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Random-field instability of the ferromagnetic state

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We consider an Ising model with infinite-ranged interaction with statistically independent site fields with Gaussian distribution. The model is solved exactly and exhibits both an independent spin phase and a ferromagnetic phase, separated by a line of second-order phase transitions. We also establish that the replica technique yields exact results in the present model but not in the related random exchange system.

The question of the influence of random fields and random coupling constants on the phase diagram and the critical phenomena has recently attracted considerable attention.[1-9] It was demonstrated that, in Ising and Heisenberg models with random interactions, ferromagnetic, paramagnetic, and spin-glass phases are likely.[5-7,9] Moreover, it was shown, that when the order parameter has a continuous symmetry, the ordered state is unstable against an arbitrarily weak random field in less than four dimensions.[8]

Here we present a simple exactly soluble model exhibiting an instability of the ferromagnetic phase to an independent spin phase that is driven by a random field. In recent work on spin glasses,[5-7,9] it was the interaction that were considered random. In the Ising model studied here, the local field conjugate to the spins is random with Gaussian distribution. The exchange interaction is assumed to be infinite ranged and only static random fields are considered. It will be established that an independent spin and a ferromagnetic phase occur. The independent spin phase will be characterized by the order parameter introduced by Edwards and Anderson.[5] The phase diagram and the critical properties will be discussed in some detail.

We consider N Ising spins interacting through an infinite ranged exchange interaction. The Ising spins are coupled to a random field with a Gaussian distribution. The Hamiltonian is

$$\mathcal{H} = -\frac{J}{N}\sum_{i\neq j} S_i S_j - \sum_i h_i S_i , \tag{1}$$

where the h_i are distributed according to

$$P(h) = [(2\pi)^{1/2}\sigma]^{-1}\exp(-h^2/2\sigma^2) , \tag{2}$$

with the same distribution for every site. By contrast the infinite ranged model considered in Ref. 7, had random exchange interactions rather than a random field. The Hamiltonian and the random exchange distribution were defined by[7]

$$\mathcal{H} = -\frac{J}{N}\sum_{i\neq j} S_i S_j - \frac{\sigma}{\sqrt{N}}\sum_{i\neq j} Z_{ij} S_i S_j , \tag{3}$$

where the Z_{ij} are distributed according to

$$P(Z_{ij}) = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}Z_{ij}^2) . \tag{4}$$

In both models only a quenched system is considered in which case the free energy rather than the partition function is to be averaged.

We first calculate the free energy for random-field model using the replica method.[5,7] The derivation is closely analogous to that for the random-J case[7] but is simpler as only quadratic couplings are obtained between different replicas. The result is

$$\frac{1}{N}\langle F\rangle_h = j\,m^2 - \frac{1}{\beta}\int dh\,P(h)\ln 2\cosh[\beta(2Jm+h)] , \tag{5}$$

where m is given by

$$\langle\langle S_i\rangle\rangle_h = m = \int dh\,P(h)\tanh[\beta(2mJ+h)] . \tag{6}$$

The details of the calculation are outlined in the Appendix. From Eq. (5) the energy is calculated to be

$$\langle\langle\mathcal{H}\rangle\rangle_h (1/N) = -Jm^2 + \beta\sigma^2(1-q_h) , \tag{7}$$

where

$$q_h = \langle\langle S_i\rangle^2\rangle_h = \int dh\,P(h)\tanh^2[\beta(2mJ+h)] . \tag{8}$$

To prove that these results are exact we introduce an effective Hamiltonian $\mathcal{H}_0$, linear in S_i, of the form

$$\mathcal{H}_0 = Jm^2 N - 2mJ\sum_i S_i - \sum_i h_i S_i \tag{9}$$

and define $\mathcal{H}_1 = \mathcal{H} - \mathcal{H}_0$. $\mathcal{H}_0$ is a mean-field Hamiltonian which may be expected to give exact results because of the infinite ranged interaction. The thermodynamic variational principle[10] (Gibbs-Bogoliubov inequality) then gives

$$\begin{aligned}(1/N)\langle\langle\mathcal{H}_1\rangle_0\rangle_h + (1/N)\langle F_0\rangle_h &\geq (1/N)\langle F\rangle_h \\ &\geq (1/N)\langle F_0\rangle_h + (1/N)\langle\langle\mathcal{H}_1\rangle\rangle_h ,\end{aligned} \tag{10}$$

where $(1/N)\langle F_0\rangle_h$ is given by the right-hand side of Eq. (5) and $\langle\ \rangle_0$ denotes the thermal average with respect to $\mathcal{H}_0$. To show that $(1/N)\langle F_0\rangle_h$ is both an upper and lower bound for $(1/N)\langle F\rangle_h$, we add to the Hamiltonian the term $\lambda\sum_i A_i$, where A_i is specified below.[11] Taking the second derivative of the free energy with respect to λ, we obtain

$$\frac{1}{N}\frac{\partial^2}{\partial\lambda^2}\langle F\rangle_{h,z/\lambda=0}=-N\beta\left\langle\!\!\left\langle\left(\frac{1}{N}\sum_i(A_i-\langle A_i\rangle)\right)^2\right\rangle\!\!\right\rangle_{h,z}. \tag{11}$$

It then follows, that

$$\left\langle\!\!\left\langle\left(\frac{1}{N}\sum_i(A_i-\langle A_i\rangle)\right)^2\right\rangle\!\!\right\rangle_{h,z}=O\left(\frac{1}{N}\right). \tag{12}$$

$\langle\ \rangle_{h,z}$ denotes on average with respect to the random variables h and z, respectively. With the aid of this equation it is easily verified, by setting $A_i=S_i$, that $(1/N)\langle\langle\mathcal{H}_1\rangle_0\rangle_h$ and $(1/N)\langle\langle\mathcal{H}_1\rangle\rangle_h$ both vanish in the thermodynamic limit. Thus for the random-field model, the replica technique, as used in Ref. 7 yields the exact result. This is a nontrivial result, because the method involves analytic continuation of an integer variable into a continuous variable, and in addition, an interchange of the limits $N\to\infty$ and $n\to 0$ with the large N limit taken before $n\to 0$ where n is the number of replicas. In fact, for the random exchange model the replica method does not yield the exact result. To show this we consider the internal energy. From Eq. (3) we obtain

$$\frac{1}{N}\langle\langle\mathcal{H}\rangle\rangle_z=-J\left\langle\!\!\left\langle\left(\frac{1}{N}\sum_i S_i\right)^2\right\rangle\!\!\right\rangle_z-\frac{\beta\sigma^2}{N^2}\sum_{i\neq j}(1-\langle\langle S_iS_j\rangle^2\rangle_z)+\frac{J}{N}. \tag{13}$$

Here, $\langle\ \rangle_z$ denotes the average over the random exchange interaction. The Z_{ij} have been eliminated by partial integration. Making use of Eq. (12) twice, with $A_i=S_i$ and $A_i=\langle S_i\rangle\ S_i$, respectively, the internal energy reduces to

$$\frac{1}{N}\langle\langle\mathcal{H}\rangle\rangle_z=-\frac{J}{N^2}\sum_{i,j}\langle\langle S_i\rangle\langle S_j\rangle\rangle_z-\frac{\beta\sigma^2}{N^2}\sum_{i,j}(1-\langle\langle S_i\rangle^2\langle S_j\rangle^2\rangle_z-\langle\langle\delta S_i\delta S_j\rangle^2\rangle_z), \tag{14}$$

where

$$0\leq\frac{1}{N^2}\sum_{i,j}(\langle\langle S_i\rangle^2\langle S_j\rangle^2\rangle_z+\langle\langle\delta S_i\delta S_j\rangle^2\rangle_z)\leq 1, \qquad \delta S_i=S_i-\langle S_i\rangle. \tag{15}$$

This expression for the energy per site differs from that in Ref. 7, derived by means of the replica technique. That approach yields[7]

$$(1/N)\langle\langle\mathcal{H}\rangle\rangle_z=-J\langle\langle S_i\rangle\rangle_z^2-\beta\sigma^2(1-q_z^2), \tag{16}$$

where

$$0\leq q_z=\langle\langle S_i\rangle^2\rangle_z\leq 1. \tag{17}$$

This result may be obtained from Eq. (14) by neglecting $\langle\langle\delta S_i\delta S_j\rangle^2\rangle_z$ and assuming the factorizations $\langle\langle S_i\rangle\langle S_j\rangle\rangle_z\to\langle\langle S_i\rangle\rangle_z\langle\langle S_j\rangle\rangle_z$ and $\langle\langle S_i\rangle^2\langle S_j\rangle^2\rangle_z\to\langle\langle S_i\rangle^2\rangle\times\langle\langle S_j\rangle^2\rangle_z$. These are exact in the random-field case. However, in the random exchange model these factorizations are not correct, because the z average involves uncorrelated pairs rather than uncorrelated single spins. As noted in Ref. 7, the replica method gives rise to a negative entropy in the spin-glass phase at $T=0$ for the random exchange model.

Next we discuss the thermodynamic properties of the random field model. The function m introduced in Eq. (6) is the magnetization and q_h [Eq. (8)] is the Edwards and Anderson spin-glass order parameter.[5,7] Nonzero q_h indicates the existence of magnetic moments, while $m\neq 0$ in addition to $q_h\neq 0$ indicates that the moments are ferromagnetically ordered. For $m=0$ but $q_h\neq 0$ the sign of the individual moments is determined by the local fields. From Eqs. (6) and (8) the boundary between the ferromagnetic and the independent spin phases are given by

$$2\beta J(1-q_h)=1. \tag{18}$$

The full phase diagram is plotted in Fig. 1 in terms of the dimensionless parameters k_BT/σ and $2J/\sigma$, where σ is the variance of the random field dis-

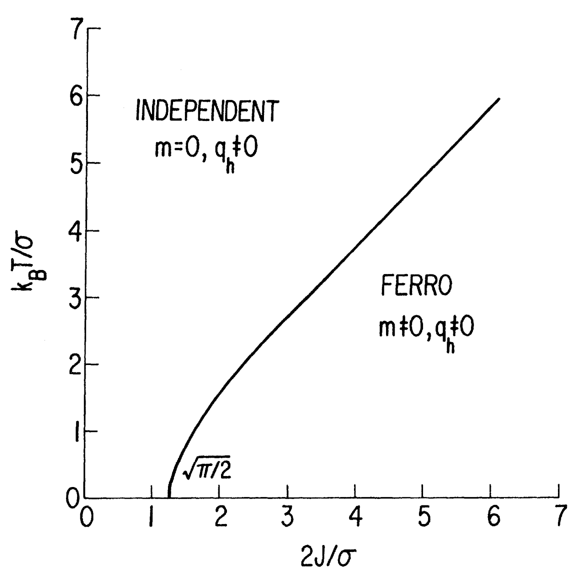

FIG. 1. Phase diagram of an Ising system with infinite-ranged exchange interaction in a random field with Gaussian distribution.

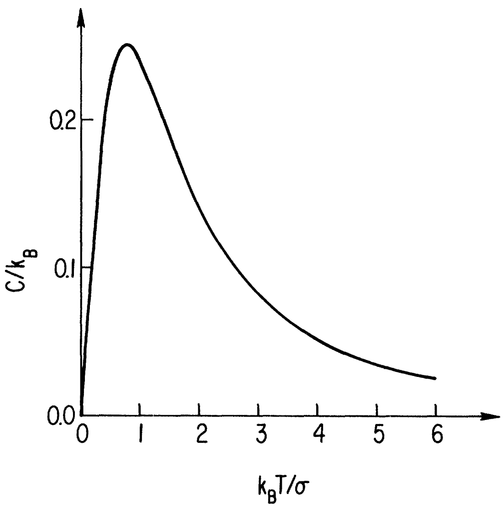

FIG. 2. Temperature dependence of the specific heat in the independent spin phase for $2J/\sigma < (\frac{1}{2}\pi)^{1/2}$.

tribution [Eq. (2)]. It is clear on physical grounds that for sufficiently strong random fields, the ferromagnetic order will be suppressed. The paramagnetic phase $m = q_h = 0$ is obtained only in the limit of infinite temperature, where the "spin-glass" order parameter q_h vanishes as $(\sigma/k_BT)^2$. Although the disordered phase is characterized by the Edwards and Anderson spin-glass order parameter, this phase is fundamentally different from the "spin-glass" phase discussed for the random exchange models. In the spin-glass phase the spins are correlated, whereas in the random field model the spins are completely independent in the $q_h \neq 0$, $m = 0$ phase. That is, the free energy as given by Eq. (5) is independent of the exchange coupling J.

The susceptibility is obtained by adding a uniform field term $H\sum_i S_i$ to the Hamiltonian. This simply adds on extra term βH to the argument of tanh in Eq. (6). Differentiating m with respect to H and taking the limit $H \to 0$, we obtain

$$\chi = \beta(1 - q_h)/[1 - 2\beta J(1 - q_h)]. \tag{19}$$

The susceptibility diverges at the phase boundary as seen from Eq. (18). By approaching the phase boundaries at $T = \text{const}$ we obtain the power laws

$$m^2 \sim J^* - J, \quad J^* > J \tag{20}$$

and

$$\chi \sim |J^* - J|^{-1}, \tag{21}$$

where J^* is fixed by the phase boundary (Fig. 1). Approaching the phase boundary at $J = \text{const}$, the same power laws are found again, with J^* and J replaced by T^* and T, respectively. The mean-field exponents associated with these power laws are a consequence of the infinite-ranged interactions.

In the independent spin phase, for $2J/\sigma < (\frac{1}{2}\pi)^{1/2}$, the specific heat and susceptibility are smooth functions of T. In fact at low temperature the specific heat is given by

$$C = (2/\pi)^{1/2} \tfrac{1}{12}\pi^2 k_B^2 T/\sigma \tag{22}$$

and vanishes for large T as $k_B(\sigma/k_BT)^2$, reaching a maximum in between. The full temperature dependence of the specific heat is shown in Fig. 2. At $T = 0$, the susceptibility goes to a finite value $[\sigma(\frac{1}{2}\pi)^{1/2} - J]^{-1}$, decreases for $2J/\sigma < (\frac{1}{2}\pi)^{1/2}$ smoothly with T and vanishes for large T as $(k_BT)^{-1}$. We also note that the entropy vanishes as $T \to 0$ linearly with T, and is equal to $k_B \ln 2$ at infinite temperature. The ground state is nondegenerate. At $T = 0$ the spins are all aligned by their respective local fields. As the temperature is increased only those spins for which the thermal energy is sufficiently large to overcome the local field, contribute to the thermal properties.

To summarize, we have shown that the replica[7] technique yields the exact result for the random field model, but that it fails for the random exchange model defined by Eqs. (3) and (4). Moreover, we have demonstrated that the ferromagnetic phase is unstable at all temperatures against a local random field with a sufficiently broad Gaussian distribution. Thus for $2J/\sigma \leq (\frac{1}{2}\pi)^{1/2}$ there is no ferromagnetic phase. On the otherhand for $2J/\sigma > (\frac{1}{2}\pi)^{1/2}$ both independent spin and ferromagnetic phases occur, depending on the temperature, separated by a line of second-order transitions.

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APPENDIX

We consider the Hamiltonian

$$H = -\frac{J}{N}\sum_{i \neq j} S_i S_j - \sum_i h_i S_i, \tag{A1}$$

where the random fields are distributed according to

$$p(h_i) = [(2\pi)^{1/2}\sigma]^{-1} \exp(-h_i^2/2\sigma^2). \tag{A2}$$

Using the identity

$$\ln x = \lim_{n \to 0} \frac{1}{n}(x^n - 1), \tag{A3}$$

the averaged free energy may be expressed as

$$F=-kT\lim_{n\to 0}\frac{1}{n}\left(\mathrm{Tr}_n\left\{\left(\exp\sum_{\alpha=1}^{n}\frac{J}{N}\sum_{i\neq j}S_i^\alpha S_j^\alpha\right)\left[\exp\frac{1}{2}\left(\frac{\sigma}{kT}\right)^2\sum_i\left(\sum_\alpha S_i^\alpha\right)^2\right]\right\}-1\right). \tag{A4}$$

Making use of the formula

$$e^{\lambda a^2}=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dx\,\exp[-\tfrac{1}{2}x^2+(2\lambda)^{1/2}ax] \tag{A5}$$

and the identity

$$\mathrm{Tr}\exp\left(\sum_i f_i\right)=(\mathrm{Tr}\exp f_i)^N=[\exp(\ln\mathrm{Tr}\exp f_i)]^N=\exp(N\ln\mathrm{Tr}\exp f_i), \tag{A6}$$

we obtain (apart from terms which vanish in the thermodynamic limit)

$$F=-kT\lim_{n\to 0}\frac{1}{n}\left(\prod_\alpha\left[\int dx_\alpha\left(\frac{N}{2\pi}\right)^{1/2}\right]\exp N\left\{-\sum\frac{x_\alpha^2}{2}+\ln\mathrm{Tr}\exp\left[\left(\frac{2J}{kT}\right)^{1/2}\sum_\alpha S_\alpha x_\alpha+\frac{1}{2}\left(\frac{\sigma}{kT}\right)^2\left(\sum S_\alpha\right)^2\right]\right\}^{-1}\right), \tag{A7}$$

where the trace is now over n replicas at a single spin site. In the thermodynamic limit the integrals may be evaluated by steepest descent in the usual way. At the maximum all x_α are equal. Then, if we further define $m=(kT/2J)^{1/2}x_{\max}$, the free energy and the extremum condition may be written

$$F=-kT\lim_{n\to 0}\frac{1}{n}\left[\exp N\left(-n\frac{J}{kT}m^2+\ln\mathrm{Tr}\exp A\right)-1\right], \tag{A8}$$

$$m=\frac{\mathrm{Tr}S_\alpha e^A}{\mathrm{Tr}\,e^A}, \tag{A9}$$

where

$$A=\frac{2J}{kT}m\sum_\alpha S_\alpha+\frac{1}{2}\left(\frac{\sigma}{kT}\right)^2\left(\sum_\alpha S_\alpha\right)^2. \tag{A10}$$

Making use of Eq. (A5), once more, e^A may be expressed as

$$e^A=\int\frac{ds}{(2\pi)^{1/2}}\exp(-\tfrac{1}{2}s^2)\prod_\alpha\exp\left(\frac{2J}{kT}m+s\frac{\sigma}{kT}\right)S_\alpha. \tag{A11}$$

Then evaluating the traces in Eqs. (A8) and (A9) and taking the limit $n\to 0$ we finally obtain

$$\frac{F}{N}=Jm^2-\frac{1}{\beta}\int dh\,p(h)\ln 2\cosh(2\beta Jm+h), \tag{A12}$$

$$m=\int dh\,p(h)\tanh(2\beta Jm+h), \tag{A13}$$

as given in the text.

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