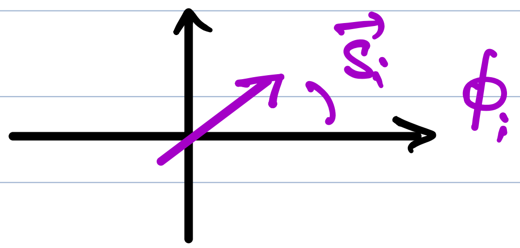
2.10 MODELS WITH CONTINUOUS SYMMETRIES

2d XY MODEL

$$\vec{S}_i = \|\vec{S}_i\| (\cos\phi_i, \sin\phi_i) \quad \forall i$$



TWO COMPONENT SPINS LIVE ON A PLANE

$n=2$ DIMENSION OF VARIABLES & OF POTENTIAL ORDER PARAMETER

$$H(\{\vec{S}_i\}) = -\frac{J}{2}\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j$$

SCALAR PRODUCT

$$= -\frac{J}{2}\sum_{\langle ij\rangle} \|S_i\| \, \|S_j\| \cos(\phi_i - \phi_j)$$

USUALLY $\|S_i\| = 1 \quad \forall i$ FIXED MODULUS

(SUM CONTRIB. FROM EACH LINK ONCE)

EXPERIMENTAL REALIZATIONS

- CLASSICAL MAGNETISM

FURTHER IMPORTANCE OF 2d XY - LIKE PHYSICS IN CLASSICAL SYST:

- MELTING IN 2d

DRIVEN BY TOPOLOGICAL DEFECTS

SOLID $\rightarrow$ HEXATIC $\rightarrow$ LIQUID

- QUANTUM PHYSICS

$$\psi(\vec{x}) = |\psi(\vec{x})| \, e^{i\phi(\vec{x})}$$

MODULUS & PHASE OF WAVE FUNCTION

~ MODULUS $\|\vec{S}_i\|$ & ANGLE ϕ_i.

SUPERFLUIDITY - JOSEPHSON JCTS. ETC.

SPECIALLY INTERESTING IN $d=2$
WHEN SPACE IS ALSO PLANAR

See, e.g. J. DALIBARD'S LECTURES 2016
AT COLLÈGE DE FRANCE
FOR REAL. IN. ATOMIC PHYSICS

SPACE & SPIN PLANES DON'T NEED TO BE THE SAME
BUT WE'LL TAKE THEM TO BE (SIMPLER)

MODEL HAMILTONIAN

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)$$

$$\|\vec{S}_i\| = 1 \quad \forall i$$

SYMMETRIES

GLOBAL CONTINUOUS ROTATIONAL SYMM

$$\vec{S}_i' = R\,\vec{S}_i \qquad \phi_i' = \phi_i + \Delta \qquad \Delta \in \mathbb{R}$$

LOCAL DISCRETE SYMM

$$\phi_i - \phi_j \longrightarrow \phi_i - \phi_j + 2\pi q_{ij} \qquad q_{ij} \in \mathbb{Z}$$

GROUND STATES

INFINITE DEGENERACY AS ALL

$\vec{S}_i = \vec{S}$ WITH $\|\vec{S}\| = 1$

ARE POSSIBLE

LOW ENERGY EXCITATIONS

IN ISING $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow$

GAPPED

FROM $-J$ TO J $\Rightarrow$ $2J > 0$ COST

IN XY $\uparrow \nearrow \nearrow \nearrow \rightarrow \searrow \searrow \searrow \downarrow \downarrow$

CAN BE MADE INFINITESIMAL

GAPLESS SOFT MODES

EASY TO SUSTAIN LONG-WAVE LENGTH EXCITATIONS $\Rightarrow$

SPIN-WAVES

NO SPONTANEOUS SYMMETRY BREAKING

ADD PINNING FIELD $\vec{h}$

$$H \to H - \sum_i \vec{h}_i \cdot \vec{S}_i$$

AND COMPUTE

$$\lim_{\vec{h} \to \vec{0}} \lim_{N \to \infty} \vec{m}_h$$

$$\frac{1}{N} \sum_i \langle \vec{S}_i \rangle_{\vec{h}}$$

MERMIN WAGNER

BOUND THIS QUANTITY BY

$$f(N) \xrightarrow[N \to \infty]{} 0 \qquad \forall T > 0$$

NO LONG-RANGE ORDER AT ANY $T > 0$ IN $d = 2$

(BUT IMPORTANT FINITE SIZE EFFECTS)

2.10.1

LOW ENERGY EXCITATIONS SPIN WAVES

$$\phi_i - \phi_j \approx \varepsilon \Rightarrow \cos(\phi_i - \phi_j) \sim 1 - \frac{(\phi_i - \phi_j)^2}{2}$$

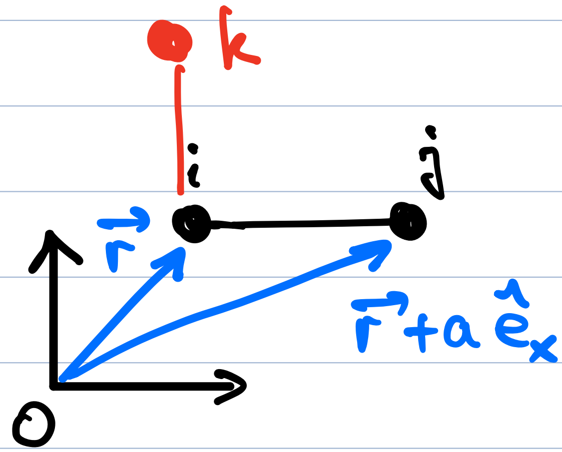
$$H(\{\vec{S}_i\}) \simeq E_0 + \frac{J}{4} \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2$$

DROPPING h.o.t.

CONTINUOUS FIELD THEORY LIMIT

$$\phi_i - \phi_j = \phi(\vec{r} + \vec{a}) - \phi(\vec{r})$$

INDICES i, j IDENTIFY TWO N.N. SITES



$$\phi_i - \phi_j = \phi(\vec{r}+\vec{a}) - \phi(\vec{r}) \qquad \vec{a} = a \cdot e_x$$

$$= \cancel{\phi(\vec{r})} + \frac{d\phi(\vec{r})}{dx}\, a - \cancel{\phi(\vec{r})} + h.o.t$$

$$\simeq \frac{d\phi}{dx} \cdot a$$

NEGLECTING

$$\phi_i - \phi_k = \phi(\vec{r}+\vec{a}) - \phi(\vec{r}) \qquad \vec{a} = a\,\hat{e}_y$$

$$= \cancel{\phi(\vec{r})} + \frac{d\phi(\vec{r})}{dy} \cdot a - \cancel{\phi(\vec{r})} + h.o.t$$

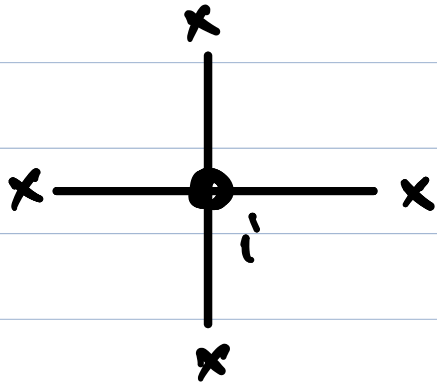
$$\simeq \frac{d\phi(\vec{r})}{dy}\, a$$

$$\sum_{\langle ij \rangle} (\phi_i - \phi_j)^2 = \sum_i \sum_{j \in \partial i} (\phi_i - \phi_j)^2$$

EACH PAIR (ij) IS ADDED TWICE

FOCUS ON ONE TERM OF SUM:

i FIXED $\quad j \in \partial i \Rightarrow$ 4 CONTRIB.



THE SIGN WHEN ADDING LEFT & RIGHT DOESN'T MATTER

$$= \left\{ \left(\frac{d\phi}{dx}\right)^2 a^2 + \left(\frac{d\phi}{dy}\right)^2 a^2 \right\} 2$$

$$= 2a^2 \, \|\vec{\nabla}\phi\|^2$$

NOW, INSERT IN THE FULL SUM

i.e. IN H:

$$H(\{s_i\}) \to H(\{\phi_i\}) \to H(\{\phi(r)\}) \qquad (1)$$

FACTORS a :

THERE'S AN a^2 COMING FROM CONT VERSION OF DERIVATIVE

AND

$$\sum_{\langle ij \rangle} \longrightarrow \sum_{\vec{r}} \sum$$

directions on lattice (nn)

STILL DISCRETE SUM OVER LATTICE SITES

ALREADY TAKEN INTO ACCOUNT IN (1)

$$\xrightarrow[\text{CONT LIMIT}]{} \frac{1}{a^d} \int d^d r \quad \dots$$

$d=2$ DEN $= a^2$

$$\boxed{H(\{\phi(\vec{r})\}) = E_0 + \frac{Ja^2}{2a^d}\int d^d r \, \|\vec{\nabla}\phi(\vec{r})\|^2}$$

FACTOR 1/2 FROM H
SINCE NOT COUNTING TWICE
EACH COUPLING AND
ANOTHER 1/2 TAYLOR EXP.

$\phi(\vec{r})$ IS A LOCAL ANGLE

$$0 \leq \phi(\vec{r}) \leq 2\pi$$

BUT LIFT THE CONSTRAINT

$$-\infty \leq \phi(\vec{r}) \leq \infty$$

WHICH ARE THE GROUND STATES?

$$\frac{\delta H(\{\phi(\vec{r})\})}{\delta\phi(\vec{r})} = \frac{J}{2}a^{2-d}\int d^d r' \, \frac{\delta}{\delta\phi(\vec{r})}\left(\vec{\nabla}\phi(\vec{r}')\cdot\vec{\nabla}\phi(\vec{r}')\right)$$

$$= \cancel{2}\,\frac{Ja^{2-d}}{\cancel{2}}\int d^d r' \left(\vec{\nabla}_{\vec{r}'}\,\delta(\vec{r}'-\vec{r})\right)\cdot\vec{\nabla}\phi(\vec{r}')$$

$$= -\,Ja^{2-d}\int d^d r'\,\delta(\vec{r}'-\vec{r})\,\nabla^2\phi(\vec{r}') + \underbrace{\delta(\vec{r}'-\vec{r})\,\vec{\nabla}\phi(\vec{r})}_{=0}\Big|_{\text{BORDER}}$$

$$= -\,Ja^{2-d}\,\nabla^2\phi(\vec{r})$$

NOTE CHANGE IN sgn DUE TO INT. BY PARTS

AS FOR THE SCALAR FIELD IN THE GINZBURG-LANDAU TH.

HERE $\nabla^2\phi(\vec{r})$ HAS TO BE SET TO ZERO

GROUND STATES – MINIMA OF ENERGY GIVEN BY THE

LAPLACE EQ

$$\boxed{\nabla^2 \phi(\vec{r}) = 0}$$

REWRITING THE ENERGY AS UNCOUPLED MODES

FOURIER TRANSF $\Rightarrow$ UNCOUPLED ANGLES - OPEN UP $\vec{\nabla}\phi(\vec{r})$

$$\tilde{\phi}(\vec{k}) = \int d^d r \; e^{i\vec{k}\cdot\vec{r}} \, \phi(\vec{r}) \in \mathbb{C}$$

$$\phi(\vec{r}) = \int \frac{d^d k}{(2\pi)^d} \, \tilde{\phi}(\vec{k}) \, e^{-i\vec{k}\cdot\vec{r}} \in \mathbb{R}$$

BOTH INTEGRALS IF MODEL DEFINED IN

V : VOLUME OF SPACE $= L^d \to \infty$

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \qquad (\text{SINCE } \phi(\vec{r}) \in \mathbb{R})$$

$$H(\{\phi(\vec{r})\}) \longrightarrow H(\{\tilde{\phi}(\vec{k})\})$$

$$\boxed{H(\{\tilde{\phi}(\vec{k})\}) = E_0 + \frac{J}{2a^{d-2}} \int \frac{d^d k}{(2\pi)^d} k^2 |\tilde{\phi}(\vec{k})|^2}$$

GIVE IT AS AN EXCERC.
AND TELL THEM IT'S HERE

CHECK

$$\int d^d r \left(\vec{\nabla}\phi(\vec{r})\right) \cdot \left(\vec{\nabla}\phi(\vec{r})\right) =$$

$$= \int d^d r \left(\vec{\nabla} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(\vec{k}) e^{-i\vec{k}\vec{r}}\right) \cdot$$

$$\left(\vec{\nabla} \int \frac{d^d k'}{(2\pi)^d} \tilde{\phi}(\vec{k}') e^{-i\vec{k}'\vec{r}}\right)$$

$$= \int d^d r \left(\int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(\vec{k}) (-i\vec{k}) e^{-i\vec{k}\cdot\vec{r}} \right) \cdot \left(\int \frac{d^d k'}{(2\pi)^d} \tilde{\phi}(\vec{k}') (-i\vec{k}') e^{-i\vec{k}'\cdot\vec{r}} \right)$$

$$= \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(\vec{k}) \int \frac{d^d k'}{(2\pi)^d} \tilde{\phi}(\vec{k}') (-\vec{k}\cdot\vec{k}') \underbrace{\int d^d r \, e^{-i(\vec{k}+\vec{k}')\cdot\vec{r}}}_{(2\pi)^d \delta(\vec{k}+\vec{k}')}$$

$$= \int \frac{d^d k}{(2\pi)^d} k^2 \tilde{\phi}(\vec{k}) \phi(-\vec{k})$$

$$= \int \frac{d^d k}{(2\pi)^d} k^2 |\tilde{\phi}(\vec{k})|^2$$

ok

CORRELATION FUNCTIONS WITHIN SW TH.

(TD & SCALAR FIELD THEORY)

$$G(\vec{r}) \equiv \langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle$$

$$= \mathrm{Re} \langle \exp i\left(\phi(\vec{r}) - \phi(\vec{0})\right) \rangle$$

$$\boxed{G(\vec{r}) = e^{-\frac{1}{2} \langle \left(\phi(\vec{r}) - \phi(\vec{0})\right)^2 \rangle}}$$

SPIN-SPIN CORR

SINCE $\phi(\vec{r})$ HAS A QUADRATIC HAMILT

$$\boxed{G(\vec{r}) = e^{-\frac{1}{2} g(\vec{r})}}$$

AVERAGE OF GAUSSIAN VARIABLES

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2} + iz}$$

$$= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{z - i\sigma}{\sigma}\right)^2 - \frac{\sigma^2}{2}} = e^{-\frac{\sigma^2}{2}}$$

EVEN CLEARER IN $\tilde{\phi}(\vec{k})$ REPRES.

HAVE TO COMPUTE THIS CORREL

$$g(\vec{r}) = \left\langle \left(\phi(\vec{r}) - \phi(\vec{0}) \right)^2 \right\rangle$$

AND THEN EXPONENTIATE IT.

$$P(\{\tilde{\phi}(\vec{k})\}) = \frac{1}{Z} e^{-\frac{K}{2}\int \frac{d^d k}{(2\pi)^d} k^2 |\tilde{\phi}(\vec{k})|^2}$$

ALL PREFACTORS INCLUDING β

IT'S NOT THE WAVE VECTOR ! JUST THE PARAMETERS

RECALL $\tilde{\phi}(\vec{k}) \in \mathbb{C}$

$$K = \frac{\beta J}{a^{d-2}}$$

$$\boxed{K = \beta J \quad \text{in } d=2}$$

RECALL $\tilde{\phi}^*(\vec{k}) = \tilde{\phi}(-\vec{k})$

WEIGHT $|\tilde{\phi}(\vec{k})|^2 = \tilde{\phi}(\vec{k})\,\tilde{\phi}(-\vec{k})$

QUADRATIC NO LINEAR TERM

$$\Rightarrow \boxed{\langle \tilde{\phi}(\vec{k}) \rangle = 0 \Rightarrow \langle \phi(\vec{r}) \rangle = 0 \text{ TOO.}}$$

$$\langle \tilde{\phi}(\vec{k})\,\tilde{\phi}(\vec{k}') \rangle = (2\pi)^d \frac{1}{K k^2} \delta(\vec{k}+\vec{k}')$$

BECAUSE OF GAUSSIAN WEIGHT.

$$\langle \phi(\vec{x})\,\phi(\vec{y}) \rangle =$$

$$= \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\vec{x}} \int \frac{d^dk'}{(2\pi)^d} e^{i\vec{k}'\vec{y}} \langle \tilde{\phi}(\vec{k})\,\tilde{\phi}(\vec{k}') \rangle$$

$$= \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\vec{x} - i\vec{k}\vec{y}} \frac{1}{K k^2}$$

$$= \frac{1}{K} \underbrace{\int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2}}$$

DEF. AS $-C_d(\vec{x}-\vec{y})$

$$\boxed{\begin{aligned} \langle \phi(\vec{x})\,\phi(\vec{y}) \rangle &= \frac{1}{K} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2} \\ &\equiv -\frac{1}{K} C_d(\vec{x}-\vec{y}) \end{aligned}}$$

HAVE TO COMPUTE THE INTEGRAL OVER k.

$$\nabla^2 \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{k^2} = \delta^d(\vec{x}-\vec{y})$$

JUST BY APPLYING ∇^2 ONE RECOVERS THE $\delta^d(\vec{x}-\vec{y})$ REPRESENT.

ALREADY DONE FOR SCALAR THEORY $\Longrightarrow$ JUMP TO RESULT

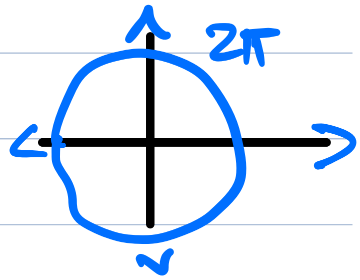
INT. OVER SPACE

BORDER

$$\int d^d r \, \nabla^2 C_d(\vec{r}) = \int d\vec{S} \cdot \vec{\nabla} C_d(\vec{r}) = 1$$

SPH SYMM $\quad \vec{\nabla} C_d(r) = \frac{dC_d(r)}{dr} \, \vec{e}_r$

$\Rightarrow \quad 1 = \Omega_d \, r^{d-1} \, \frac{dC_d(r)}{dr}$



$$\Omega_d = \frac{2\pi^{d/2}}{\left(\frac{d}{2}-1\right)!}$$

d-dim UNIT RADIUS SURFACE ; e.g. $\Omega_2 = 2\pi$

$$\frac{dC_d(r)}{dr} = \frac{1}{\Omega_d} r^{1-d} \Rightarrow$$

$$\boxed{C_d(r) \to \frac{1}{\Omega_d} \begin{cases} c_0 & d>2 \\ \ln r/a & d=2 \\ r^{2-d} & d<2 \end{cases}}$$

BACK TO RELATION BETWEEN

$C_d(r)$ AND $g(\vec{r})$

$$g(\vec{r}) = \langle \left(\phi(\vec{r}) - \phi(\vec{0}) \right)^2 \rangle$$

$$= \langle \phi^2(\vec{r}) \rangle + \langle \phi^2(\vec{0}) \rangle$$

$$- 2 \langle \phi(\vec{r}) \, \phi(\vec{0}) \rangle$$

$$= 2 \text{ CONST} - 2 \langle \phi(\vec{r}) \, \phi(\vec{0}) \rangle$$

$$= 2 \text{ CONST} + \frac{2}{K} C_d(\vec{r})$$

$d = 2$

$$g(\vec{r}) = 2 \text{ CONST} + \frac{2}{K} \frac{1}{\Omega_d} \ln(r/a)$$

$$G(\vec{r}) = e^{-\frac{1}{2} g(\vec{r})}$$

$$\propto e^{-\cancel{\frac{1}{2}} \frac{\cancel{2}}{K} \frac{1}{\Omega_2} \ln(r/a)}$$

$$= \left(\frac{r}{a}\right)^{-\eta(K)}$$

$$\eta(K) = \frac{1}{K\Omega_2} = \frac{1}{\beta J} \frac{1}{2\pi}$$

$$\boxed{\eta = \frac{k_B T}{2\pi J}}$$

$$\boxed{G(r) \sim \left(\frac{r}{L}\right)^{-\eta(k)}}$$

SPACE LINEAR SCALE

WITH A PARAM DEP EXPONENT $\eta(k)$

$$\boxed{\eta(k) = \frac{k_B T}{2\pi J}}$$

LIKE CRITICAL SCALING AT ALL TEMP.
(HAVING ARRANGED THE NUMERICAL PREFACTORS)

NO PHASE TRANSITION ?

IT CAN'T BE, THERE SHOULD BE ONE

2.10.2 HIGH T EXPANSION IN TD

$$\Rightarrow \quad G(r) \sim e^{-r/\xi}$$

AT HIGH T

$\Rightarrow$ THERE MUST BE A PHASE TRANS.

MORE DETAILS IN TD

PHASE TRANSITION
DRIVEN BY VORTICES

2.10.3 VORTICES

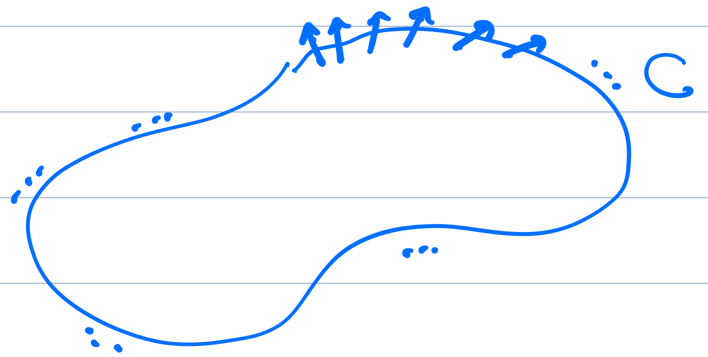
$$0 = \sum_{\substack{nn \\ \vec{r}, \vec{y} \in C}} \left[\phi(\vec{r}) - \phi(\vec{y}) \right] \longrightarrow \oint_C d\vec{\ell} \cdot \vec{\nabla} \phi(\vec{r})$$

ON THE CONTOUR

IF NO VORTICES

BUT

SUM OVER ANGLE DIFF ON A CLOSED CONTOUR IN REAL SPACE



ONLY VERY TINY DIFF. ALLOWED & ANGLE COMES BACK TO SAME VALUE

BUT THE ANGLE COULD COME BACK TO ANY $\boxed{\pm 2\pi q}$ WITH $q \in \mathbb{Z}$

AND KEEP THE SPIN $\vec{S}$ SINGLE VALUED

MAKING A TURN (OR MORE) OF THE SPIN

JUST CONSEQUENCE OF

$$\cos(\phi \pm 2\pi q) = \cos\phi \cos(\pm 2\pi q) - \sin\phi \sin(\pm 2\pi q)$$
$$= \cos\phi$$

TOPOLOGICAL DEFECTS CONFS. WHICH ARE LOCAL MIN. (STABLE) of H BUT CANNOT BE CONTINUALLY & SMOOTHLY TRANSFORMED TO THE GROUND STATES

$\nexists$ NO $R(\vec{r})$ SUCH THAT $\vec{S}'_{G.S.}(\vec{r}) = R(\vec{r})\,\vec{S}(\vec{r})$

ROTATION / TOP. DEFECT

$$\frac{\delta H}{\delta \phi(\vec{r})} = 0 \qquad \left.\frac{\delta^2 H}{\delta \phi(\vec{r})\,\delta \phi(\vec{r}\,')}\right|_{\phi_{\text{TOP DEF.}}} \quad \text{POSITIVE DEF (STABLE)}$$

EXPLICIT VORTEX CONF

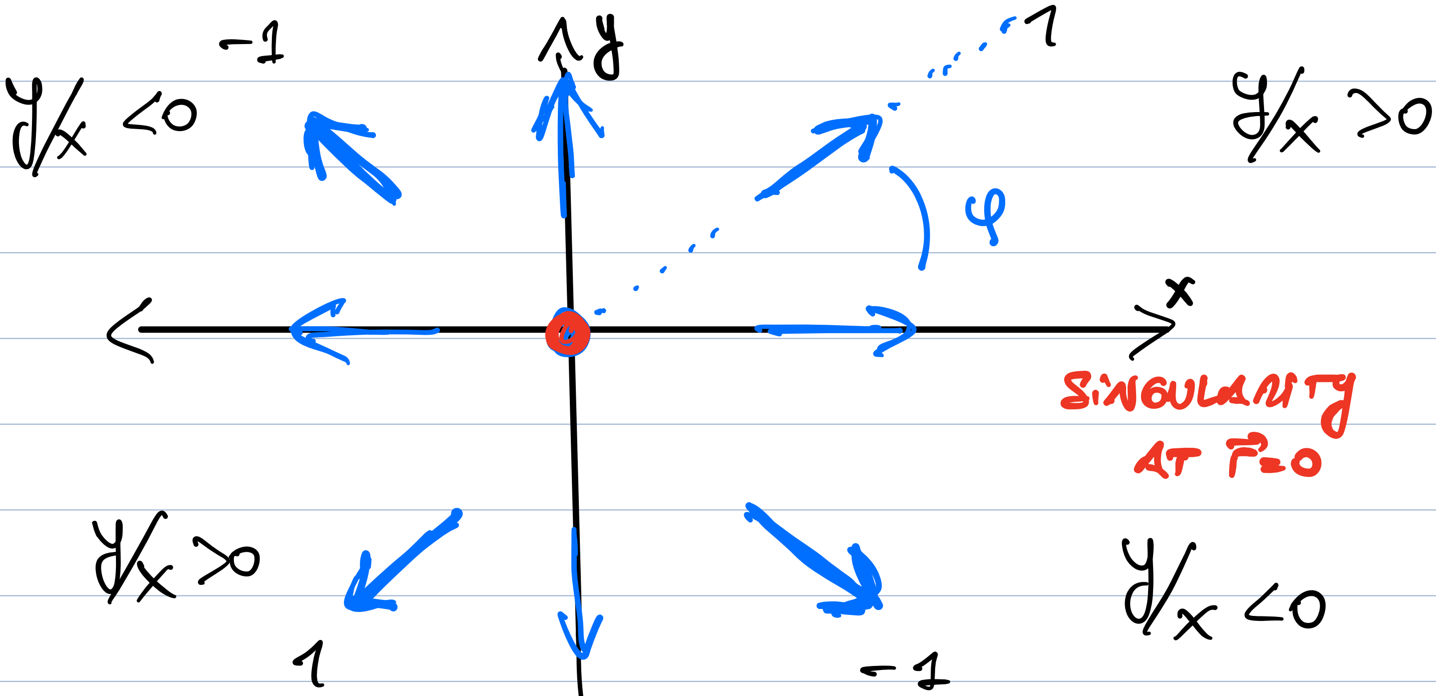
$$\phi(\vec{r}) = q\,\varphi(\vec{r}) + \phi_0 \qquad \phi_0 \to \text{CONST.}$$

AT $\vec{r} = (x, y)$

$$q \in \mathbb{Z} \qquad \varphi(\vec{r}) = \operatorname{arctg}\left(\frac{y}{x}\right)$$

eg. $q = 1$ $\quad \phi_0 = 0$ $\quad \phi(\vec{r}) = \operatorname{arctg} y/x$

NB in $\mathbb{R}^2$ SPACE PARAM. BY (x, y)



$\vec{S} = (\cos\phi, \sin\phi)$

SPIN ROTATION

$\arctan 1 = \pi/4 \qquad \arctan \infty = \pi/2$

$\arctan(-1) = 3\pi/4 \qquad \arctan 0 = \pi$

q IS THE "CHARGE" OF THE VORTEX

VORTICES WITH SAME q CAN BE TRANSFORMED

" " DIFFERENT q CANNOT " "

INTO EACH OTHER

e.g. FOR $q = 1$ ANOTHER WAY OF WRITING

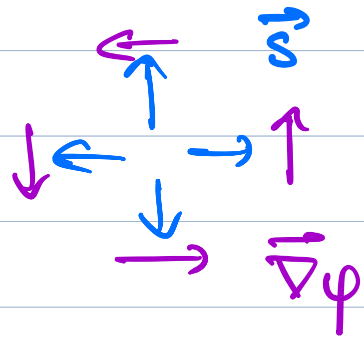
$$\vec{S}(t) = \Big(\cos(\varphi + t), \sin(\varphi + t)\Big)$$

t PLAYING ROLE OF ϕ_0

SINGULARITY AT THE ORIGIN in EQUATIONS

$$\vec\nabla\phi(\vec r) = q\vec\nabla\varphi(\vec r) = q\vec\nabla \operatorname{arctg}\left(\frac{y}{x}\right)$$

$$= \dots = \frac{q}{r}\hat e_\varphi$$



$$\hat e_\varphi = \cos\varphi\,\hat e_y - \sin\varphi\,\hat e_y$$

EQ OF STATE

$$\nabla^2\phi(\vec r) = \vec\nabla\cdot\vec\nabla\phi(\vec r) = 0$$ ok

LOCAL MINIMUM

CHECK IT. $\dfrac{\delta^2 H}{\delta\phi(\vec x)\,\delta\phi(\vec y)}$ POS. DEF.

CIRCULATION OF ANGULAR FIELD

$$\oint_C d\phi(\vec r) = \oint_C d\vec\ell\cdot\vec\nabla\phi(\vec r)$$

TAKE A CIRCLE W/RADIUS R

$$= \int_0^{2\pi} R\,d\varphi\,\hat e_\varphi\cdot\frac{q}{R}\hat e_\varphi$$

$$= q \int_0^{2\pi} d\varphi = \boxed{2\pi q = \oint_C d\phi(\vec{r})}$$

RESULT INDEP OF C

IT'S $\neq 0$

PROVE IT (GAUSS THM.)

MANY VORTEX CONF

$$\phi(\vec{r}) = \sum_{i=1}^{M} q_i \, \operatorname{arctg} \frac{(\vec{r}-\vec{r}_i)\cdot \vec{e}_x}{(\vec{r}-\vec{r}_i)\cdot \vec{e}_y}$$

ENERGY OF A VORTEX

$$E_{1\,\mathrm{VORTEX}} = \frac{J}{2} \int d^2 r \, \| \vec{\nabla} \phi(\vec{r}) \|^2$$

$$= \frac{J}{2} q^2 \int_0^{2\pi} d\varphi \int_a^L dr \, r \, \frac{1}{r^2}$$

$$\boxed{E_{1\,\mathrm{VORT}} = \pi J q^2 \ln L/a}$$

IT ACTUALLY DIVERGES AS $\ln L$

WELL ABOVE GROUND STATE! BUT:

ENTROPY OF 1 VORTEX CONF

$$S = k_B \ln \left(\frac{L}{a}\right)^2 = 2k_B \ln \frac{L}{a}$$

FREE ENERGY - PEIERLS ARG

$$\Delta F = F_{1\,\text{VORTEX}} - F_0$$

F_0 : FINITE

$$= \underbrace{\left(\pi J q^2 - 2k_B T\right)} \ln \frac{L}{a}$$

COULD BE A $0 < T_C < \infty$

$$\boxed{k_B T_{KT} = \frac{\pi J}{2}}$$

$q = 1$
LOWEST PRED
FOR T_{KT}

MECANISM

UNBINDING OF VORTEX PAIRS

BELOW T_c VORTICES $\exists$ BUT THEY ARE BOUNDED TOGETHER

AND DON'T HAVE AN EFFECT FAR AWAY

$q^{\circ} \quad {}^{\circ}\!-\!|q|$

AT $T_{KT} \rightarrow$ UNBIND.!

CORR. FUNCTIONS & CORR LENGTH

$$G(r) = \langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle$$

$$\sim e^{-r/\xi}$$

$$\xi \sim e^{\left(\frac{T_{KT}}{T - T_{KT}}\right)^{\nu}} \qquad \nu = 1/2$$

RG CALCULATION w/ VILLAIN'S MODEL

TD

1ST ORDER PHASE TRANSITIONS

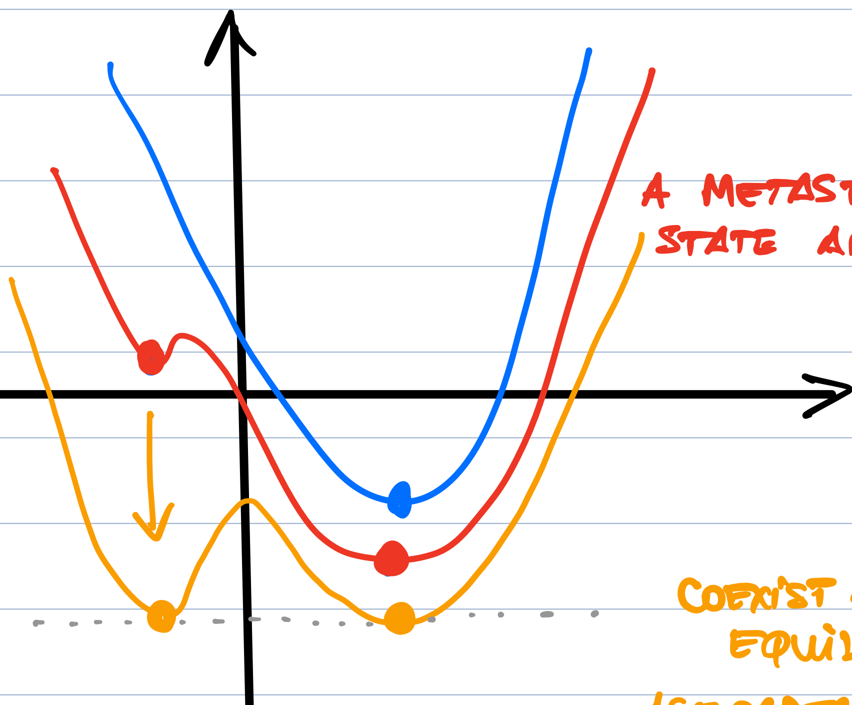
THE ORDER PARAMETER JUMPS AT THE TRANSITION

- eg. FIELD DRIVEN

$m(T, h)$ AT $T < T_c(h=0)$

$$\left.\begin{aligned} \lim_{h \to 0^+} m &= m^+ > 0 \\ \lim_{h \to 0^-} m &= m^- < 0 \end{aligned}\right\} \text{JUMP}$$

eg. CURVES FOR DIFF FIELDS $h>0$ CHANGES PROGRESSIVELY TO $h<0$ AT $T<T_c(h=0)$



A METASTABLE STATE APPEARS

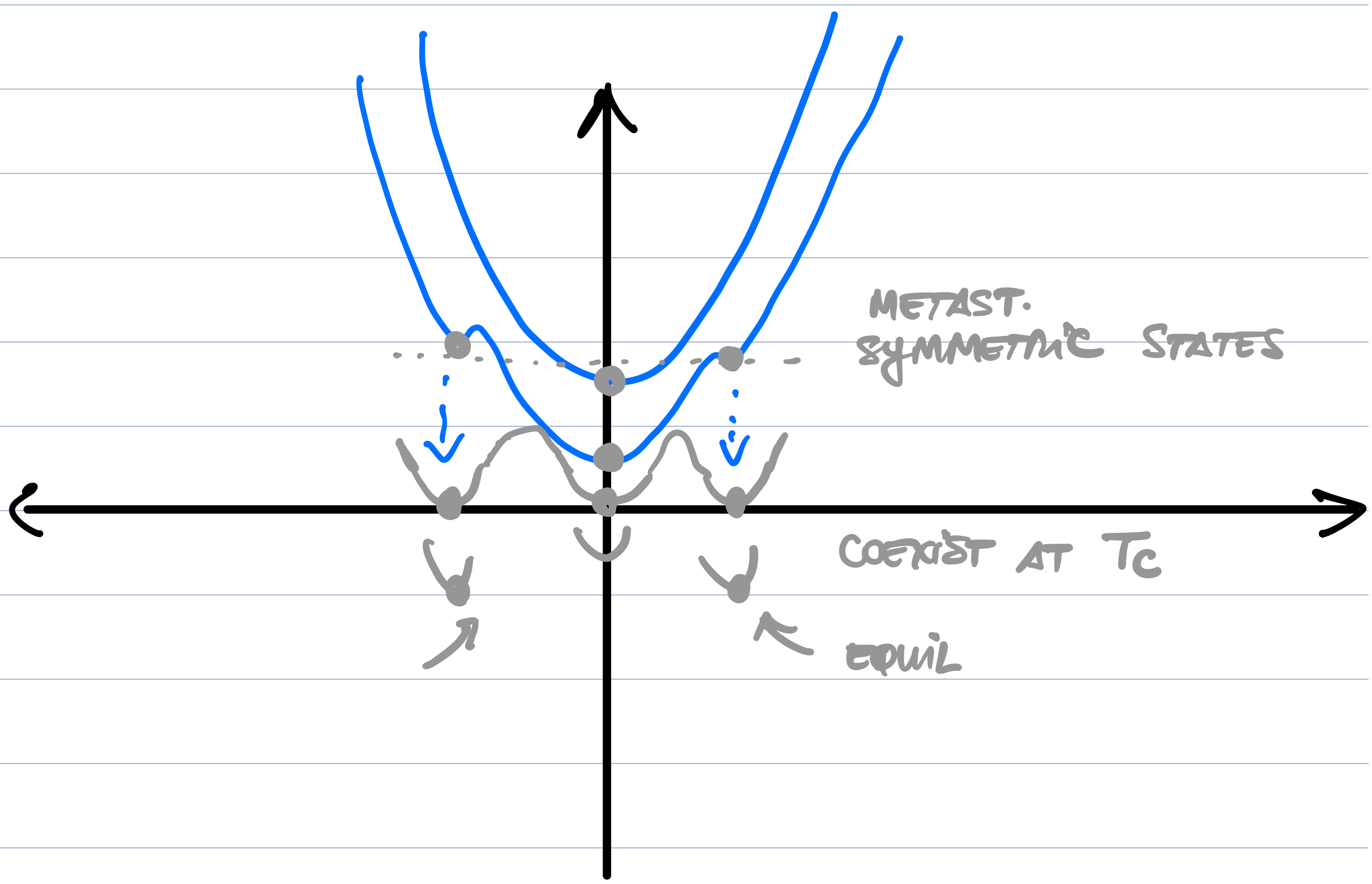
COEXIST AT EQUIL

1ST ORDER PH. TR.

- e.g. TEMP. DRIVEN

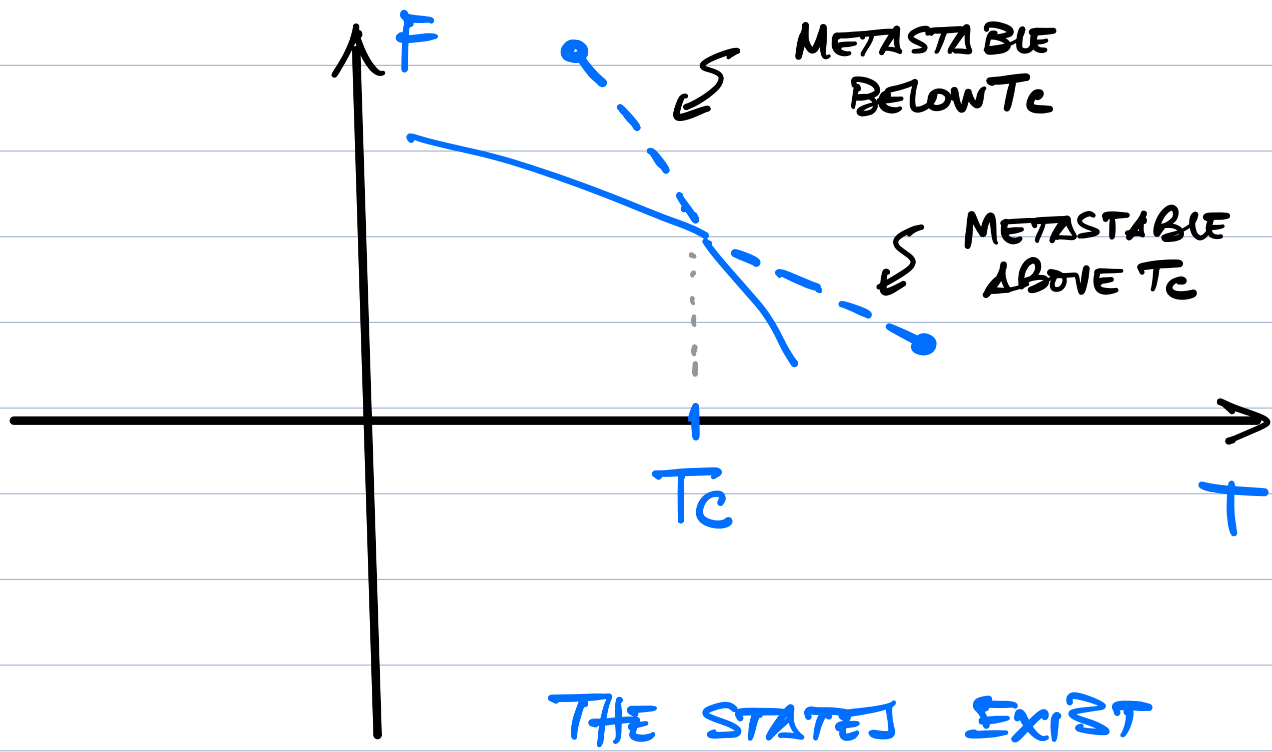
MF MODEL $H = -\frac{J}{N^2} \sum_{i \neq j \neq k \neq l} S_i S_j S_k S_l$

IN A LANDAU PICTURE



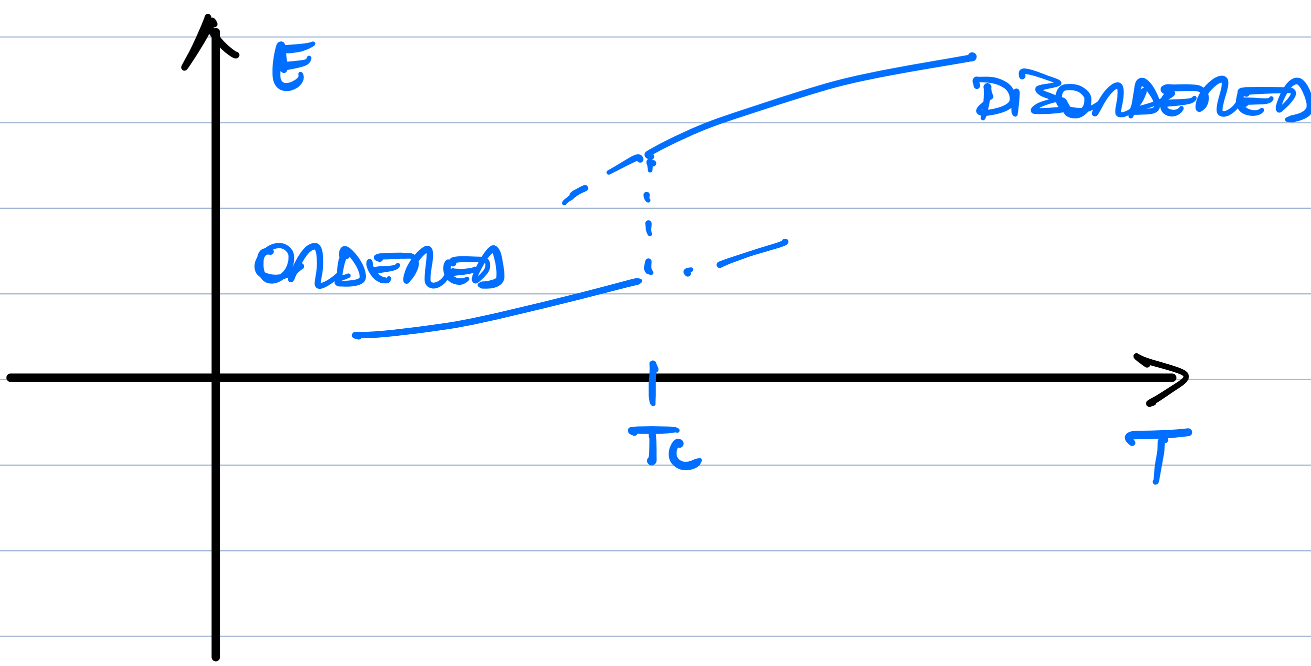
- THE FREE-ENERGY AS A fct OF THE CONTROL PARAMETER

TWO "BRANCHES" THAT CROSS AT T_c



THE STATES EXIST BEYOND T_c AS METASTABLE ONES UNTIL SOME T^*

- THE DISCONT. IN THE ORDER PARAM $\Rightarrow$ DISCONT. IN THE ENERGY



DISCONTINUITY

$\Rightarrow$ LATENT HEAT L

- DISTRIBUTION OF ORDER PARAMETER

$$P(\theta) = \frac{1}{\mathcal{N}} \exp\left[-\beta \sum_{\alpha=1}^{N_p} \frac{\overbrace{(\theta - \langle\theta\rangle_\alpha)^2}^{\text{VARIANCE OF } \theta}}{2\chi_\alpha L^{-d}} \right]$$

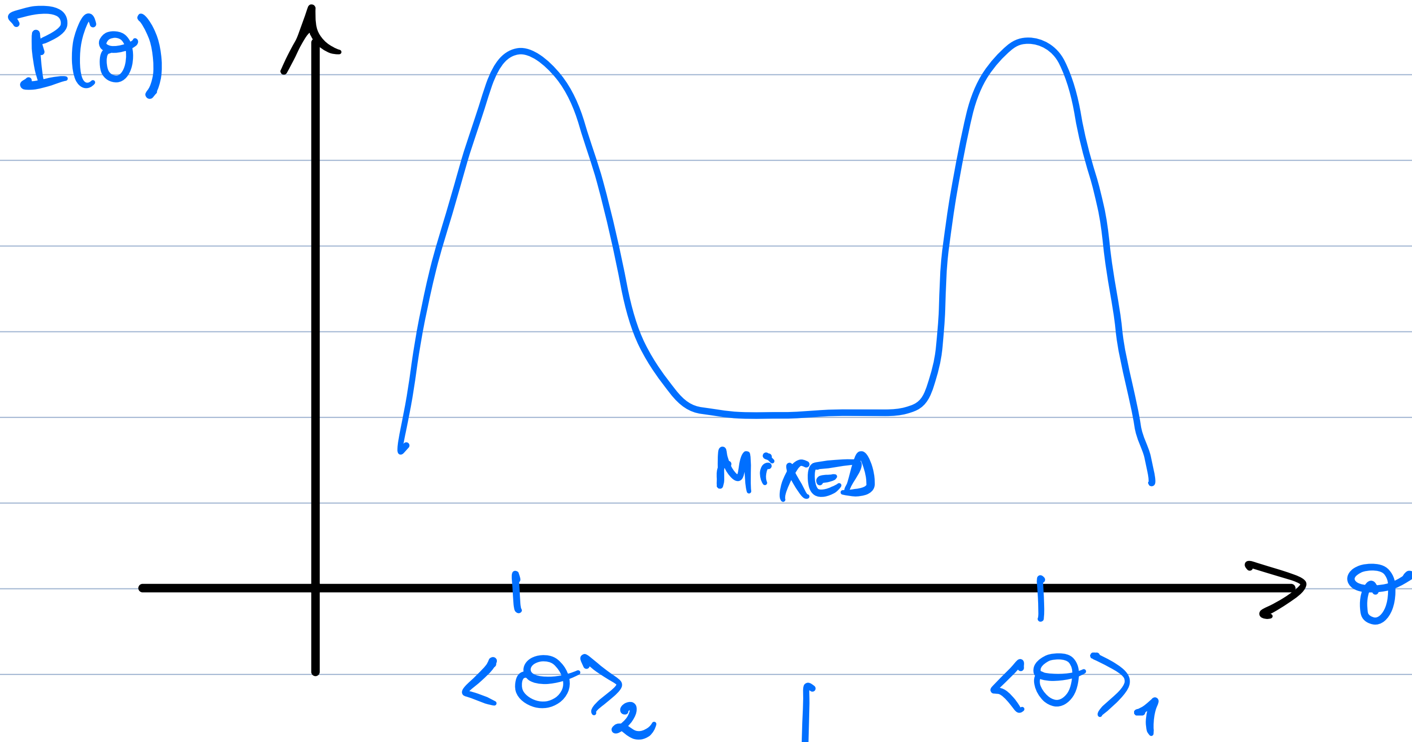
FINITE SUSCEPT.

θ ORDER PARAMETER

$\langle\theta\rangle_\alpha$ AVERAGE IN THE αth PHASE

χ_α SUSCEPTIBILITY OF αth PHASE (FROM FDT)

$$\chi_\alpha = \beta L^d \langle (\theta - \langle\theta\rangle_\alpha)^2 \rangle_\alpha \quad \text{FDT}$$



$$P(\theta) \sim e^{-2\sigma L^{d-1}}$$

INTERFACIAL ENERGY PER UNIT SURFACE

$\to 0$
$L \to \infty$

σ FINITE

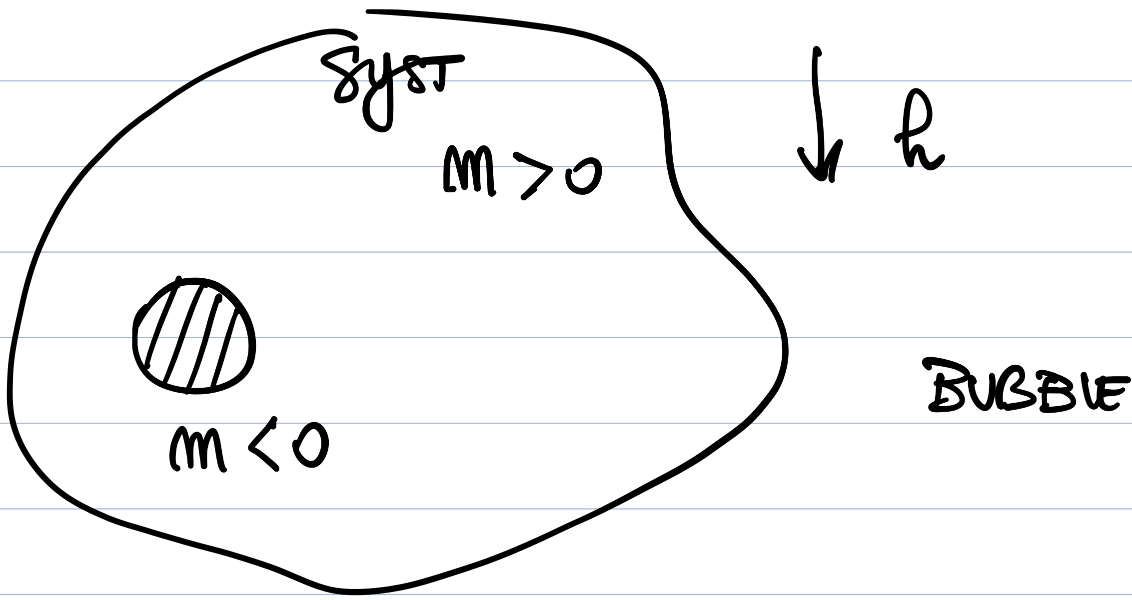
- IN A SYST WITH CO-EXISTENCE

$$\langle \theta \rangle = \langle \theta \rangle_1 \frac{V_1}{V} + \langle \theta \rangle_2 \frac{V_2}{V}$$

$V_{1,2}$ VOLUME OCCUPIED BY EACH PHASE. EX $\alpha = 1, 2$

- ξ IS FINITE AT THE TRANSITION

STABILITY OF A PHASE
NUCLEATION ARGUMENT



$T = 0$

$$\Delta F = F_{\text{WITH BUBBLE}} - F_{\text{WITHOUT}} \gtrless 0 \; ?$$

THERE'S A "BULK" ENERGY GAIN
AND A "SURFACE" COST

NO ENTROPIC CONTRIB TO ΔF $(T=0)$

$$\Delta E_{\text{BULK}} = -\,2h|m|\,L^d$$

$$\Delta E_{\text{SURF}} = \sigma\, L^{d-1}$$

$$\Delta F = \Delta E = -2h\,|m|\,L^d + \sigma L^{d-1}$$

THIS JUST TAKE INTO ACCOUNT THE LINEAR SCALE

SPHERICAL BUBBLE

$$\Delta F = -2h\,|m|\,\Omega_d\, R^d + \sigma\, \Omega_{d-1}\, R^{d-1}$$

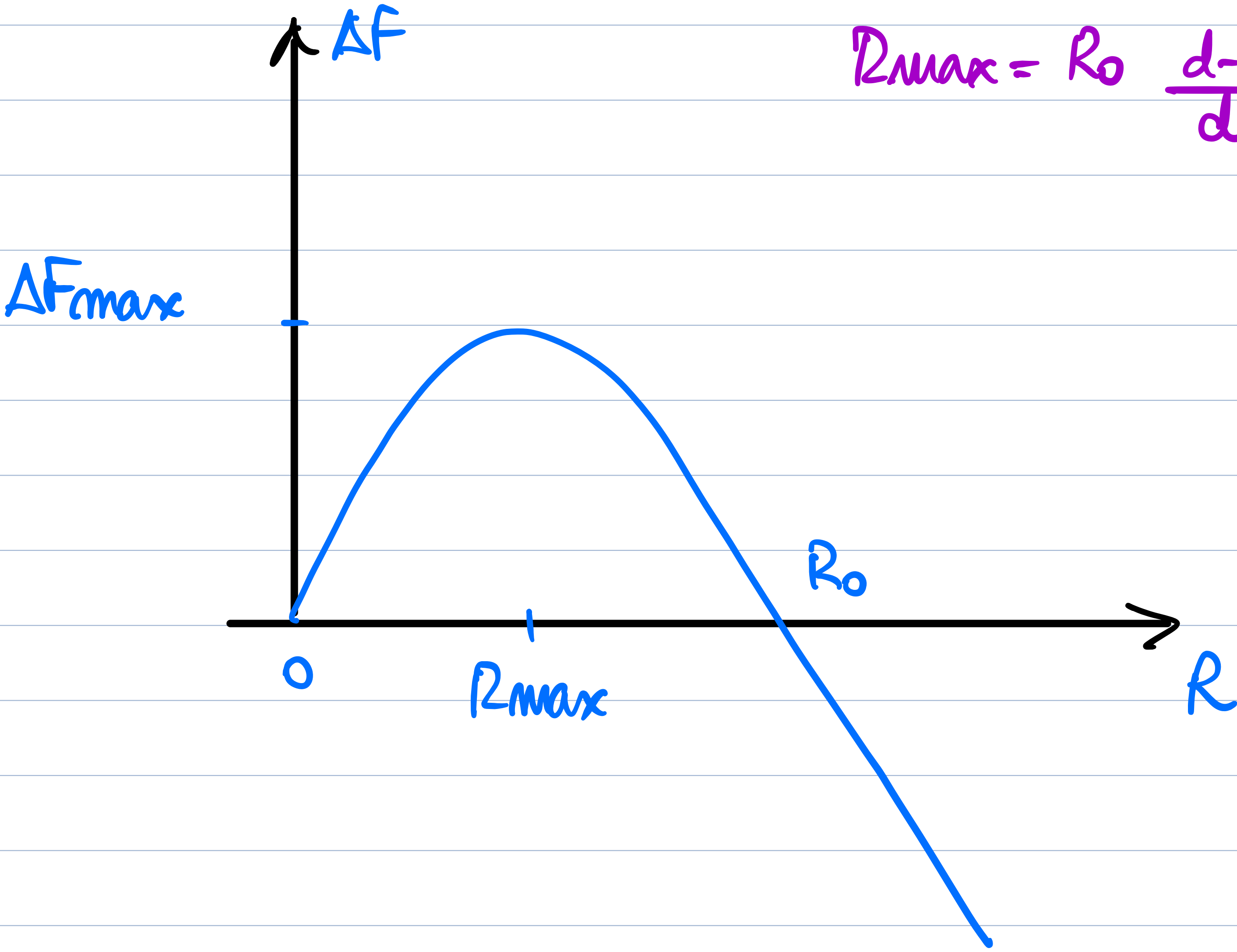
$$\frac{d\Delta F}{dR} = -2h\,|m|\,\Omega_d\, d\, R^{d-1} + \sigma\, \Omega_{d-1}\,(d-1)\, R^{d-2} = 0$$

$$\Rightarrow \quad R_{max} = \frac{\sigma\, \Omega_{d-1}\,(d-1)}{2h\,|m|\,\Omega_d\, d}$$

$$\boxed{R_{max} \propto \frac{\sigma}{h}} \qquad d \neq 1$$

$$\Delta F = 0 \Rightarrow \boxed{R_0 = \frac{\sigma\, \Omega_{d-1}}{2h\,|m|\,\Omega_d}}$$

$$R_{max} = R_0 \frac{d-1}{d}$$



$$\Delta F_{max} = -2h|m|\,\Omega_d \left(\frac{\sigma \Omega_{d-1} \,(d-1)}{2h|m|\,\Omega_d \, d} \right)^{d}$$

$$+ \, \sigma \Omega_{d-1} \left(\frac{\sigma \Omega_{d-1} \,(d-1)}{2h|m|\,\Omega_d \, d} \right)^{d-1}$$

$$\propto \quad h^{1-d} \quad \sigma^{d} = \frac{\sigma^{d}}{h^{d-1}}$$

WE NOTE $R_{max} = R_0 \frac{d-1}{d}$ SAME PARAM DEPENDENCE

$$(R_0, R_{max}) \xrightarrow[\substack{\sigma \to \infty \\ \text{or } h \to 0}]{} \infty \longleftarrow \Delta F_{max}$$

$$(R_0, R_{max}) \xrightarrow[\substack{\sigma \to 0 \\ h \to \infty}]{} 0 \longleftarrow \Delta F_{max}$$

NEEDS ACTIVATION TO JUMP OVER BARRIER

ARRENHIUS ARGUMENT YIELD TIME NEEDED TO JUMP

SIMILAR TO TUNNELING

MULTICRITICAL POINTS

TWO RELEVANT PARAM CAN BE TUNED TO A CRIT POINT

$(T_c, H=0)$ IN ISING IS NOT CONS. SO.

2nd ORDER MEETS 1st ORDER

e.g. BLUME CAPEL