

[83] F. Tanaka and S. F. Edwards, J. Phys. F **10**, 2769 (1980).

[84] A. J. Bray and M. A. Moore, J. Phys. C **13**, L469 (1980).

[85] T. Aspelmeier, R. A. Blythe, A. J. Bray, and M. A. Moore, *Free energy landscapes, dynamics and the edge of chaos in mean-field models of spin glasses*, Phys. Rev. B **74**, 184411 (2006).

[86] R. Monasson, *Structural glass transition and the entropy of metastable states*, Phys. Rev. Lett. **75**, 2847 (1995).

[87] S. Franz and G. Parisi, *Recipes for metastable states in spin-glasses*, J. Phys. I **5**, 1401 (1995).

[88] L. F. Cugliandolo and J. Kurchan, *Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model*, Phys. Rev. Lett. **71**, 173 (1993).

[89] D. Mehta, D. A. Stariolo, and M. Kastner, *Energy landscape of the finite-size mean-field 3-spin spherical model*, Phys. Rev. E **87**, 052143 (2013).

[90] J. L. van Hemmen and R. G. Palmer, *The Replica Method and a Solvable Spin Glass Model*, J. Phys. A **12**, 563 (1979).

[91] G. Parisi, *The order parameter for spin glasses: a function on the interval 0-1*, J. Phys. A: Math. Gen. 13: 11011112 (1980).

[92] F. Guerra and F. L. Toninelli, *The thermodynamic limit in mean field spin glass models*, Comm. Math. Phys. **230** 71 (2002).

[93] M. Talagrand, *Replica symmetry breaking and exponential inequalities for the Sherrington Kirkpatrick model*, Annals of Probability **28**, 1018 (2000).

[94] J. R. L. de Almeida and D. J. Thouless, *Stability of the Sherrington-Kirkpatrick solution of a spin-glass model*, J. Phys. A. **11**, 983 (1978).

[95] J.-P. Bouchaud, L. F. Cugliandolo, J. Kurchan and M. Mézard, *Out of equilibrium dynamics in spin-glasses and other glassy systems*, in *Spin glasses and random fields*, A. P. Young ed. (World Scientific, 1997).

[96] T. D. Lee and C. N. Yang, *Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model*, Phys. Rev. **87**, 410 (1952).

[97] T. Schneider and E. Pytte, *Random-field instability of the ferromagnetic state*, Phys. Rev. B **15**, 1519 (1977).