TD 6 : Quantum-Classical Mapping

1) The correlation length and the energy gap

$$\mathcal{H} = -J \sum_i S_i S_{i+1} - h \sum_i S_i$$

$$Z = \sum_{\{S_i = \pm 1\}} e^{K \sum_i S_i S_{i+1} + H \sum_i S_i}$$

$$Z = \sum_{\{S_i = \pm 1\}} e^{\sum_i \left(K S_i S_{i+1} + H (S_i + S_{i+1})/2\right)}$$

$$T(S_1, S_2) \equiv e^{K S_1 S_2 + H (S_1 + S_2)/2}$$

$$Z = \sum_{\{S_i = \pm 1\}} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_N S_1} = \mathrm{Tr}\, T^N$$
$$= \lambda_+^N + \lambda_-^N$$

$$\det T = \begin{vmatrix} e^{K+H} - \lambda & e^{-K} \\ e^{-K} & e^{K-H} - \lambda \end{vmatrix} = e^{2K} + \lambda^2 - \lambda e^K \left(e^H + e^{-H}\right) + e^{-2K}$$

$$\lambda^2 - 2\lambda e^K ch(H) + 2 sh(2K) = 0$$

$$\lambda_\pm = e^K ch(H) \pm \sqrt{\underbrace{e^{2K} ch^2(H) - 2 sh(2K)}}$$

$$\| $$

$$\underbrace{e^{2K}(ch^2 H - 1)}_{\substack{\geq \\ 0}} + e^{-2K} \geq 0$$

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

$$F = -k_B T \ln Z = -k_B T N \ln \lambda_+ - k_B T \ln \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

$$\simeq -k_B T N \ln \lambda_+ - k_B T e^{-N/\xi}$$

$$\xi = -\frac{1}{\ln(\lambda_-/\lambda_+)}$$

$$\langle S_i \rangle = m = \frac{\partial \ln Z}{\partial H}$$

$$\langle S_1 \rangle = \frac{1}{Z} \sum_{\{S_i\}} S_1 T_{S_1 S_2} \cdots T_{S_N S_1}$$

$$S \delta_{S S_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\sigma^z)_{S S_1}$$

$$S_1 = \sum_{S=\pm 1} \delta_{S S_1} S$$

$$\langle S_1 \rangle = \frac{1}{Z} \sum_{\substack{\{S_i\} \\ S=\pm 1}} (\sigma_z)_{SS_1} T_{S_1 S_2} \cdots T_{S_N S}$$

$$\langle S_1 \rangle = \frac{1}{Z} \sum_{\substack{S=\pm 1 \\ S_1=\pm 1}} (\sigma_z)_{SS_1} (T^N)_{S_1 S} = \frac{1}{Z} \mathrm{Tr}\left(\sigma_z T^N\right)$$

$$T|u\rangle = \lambda_+ |u\rangle \qquad T|v\rangle = \lambda_- |v\rangle$$

$$T = \lambda_+ |u\rangle\langle u| + \lambda_- |v\rangle\langle v|$$

$$T^N = \lambda_+^N |u\rangle\langle u| + \lambda_-^N |v\rangle\langle v|$$

$$\mathrm{Tr}\left(T^N \sigma^z\right) = \mathrm{Tr}\left[\left(\lambda_+^N |u\rangle\langle u| + \lambda_-^N |v\rangle\langle v|\right)\sigma^z\right]$$

$$= \langle u|\left[\left(\lambda_+^N |u\rangle\langle u| + \lambda_-^N |v\rangle\langle v|\right)\sigma^z\right]|u\rangle$$

$$+ \langle v|\left[\left(\lambda_+^N |u\rangle\langle u| + \lambda_-^N |v\rangle\langle v|\right)\sigma^z\right]|v\rangle$$

$$= \lambda_+^N \langle u|\sigma^z|u\rangle + \lambda_-^N \langle v|\sigma^z|v\rangle$$

$$\langle S_1 \rangle = m = \frac{\lambda_+^N \langle u|\sigma^z|u\rangle + \lambda_-^N \langle v|\sigma^z|v\rangle}{\lambda_+^N + \lambda_-^N}$$

$$\simeq \langle u|\sigma^z|u\rangle + O\left(\left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

Eigenvectors $|u\rangle, |v\rangle$

$\langle u|u\rangle = 1$ $\qquad u = (\cos\theta, \sin\theta)$

$\langle v|v\rangle = 1$ $\qquad v = (\sin\theta, -\cos\theta)$

$\langle u|v\rangle = 0$

$$T = \alpha \mathbb{1} + \underbrace{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}}_{M \to \text{same eigenvectors}}$$

$$(a-\lambda)(-a-\lambda) - b^2 = 0$$

$$(\lambda+a)(\lambda-a) - b^2 = 0$$

$$\lambda^2 - a^2 - b^2 = 0$$

$$\lambda_\pm = \pm\sqrt{a^2+b^2} \qquad M|u\rangle = \lambda_+|u\rangle$$

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos\theta$$

$$b\cos\theta - a\sin\theta = \sqrt{a^2+b^2}\sin\theta$$

$$\cancel{a^2\cos^2\theta} + b^2\sin^2\theta + 2ab\cos\theta\sin\theta = (\cancel{a^2} + b^2)\cos^2\theta$$

$$2ab\sin\theta\cos\theta + b^2(\sin^2\theta - \cos^2\theta) = 0$$

$$a\cancel{b}\sin(2\theta) - b^{\cancel{2}}\cos(2\theta) = 0$$

$$\frac{\sin(2\theta)}{\cos(2\theta)} = \frac{b}{a} \implies \boxed{\tan(2\theta) = \frac{b}{a}}$$

- $\langle u|\sigma^z|u\rangle = (\cos\theta, \sin\theta)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

$$= (\cos\theta, \sin\theta)\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix} = \cos^2\theta - \sin^2\theta$$

$$= \cos(2\theta)$$

- $\langle v|\sigma^z|v\rangle = (\sin\theta, -\cos\theta)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}$

$$= (\sin\theta, -\cos\theta)\begin{pmatrix}\sin\theta\\ +\cos\theta\end{pmatrix} = \sin^2\theta - \cos^2\theta$$

$$= -\cos(2\theta)$$

- $\langle u|\sigma^z|v\rangle = (\cos\theta, \sin\theta)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}$

$$= (\cos\theta, \sin\theta)\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix} = 2\sin\theta\cos\theta$$

$$= \sin(2\theta)$$

$$a \sin(2\theta) = b\cos(2\theta)$$

$$a^2 \sin^2(2\theta) = b^2 \cos^2(2\theta)$$

$$a^2 \sin^2(2\theta) = b^2\left(1-\sin^2(2\theta)\right)$$

$$\left(a^2+b^2\right)\sin^2(2\theta) = b^2$$

$$\boxed{\sin(2\theta) = \pm \frac{b}{\sqrt{a^2+b^2}}}$$

$$a^2\left(1-\cos^2(2\theta)\right) = b^2\cos^2(2\theta)$$

$$\left(a^2+b^2\right)\cos^2(2\theta) = a^2$$

$$\boxed{\cos(2\theta) = \pm \frac{a}{\sqrt{a^2+b^2}}}$$

$$T = \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix} = \begin{pmatrix} d+a & b \\ b & d-a \end{pmatrix}$$

$$\boxed{b = e^{-K}}$$

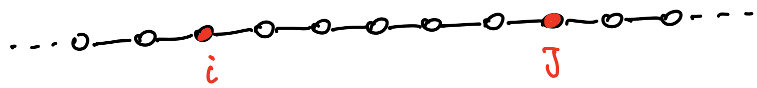
$$\left.\begin{aligned} d+a &= e^{K+H} \\ d-a &= e^{K-H} \end{aligned}\right\} \Rightarrow 2d = e^{K}\left(e^{H}+e^{-H}\right)$$

$$d = e^{K}\cosh(H)$$

$$2a = e^K\left(e^H - e^{-H}\right) \Rightarrow \boxed{a = e^K \operatorname{sh}(H)}$$

$$m = \langle u|\sigma^z|u\rangle = \cos(2\theta) = \frac{e^K \operatorname{sh}(H)}{\sqrt{e^{2K}\underbrace{\operatorname{sh}^2(H)}_{=\operatorname{ch}^2(H)-1} + e^{-2K}}}$$

1. $C(|r_i - r_J|)$



$$\langle S_i S_J\rangle = \frac{1}{Z}\sum_{\{S_m = \pm 1\}} T_{S_1S_2} T_{S_2S_3} \times \dots \times T_{S_{i-1}S_i} S_i T_{S_iS_{i+1}} \times \dots$$
$$\times T_{S_{J-1}S_J} S_J T_{S_JS_{J+1}} \times \dots$$
$$\times T_{S_NS_1}$$

$$\sum_S \delta_{SS_i} = 1$$

$$\sum_{S'} \delta_{S'S_J} = 1$$

$$S_i \delta_{SS_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left(\sigma^z\right)_{SS_i}$$

$$\langle S_i S_J\rangle = \frac{1}{Z}\sum_{\substack{\{S_m\} \\ S=\pm 1 \\ S'=\pm 1}} T_{S_1S_2} \times \dots \times T_{S_{i-1}S_i} \left(\sigma^z\right)_{S_iS} T_{SS_{i+1}} \times \dots$$
$$\times T_{S_{J-1}S_J} \left(\sigma^z\right)_{S_JS'} T_{S'S_{J+1}} \times \dots$$
$$\times T_{S_NS_1}$$

$$\langle S_i S_J \rangle = \frac{1}{Z} \sum_{\substack{\{S_m\} \\ S=\pm 1 \\ S'=\pm 1}} \left(T^{i}\right)_{S_1 S_i} \left(\sigma^z\right)_{S_i S} \left(T^{|J-i|}\right)_{S S_J} \left(\sigma^z\right)_{S_J S'} \left(T^{N-J}\right)_{S' S_1}$$

$$\mathrm{Tr}\left(T^{i} \sigma^z T^{|J-i|} \sigma^z T^{N-J}\right)$$

$$= \mathrm{Tr}\Big[\left(\lambda_+^{i} |u\rangle\langle u| + \lambda_-^{i} |v\rangle\langle v|\right) \sigma^z \left(\lambda_+^{|J-i|} |u\rangle\langle u| + \lambda_-^{|J-i|} |v\rangle\langle v|\right) \times$$

$$\times \sigma^z \left(\lambda_+^{N-J} |u\rangle\langle u| + \lambda_-^{N-J} |v\rangle\langle v|\right)$$

$$\langle u| T^{i} \sigma^z T^{|J-i|} \sigma^z T^{N-J} |u\rangle = \lambda_+^{i} \langle u| \sigma^z \left(\lambda_+^{|J-i|} |u\rangle\langle u| + \lambda_-^{|J-i|} |v\rangle\langle v|\right) \times$$

$$\times \sigma^z \lambda_+^{N-J} |u\rangle$$

$$= \lambda_+^{N+i-J} \left(\langle u|\sigma^z|u\rangle \langle u|\sigma^z|u\rangle \lambda_+^{|J-i|} + \langle u|\sigma^z|v\rangle\langle v|\sigma^z|u\rangle \lambda_-^{|J-i|} \right)$$

$$= \lambda_+^{N} \left|\langle u|\sigma^z|u\rangle\right|^2 + \lambda_+^{N+i-J} \lambda_-^{|J-i|} \left|\langle u|\sigma^z|v\rangle\right|^2$$

$$\langle v| T^{i} \sigma^z T^{|J-i|} \sigma^z T^{N-J} |v\rangle = \lambda_-^{i} \langle v| \sigma^z \left(\lambda_+^{|J-i|} |u\rangle\langle u| + \lambda_-^{|J-i|} |v\rangle\langle v|\right) \times$$

$$\times \sigma^z \lambda_-^{N-J} |v\rangle$$

$$= \lambda_-^{N-|J-i|} \left(\langle v|\sigma^z|u\rangle\langle u|\sigma^z|v\rangle \lambda_+^{|J-i|} + \langle v|\sigma^z|v\rangle\langle v|\sigma^z|v\rangle \lambda_-^{|J-i|} \right)$$

$$= \lambda_-^{N} \left|\langle v|\sigma^z|v\rangle\right|^2 + \lambda_-^{N-|J-i|} \lambda_+^{|J-i|} \left|\langle v|\sigma^z|u\rangle\right|^2$$

$$\langle S_i S_J \rangle = \frac{\lambda_+^{N} \left|\langle u|\sigma^z|u\rangle\right|^2 + \left(\lambda_+^{N-|J-i|}\lambda_-^{|J-i|} + \lambda_-^{N-|J-i|}\lambda_+^{|J-i|}\right)\left|\langle u|\sigma_z|v\rangle\right|^2 + \lambda_-^{N}\left(\langle v|\sigma^z|v\rangle\right)^2}{\lambda_+^{N} + \lambda_-^{N}}$$

$$\langle S_i S_j \rangle \simeq |\langle u|\sigma^z|u\rangle|^2 + \left[\left(\frac{\lambda_-}{\lambda_+}\right)^{|j-i|} + \left(\frac{\lambda_-}{\lambda_+}\right)^{N-|j-i|}\right] |\langle u|\sigma_z|v\rangle|^2$$

$$+ \left(\frac{\lambda_-}{\lambda_+}\right)^N |\langle v|\sigma^z|v\rangle|^2 \quad + O\left(\left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

$|j-i| \ll N$

$$\langle S_i S_j \rangle \simeq m^2 + \left(\frac{\lambda_-}{\lambda_+}\right)^{|j-i|} |\langle u|\sigma_z|v\rangle|^2$$

$$\langle S_i S_j \rangle_c = C(|r_i - r_j|) \simeq |\langle u|\sigma_z|v\rangle|^2 e^{|j-i| \ln(\lambda_-/\lambda_+)}$$

$$= |\langle u|\sigma_z|v\rangle|^2 e^{-r_{ij}/\xi}$$

$$\boxed{\xi = -\frac{1}{\ln \lambda_-/\lambda_+}}$$

$$\langle u|\sigma_z|v\rangle = \sin(2\theta) = \frac{b}{\sqrt{a^2+b^2}} = \frac{e^{-K}}{\sqrt{e^{2K}\,\underset{\underset{\mathrm{ch}^2(H)-1}{\shortparallel}}{\mathrm{sh}^2(H)} + e^{-2K}}}$$

$$\langle u|\sigma_z|v\rangle = \frac{e^{-K}}{\sqrt{e^{2K}\mathrm{ch}^2(H) - 2\,\mathrm{sh}(2K)}}$$

$$C(|r_i - r_j|) = \frac{e^{-2K}}{e^{2K}\mathrm{ch}^2(H) - 2\,\mathrm{sh}(2K)}\, e^{-|r_i - r_j|/\xi}$$

2. $$\xi = -\frac{1}{\ln \frac{\lambda_-}{\lambda_+}} = -\frac{1}{\ln \frac{e^K ch(H) - \sqrt{e^{2K} ch^2(H) - 2sh(2K)}}{e^K ch(H) + \sqrt{e^{2K} ch^2(H) - 2sh(2K)}}}$$

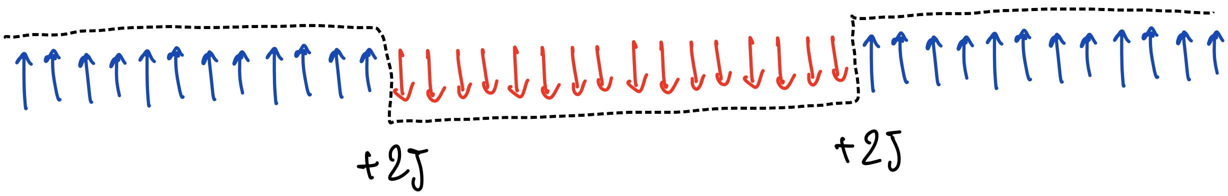
3. $H = 0$

$$\lambda_- = 2sh(K)$$
$$\lambda_+ = 2ch(K)$$

$$\xi = -\frac{1}{\ln th(K)}$$

$K \to 0 \qquad th(K) \simeq 1 - 2e^{-2K}$

$$\xi \simeq -\frac{1}{-2e^{-2K}} = \frac{e^{2K}}{2}$$



Domain walls (play the role of the topological defects in the XY model)

Peierls argument

$\Delta E = 2J \qquad\qquad \Delta F = 2J - k_B T \ln N$

$\Delta S = k_B \ln N \qquad\qquad \Delta F \leq 0$

Finite density of domain walls $\rho \sim e^{-2K}$

Distance between two neighboring DW $\sim \rho^{-1} = e^{2K} = 2\xi$

4. Scaling hypothesis

Distance from the critical point $t = e^{-2k}$, H

$m = m(t,H)$

- $m \propto |t|^\beta$
- $m \propto |H|^{1/\delta}$

$t \to \lambda t$: $\quad m \to \lambda^\beta m$, $\quad H \to (\lambda^\beta m)^\delta \Rightarrow \begin{cases} t \to \lambda t \\ m \to \lambda^\beta m \\ H \to \lambda^{\beta\delta} H \end{cases}$

Scaling Function: $\boxed{m = H^a \phi(t/H^b)}$

$t \to \lambda t \qquad m = \lambda^{\beta\delta a} H^a \phi\left(\frac{\lambda}{\lambda^{\beta\delta b}} \cdot \frac{t}{H^b}\right) = \lambda^\beta m$

- $\beta\delta a = \beta \Rightarrow a = 1/\delta$
- $\beta\delta b = 1 \Rightarrow b = 1/\beta\delta$

$$m = |H|^{1/\delta} \phi\left(t|H|^{-1/\beta\delta}\right)$$

$$|\lambda^{\beta\delta} H|^{1/\delta} \phi\left(\lambda t |\lambda^{\beta\delta} H|^{-1/\beta\delta}\right) = \lambda^\beta |H|^{1/\delta} \phi\left(t|H|^{-1/\beta\delta}\right)$$

$$= \lambda^\beta m \longrightarrow \text{it works!}$$

$$m = \frac{e^k \, sh(H)}{\sqrt{e^{2k} sh^2(H) + e^{-2k}}} \qquad \begin{matrix} t \ll 1 \\ H \ll 1 \end{matrix}$$

$$m \simeq \frac{H/\sqrt{t}}{\sqrt{H^2/t + t}} = \frac{H/\cancel{\sqrt{t}}}{\sqrt{(H^2+t^2)/\cancel{t}}} = \frac{H}{\sqrt{H^2+t^2}}$$

$$m \simeq \frac{H}{t\sqrt{1+(H/t)^2}} = \frac{H/t}{\sqrt{1+(H/t)^2}} = f(t/H) \Rightarrow \begin{matrix} \beta\delta = 1 \\ \delta = \infty \end{matrix}$$

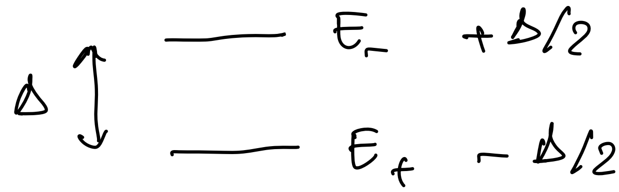
$$\xi = \frac{1}{2t} \Rightarrow \nu = 1$$

- $\alpha + 2\beta + \gamma = 2 \Rightarrow \alpha + \gamma = 2 \Rightarrow \alpha = 1$
- $\delta - 1 = \gamma/\beta \Rightarrow \gamma = \beta\delta - \beta \Rightarrow \gamma = 1$
- $2 - \alpha = d\nu \Rightarrow \alpha = 1$

$$\boxed{\beta = 0, \alpha = 1, \gamma = 1, \nu = 1, \delta = \infty}$$

Quantum model

$$\mathcal{H} = -\frac{\Delta}{2}\hat{\sigma}^x$$



$$\langle \rightarrow | \mathcal{H} | \rightarrow \rangle = -\frac{\Delta}{2} = E_+ ; \qquad \langle \leftarrow | \mathcal{H} | \leftarrow \rangle = \frac{\Delta}{2} = E_-$$

$$E_- - E_+ \equiv \Delta E = \Delta \qquad \text{Gap}$$

$$Z = \mathrm{Tr}\, e^{-\beta\hat{\mathcal{H}}} = \langle \rightarrow | e^{-\beta\mathcal{H}} | \rightarrow \rangle + \langle \leftarrow | e^{-\beta\mathcal{H}} | \leftarrow \rangle$$

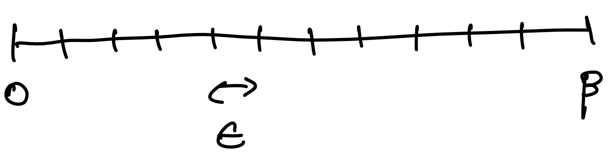
$$= e^{+\beta\Delta/2} + e^{-\beta\Delta/2}$$

$$= 2\,\mathrm{ch}\left(\frac{\beta\Delta}{2}\right)$$

General treatment when the eigenvectors and the eigenvalues of $\hat{\mathcal{H}}$ are not known:

$$Z = \sum_s \langle s | e^{-\beta\mathcal{H}} | s \rangle = \sum_s \langle s | e^{-\epsilon\mathcal{H} - \epsilon\mathcal{H} - \ldots - \epsilon\mathcal{H}} | s \rangle$$

$$\wedge \quad \sum_{s_i} |s_i\rangle\langle s_i|$$



$$\frac{\beta}{N} = \epsilon$$

$$Z = \sum_{\{s_1, s_2, \ldots s_N\}} \langle s_1 | e^{-\epsilon H} | s_2 \rangle \langle s_2 | e^{-\epsilon H} | s_3 \rangle \times \ldots$$

$$\times \langle s_N | e^{-\epsilon H} | s_1 \rangle$$

$$e^{-\epsilon \hat{H}} \equiv \hat{T}$$

$$Z = \mathrm{Tr}\, T^N$$

$$-H = A\mathbb{1} + B\sigma^x + \cancel{C\sigma^y} + D\sigma^z$$

$$e^{-\epsilon H} = e^{\epsilon A \mathbb{1} + \epsilon B \sigma^x + \epsilon D \sigma^z} = e^{\epsilon A}\, \mathbb{1}\, e^{\epsilon B \sigma^x + \epsilon C \sigma^z}$$

$$e^{\epsilon B \sigma^x + \epsilon D \sigma^z} = \sum_n \frac{(\epsilon B \sigma^x + \epsilon D \sigma^z)^n}{n!}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x^2 = \sigma_z^2 = \sigma_y^2 = 1 \qquad \sigma_i \sigma_j = -\sigma_j \sigma_i \quad \text{for } i \neq j$$

$$\sigma_1 \sigma_2 = i \sigma_3 \qquad [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$$

$$\sigma_3 \sigma_1 = i \sigma_2 \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} I$$

$$\sigma_2 \sigma_3 = i \sigma_1$$

$$\left(\epsilon B\sigma^x + \epsilon D\sigma^z\right)^2 = \epsilon^2 B^2 \mathbb{1} + \epsilon^2 D^2 \mathbb{1} + \epsilon^2 BD\underbrace{\left(\sigma^x\sigma^z + \sigma^z\sigma^x\right)}_{=0}$$

$$\left(\epsilon B\sigma^x + \epsilon D\sigma^z\right)^3 = \left(\epsilon^2 B^2 + \epsilon^2 D^2\right)\mathbb{1}\left(\epsilon B\sigma^x + \epsilon D\sigma^z\right)$$

Terms with n even:

$$\sum_{n=0}^{+\infty} \frac{\left(\epsilon^2 B^2 + \epsilon^2 D^2\right)^n}{(2n)!}\mathbb{1} = \mathbb{1}\, ch\left(\sqrt{\epsilon^2 B^2 + \epsilon^2 D^2}\right)$$

Terms with n odd:

$$\sum_{n=0}^{+\infty} \frac{\left(\epsilon^2 B^2 + \epsilon^2 D^2\right)^n}{(2n+1)!}\left(\epsilon B\sigma^x + \epsilon D\sigma^z\right) =$$

$$= \frac{\epsilon\left(B\sigma^x + D\sigma^z\right)}{\sqrt{\epsilon^2 B^2 + \epsilon^2 D^2}}\, sh\left(\sqrt{\epsilon^2 B^2 + \epsilon^2 D^2}\right)$$

$$e^{-\epsilon\hat{H}} = e^{\epsilon A}\,\mathbb{1}\left(ch\left(\epsilon\sqrt{B^2+D^2}\right)\mathbb{1} + \frac{B\sigma^x + D\sigma^z}{\sqrt{B^2+D^2}}\, sh\left(\epsilon\sqrt{B^2+D^2}\right)\right)$$

$$= e^{\epsilon A}\, ch\left(\epsilon\sqrt{B^2+D^2}\right)\mathbb{1}$$

$$+ \frac{B e^{\epsilon A}}{\sqrt{B^2+D^2}}\, sh\left(\epsilon\sqrt{B^2+D^2}\right)\sigma^x$$

$$+ \frac{D e^{\epsilon A}}{\sqrt{B^2+D^2}}\, sh\left(\epsilon\sqrt{B^2+D^2}\right)\sigma^z$$

$$\hat{H} = -\frac{\Delta}{2}\hat{\sigma}^x$$

$$\frac{\Delta}{2}\hat{\sigma}^x = A\,\mathbb{1} + B\hat{\sigma}^x + D\hat{\sigma}^z$$

$$A = D = 0$$

$$B = \Delta/2$$

$$e^{-\epsilon\hat{H}} = ch(\epsilon\Delta/2)\,\mathbb{1} + sh(\epsilon\Delta/2)\,\hat{\sigma}^x$$

$$e^{-\epsilon \hat{H}} = \begin{pmatrix} ch(\epsilon\Delta/2) & sh(\epsilon\Delta/2) \\ sh(\epsilon\Delta/2) & ch(\epsilon\Delta/2) \end{pmatrix}$$

$$= \begin{pmatrix} e^{K} & e^{-K} \\ e^{K} & e^{K} \end{pmatrix}$$

$$e^{K} = ch\left(\frac{\epsilon\Delta}{2}\right) \qquad e^{-K} = sh\left(\frac{\epsilon\Delta}{2}\right)$$

$$\begin{cases} e^{\epsilon\Delta/2} + e^{-\epsilon\Delta/2} = 2e^{K} \\ e^{\epsilon\Delta/2} - e^{-\epsilon\Delta/2} = 2e^{-K} \end{cases}$$

$$2(e^{K} + e^{-K}) = 2e^{\epsilon\Delta/2}$$

$$2(e^{K} - e^{-K}) = 2e^{-\epsilon\Delta/2}$$

$$2\,ch(K) = e^{\epsilon\Delta/2}$$

$$2\,sh(K) = e^{-\epsilon\Delta/2}$$

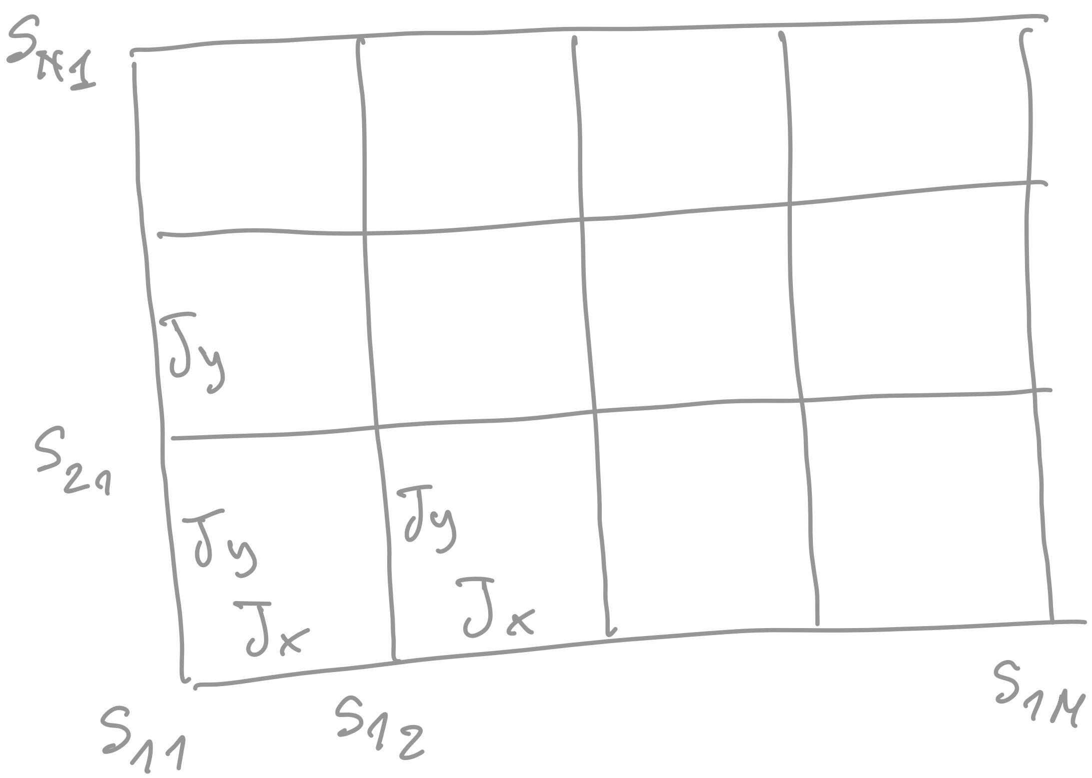
$$\frac{\epsilon\Delta}{2} = \ln 2ch(K) \qquad -\frac{\epsilon\Delta}{2} = \ln 2sh(K)$$

$$e^{-G\Delta} = \operatorname{th}(k)$$

$$-G\Delta = \ln \operatorname{th}(k)$$

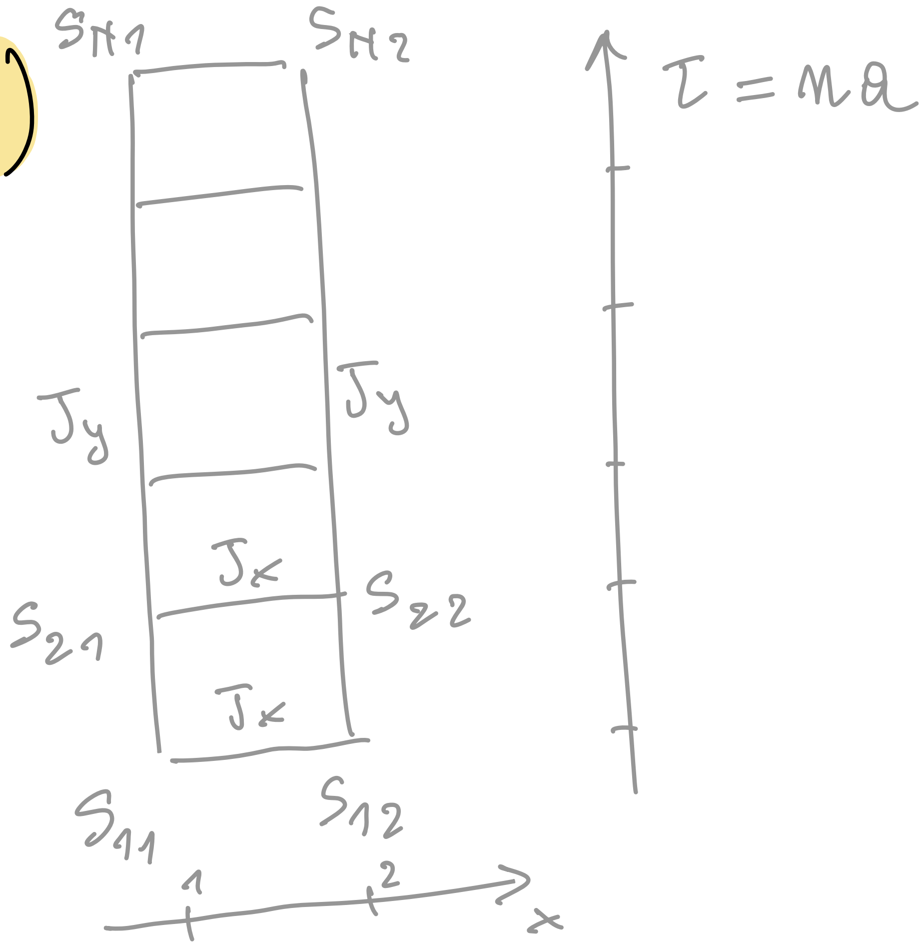
$$\boxed{\Delta = -\frac{1}{\epsilon} \ln \operatorname{th}(k) = \frac{1}{\xi\epsilon}}$$

2) 2d Ising model and the transverse field quantum chain



$$H = -J_x \sum_{n,m} S_{n,m} S_{n,m+1} - J_y \sum_{n,m} S_{n,m} S_{n+1,m}$$

1)



$$Z = \sum_{\substack{\{S_{i1} = \pm 1\} \\ \{S_{i2} = \pm 1\}}} e^{-\beta H}$$

$$K_x \equiv \beta J_x \qquad K_y \equiv \beta J_y$$

$$Z = \sum_{\{s_{i1}, s_{i2}\}} e^{K_y \sum_i s_{i,1} s_{i+1,1} + K_y \sum_i s_{i,2} s_{i+1,2}} \times$$

$$\times\, e^{K_x \sum_i s_{i1} s_{i2}}$$

$$Z = \sum_{\{s_{i1}, s_{i2}\}} \prod_i e^{K_y s_{i,1} s_{i+1,1}} \prod_i e^{K_y s_{i,2} s_{i+1,2}}$$

$$\prod_i e^{\frac{K_x}{2}(s_{i1} s_{i2} + s_{i+1,1} s_{i+1,2})}$$

2) $\sigma_1^{x,y,z} = \sigma^{x,y,z} \otimes \mathbb{1}$

$\sigma_2^{x,y,z} = \mathbb{1} \otimes \sigma^{x,y,z}$

3) $$e^{K_y s_i s_{i+1}} = \begin{pmatrix} e^{K_y} & e^{-K_y} \\ e^{-K_y} & e^{K_y} \end{pmatrix}$$

$$= a\mathbb{1} + b\sigma^x + d\sigma^z$$

$$= \begin{pmatrix} a+d & b \\ b & a-d \end{pmatrix}$$

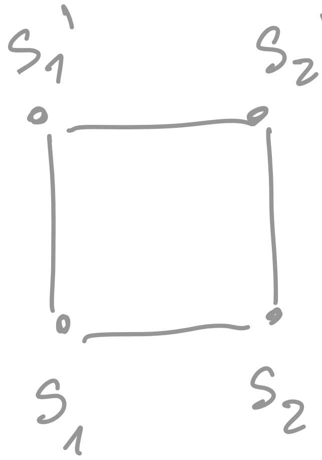
$$\begin{cases} b = e^{-K_y} \\ a+d = e^{K_y} \\ a-d = e^{-K_y} \end{cases} \Longrightarrow \begin{array}{l} b = e^{-K_y} \\ d = 0 \\ a = e^{K_y} \end{array}$$

$$e^{\frac{K_x}{2} s_1 s_2} = e^{\frac{K_x}{2} \sigma_1^z \sigma_2^z}$$

$$T = e^{\frac{K_x}{2} \sigma_1^z \sigma_2^z} \left(e^{K_y} \mathbb{1}_1 + e^{-K_y} \sigma_1^x \right)$$

$$\left(e^{K_y} \mathbb{1}_2 + e^{-K_y} \sigma_2^x \right) e^{\frac{K_x}{2} \sigma_1^z \sigma_2^z}$$

$$Z = \sum_{\substack{\{s_{1,i}\} \\ \{s_{2,i}\}}} T_{s_{1,1} s_{2,1}; s_{1,2} s_{2,2}} T_{s_{1,2} s_{2,2}; s_{13} s_{23}} \times \dots \times T_{s_{1N} s_{2N}; s_{11} s_{21}}$$



$$\langle s_1 s_2 | T | s_1' s_2' \rangle =$$

$$= e^{\frac{K_x}{2} s_1 s_2} \langle s_1 | \mathbb{1}_1 e^{K_y} + \sigma_1^x e^{-K_y} | s_1' \rangle$$
$$\langle s_2 | \mathbb{1}_2 e^{K_y} + \sigma_2^x e^{-K_y} | s_2' \rangle$$
$$e^{\frac{K_x}{2} s_1' s_2'}$$

$$\langle s | \sigma^x | s' \rangle = \begin{cases} 1 & s' = -s \\ 0 & s' = s \end{cases}$$

$$\langle s | e^{K_y} \mathbb{1} + e^{-K_y} \sigma^x | s' \rangle = \begin{cases} e^{K_y} & s = s' \\ e^{-K_y} & s = -s' \end{cases}$$

$$\langle s | e^{k_y} \mathbb{1} + e^{-k_y} \sigma^x | s' \rangle = e^{k_y s s'}$$

$$\langle s_1 s_2 | T | s_1' s_2' \rangle = e^{\frac{k_x}{2} s_1 s_2} \, e^{k_y s_1 s_1'}$$
$$e^{k_y s_2 s_2'} \, e^{\frac{k_x}{2} s_1' s_2'}$$

4) $e^{k} \mathbb{1} + e^{-k} \sigma^x = e^{-\epsilon H}$

$$-H = A \mathbb{1} + B \sigma^x + D \sigma^z$$

$$e^{-\epsilon H} = e^{\epsilon A} \operatorname{ch}\left(\epsilon \sqrt{B^2 + D^2}\right) \mathbb{1}$$
$$+ \frac{e^{\epsilon A} \operatorname{sh}\left(\epsilon \sqrt{B^2+D^2}\right) B}{\sqrt{B^2+D^2}} \sigma^x$$
$$+ \frac{e^{\epsilon A} \operatorname{sh}\left(\epsilon \sqrt{B^2+D^2}\right) D}{\sqrt{B^2+D^2}} \sigma^z$$

$$D = 0 \implies \sqrt{B^2 + D^2} = B$$

$$\begin{cases} e^{\epsilon A} \, ch(\epsilon B) = e^{k} \\ e^{\epsilon A} \, sh(\epsilon B) = e^{-k} \end{cases}$$

$$e^{\epsilon A} \frac{\left(e^{\epsilon B} + \cancel{e^{-\epsilon B}} + e^{\epsilon B} - \cancel{e^{-\epsilon B}}\right)}{2} = e^{k} + e^{-k}$$

$$e^{\epsilon(A+B)} = 2\,ch(k)$$

$$e^{\epsilon A} \frac{\left(\cancel{e^{\epsilon B}} + e^{-\epsilon B} - \cancel{e^{\epsilon B}} + e^{-\epsilon B}\right)}{2} = e^{k} - e^{-k}$$

$$e^{\epsilon(A-B)} = 2\,sh(k)$$

$$\begin{cases} \epsilon(A+B) = \ln 2\,ch(k) \\ \epsilon(A-B) = \ln 2\,sh(k) \end{cases}$$

$$2\epsilon A = \ln 4\,sh(k)\,ch(k) \implies A = \frac{1}{2\epsilon} \ln 4\,sh(k)\,ch(k)$$

$$2\epsilon B = \ln\left(\frac{1}{th(k)}\right) \implies B = -\frac{1}{2\epsilon} \ln th(k)$$

$$\hat{H} = -\Delta \hat{\sigma}^x$$

$$e^{-\epsilon \hat{H}} = e^{\Delta \epsilon \hat{\sigma}^x} = ch(\Delta\epsilon)\,\mathbb{1} + sh(\Delta\epsilon)\,\hat{\sigma}^x$$

$$ch(\Delta\epsilon) = e^{k}$$

$$sh(\Delta\epsilon) = e^{-k}$$

$$th(\Delta\epsilon) = e^{-2k}$$

$$\Delta\epsilon \simeq e^{-2K}$$

$$\Delta \simeq \frac{e^{-2K}}{\epsilon}$$

$$T = e^{\frac{K_x}{2}\sigma_1^z\sigma_2^z}\, e^{-\epsilon H_1}\, e^{-\epsilon H_2}\, e^{\frac{K_x}{2}\sigma_1^z\sigma_2^z}$$

$$e^x e^y = e^{X+Y + \frac{1}{2}[x,y] + \dots}$$

$$T = e^{\epsilon \tilde{K}_x \sigma_1^z \sigma_2^z + \epsilon A(\mathbb{1}_1 + \mathbb{1}_2) + \epsilon B(\sigma_1^x + \sigma_2^x) + O(\epsilon^2)}$$

$$\hat{H}_q = -\tilde{K}_x \sigma_1^z \sigma_2^z - \tilde{K}_y(\sigma_1^x + \sigma_2^x) + C$$

$$\boxed{\begin{aligned} \tilde{K}_x &= K_x/\epsilon \\ \tilde{K}_y &= \frac{1}{2\epsilon}\ln\left(\frac{1}{th\,K}\right) \end{aligned}}$$

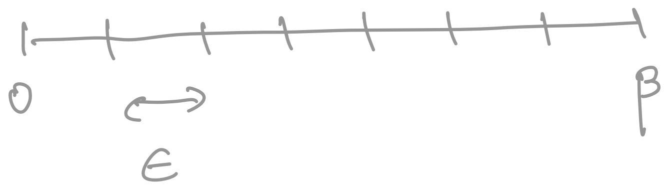
Alternative strategy: Suzuki-Trotter

1d quantum $\longrightarrow$ 2d classical

$$\mathcal{H} = -J \sum_i \sigma_i^z \sigma_{i+1}^z - \Delta \sum_i \sigma_i^x$$

$$Z = \mathrm{Tr}\, e^{-\beta \mathcal{H}} = \sum_{|\vec{s}\rangle} \langle \vec{s} | e^{-\beta \mathcal{H}} | \vec{s} \rangle$$

$$-\beta \mathcal{H} = \underbrace{-\epsilon \mathcal{H} - \epsilon \mathcal{H} - \cdots - \epsilon \mathcal{H}}_{N} \qquad \epsilon N = \beta$$



$$|\psi(t)\rangle = e^{-\frac{it}{\hbar}\hat{\mathcal{H}}} |\psi(0)\rangle$$

$$\epsilon \equiv \frac{\tau}{\hbar} \Rightarrow \tau = \epsilon \hbar = \frac{\beta \hbar}{N}$$

$$Z = \sum_{\substack{|\vec{s}_1\rangle \\ |\vec{s}_2\rangle \\ \vdots \\ |\vec{s}_N\rangle}} \langle \vec{s}_1 | e^{-\epsilon \hat{H}} | \vec{s}_2 \rangle \times \cdots \times \langle \vec{s}_N | e^{-\epsilon \hat{\mathcal{H}}} | \vec{s}_1 \rangle$$

$$\langle \vec{S}_n | e^{-\epsilon \hat{H}} | \vec{S}_{n+1} \rangle =$$

$$= \langle \vec{S}_n | \underbrace{e^{\epsilon J \sum_i \sigma_i^z \sigma_{i+1}^z + \epsilon \Delta \sum_i \sigma_i^x}}_{\parallel} | \vec{S}_{n+1} \rangle$$

$$e^{\epsilon J \sum_i \sigma_i^z \sigma_{i+1}^z} \; e^{\epsilon \Delta \sum_i \sigma_i^x} + O(\epsilon^2)$$

$$\simeq e^{\epsilon J \sum_i S_i(n) S_{i+1}(n)} \langle \vec{S}_n | e^{\epsilon \Delta \sum_i \sigma_i^x} | \vec{S}_{n+1} \rangle$$

$$e^{\epsilon \Delta \sigma_i^x} = ch(\epsilon \Delta) \mathbb{1}_i + sh(\epsilon \Delta) \sigma_i^x$$

$$\langle \vec{S}_n | \prod_i \left(ch(\epsilon \Delta) \mathbb{1}_i + sh(\epsilon \Delta) \sigma_i^x \right) | \vec{S}_{n+1} \rangle$$

$$= \prod_i \underbrace{\left(ch(\epsilon \Delta) \delta_{S_i(n), S_i(n+1)} + sh(\epsilon \Delta) \left(1 - \delta_{S_i(n), S_i(n+1)} \right) \right)}_{\parallel}$$

$$\begin{pmatrix} ch(\epsilon \Delta) & sh(\epsilon \Delta) \\ sh(\epsilon \Delta) & ch(\epsilon \Delta) \end{pmatrix} = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix}$$

$$e^{K} = ch(\epsilon\Delta) = \frac{e^{\epsilon\Delta} + e^{-\epsilon\Delta}}{2}$$

$$e^{-K} = sh(\epsilon\Delta) = \frac{e^{\epsilon\Delta} - e^{-\epsilon\Delta}}{2}$$

$$e^{-2K} = th(\epsilon\Delta)$$

$$-2K = \ln th(\epsilon\Delta)$$

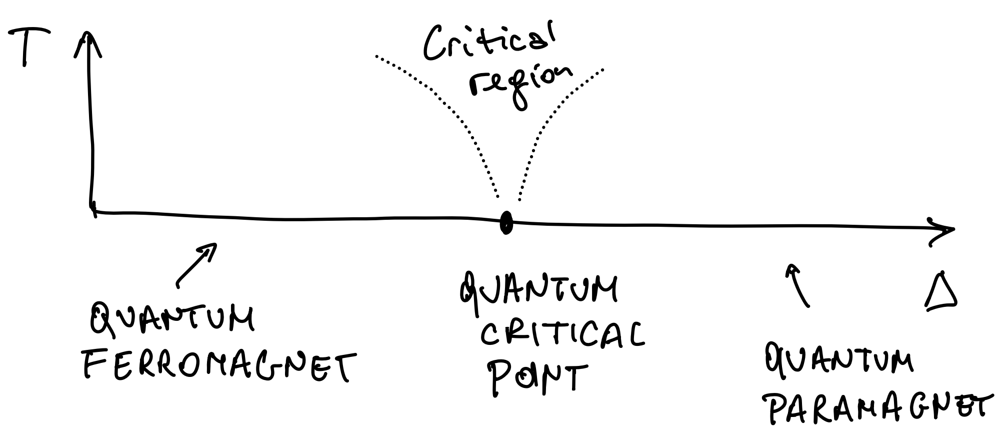
$$\boxed{K = -\frac{1}{2}\ln th(\epsilon\Delta)}$$

$$\langle \vec{S}_n | e^{-\epsilon H} | \vec{S}_{n+1} \rangle = e^{\epsilon J \sum_i S_i(n) S_{i+1}(n) + K \sum_i S_i(n) S_i(n+1)}$$

equivalent to a transfer matrix of a 2d Ising model with $\boxed{\begin{array}{l}\beta J_x = \epsilon J \\ \beta J_y = K\end{array}}$

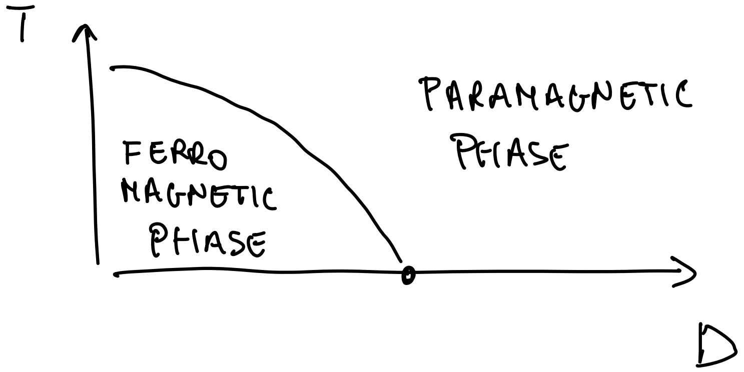
$K_x = \frac{\beta J}{N} \longrightarrow$ At finite temperature, in the $N \to \infty$ limit, $K_x \to 0$

$\Rightarrow$ No phase transition



d = 1
QUANTUM ISING MODEL

In $d>1$, instead, one can have a phase transition at finite temperature



$d>1$