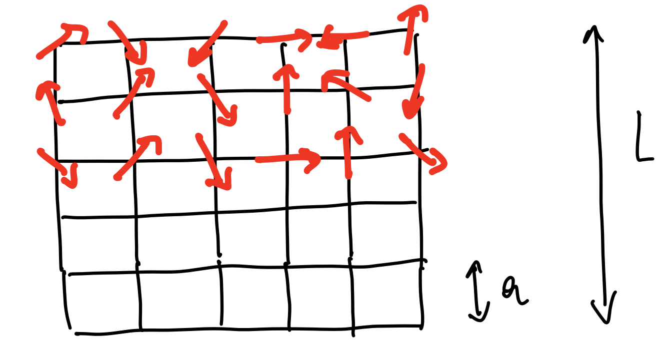
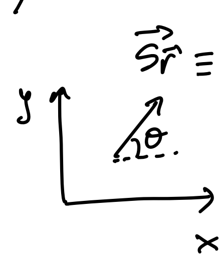
TD2: XY model and spin waves



$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}\,' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}\,'}$$

$$\vec{S}_{\vec{r}} \equiv (\cos\theta_{\vec{r}}, \sin\theta_{\vec{r}})$$

$$|\vec{S}_{\vec{r}}| = 1$$



Phenomenological analysis

1. $J > 0$ ferromagnetic interaction $\rightarrow$ favours the alignement of the spins

2. $T_c \sim J$ (if a critical temperature exists for this model)

3. $T \gg J$ $\rightarrow$ disordered configuration
   Paramagnetic phase
   Each spin points in a random direction

Can we perform a mean-field analysis ?

Fully connected approximation $\rightarrow$ $\mathcal{H} = -\frac{J}{2N} \sum_{i,j} \vec{S}_i \cdot \vec{S}_j = -\frac{J}{2N}\left(\sum_i \vec{S}_i\right)^2$

$$\frac{1}{N}\sum_i \vec{S}_i = \vec{m} = m_x \vec{e}_x + m_y \vec{e}_y$$

$$\vec{m}\cdot\vec{m} = m_x^2 + m_y^2 = \left(\frac{1}{N}\sum_i \cos(\theta_i)\right)^2 + \left(\frac{1}{N}\sum_i \sin\theta_i\right)^2$$

$$Z = \int \prod_{i=1}^{N} d\vec{S}_i \; e^{\frac{\beta J N}{2}\left(\frac{1}{N}\sum_i \cos\theta_i\right)^2 + \frac{\beta J N}{2}\left(\frac{1}{N}\sum_i \sin\theta_i\right)^2}$$

$$= \frac{\beta J N}{2\pi} \int_{-\infty}^{+\infty} dm_x \int_{-\infty}^{+\infty} dm_y \; e^{-\frac{\beta J N}{2}(m_x^2 + m_y^2)} \times$$

$$\times \int \prod_i \theta_i \quad e^{\beta J m_x \cos\theta_i + \beta J m_y \sin\theta_i}$$

$$Z = \int dm_x \, dm_y \; e^{-\beta N f(m_x, m_y)}$$

$$f(m_x, m_y) = \frac{J}{2}\left(m_x^2 + m_y^2\right) - \frac{1}{\beta} \ln \underbrace{\int_0^{2\pi} d\theta_i \; e^{\beta J \left(m_x \cos\theta_i + m_y \sin\theta_i\right)}}_{= \, 2\pi I_0\left(\beta J \sqrt{m_x^2 + m_y^2}\right)}$$

modified Bessel function of the first kind

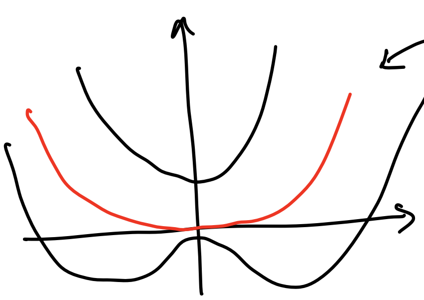
$$\boxed{\begin{aligned} I_0(x) &\simeq 1 + \frac{x^2}{4} + \frac{1}{64} x^4 \\ \ln I_0(x) &\simeq \frac{x^2}{4} + \frac{x^4}{64} - \frac{1}{2}\frac{x^4}{16} \end{aligned}}$$

$$f(|m|) = \frac{J}{2}|m|^2 - \frac{1}{\beta} \ln\left(2\pi I_0(\beta J |m|)\right)$$

expand around $|m| = 0$

$$f(|m|) \approx \frac{J}{2}|m|^2 - \frac{1}{\beta}\left(\ln 2\pi + \frac{(\beta J)^2}{4}|m|^2 - \frac{(\beta J)^4}{64}|m|^4 + \dots\right)$$

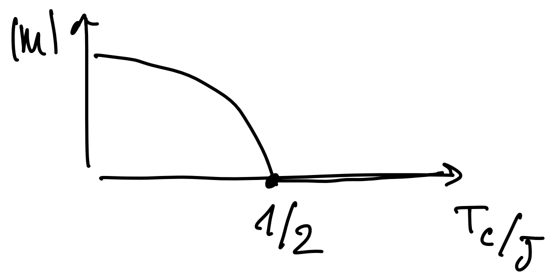
$$f(|m|) \simeq -\frac{1}{\beta}\ln 2\pi + \frac{J}{2}\left(1 - \frac{\beta J}{2}\right)|m|^2 + \frac{\beta^3 J^4}{64}|m|^4$$



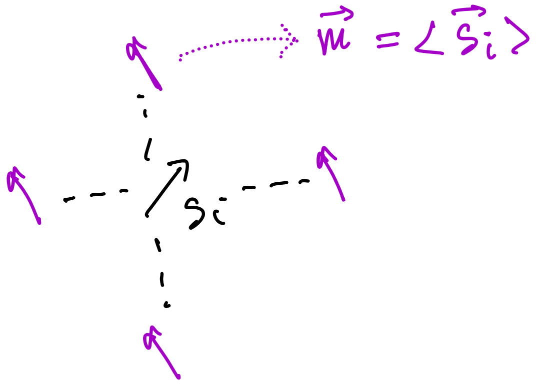
$$1 - \frac{\beta_c J}{2} = 0$$

critical temperature

$$\frac{T_c}{J} = \frac{1}{2}$$



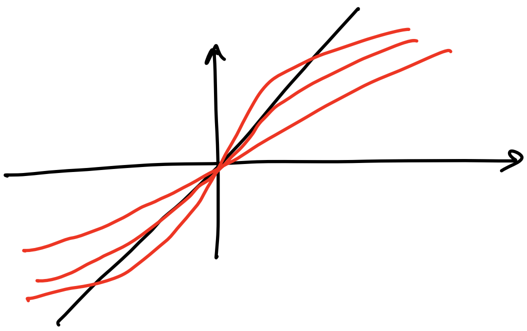
Curie-Weiss approximation



$$H_{cw} \simeq -Jz\vec{m}\cdot\vec{s}_i$$

$$Z_{cw} = \int_0^{2\pi} d\theta \, e^{\beta J z m \cos(\theta)} = 2\pi I_0(\beta J z m)$$

$$\langle|\vec{s}_i|\rangle = \langle \cos\theta \rangle = \frac{\partial \ln Z_{cw}}{\partial(\beta J z m)}$$



$$|m| = \frac{I_1(\beta J z|m|)}{I_0(\beta J z|m|)} \approx \frac{\beta J z|m|}{2} - \frac{(\beta J z|m|)^3}{16}$$

$$\frac{\beta_c J z}{2} = 1 \qquad \frac{T_c}{J} = \frac{z}{2}$$

$$\vec{s}_i = \vec{m} + \delta\vec{s}_i$$

$$\begin{aligned}
\vec{s}_i \cdot \vec{s}_j &= (\vec{m} + \delta\vec{s}_i)\cdot(\vec{m} + \delta\vec{s}_j) \\
&= m^2 + \vec{m}\cdot\delta\vec{s}_i + \vec{m}\cdot\delta\vec{s}_j + \delta\vec{s}_i\cdot\delta\vec{s}_j \\
&= m^2 + \vec{m}\cdot(\vec{s}_i - \vec{m}) + \vec{m}\cdot(\vec{s}_j - \vec{m}) + \cancel{\delta\vec{s}_i\,\delta\vec{s}_j} \\
&\simeq -m^2 + \vec{m}(\vec{s}_i + \vec{s}_j)
\end{aligned}$$

However the mean-field approximation is wrong for the XY model in 2d

- Many low-energy excitations that destroy the ferromagnetic order

Similar to the Ising model in 1d

$\uparrow\uparrow\uparrow\uparrow ---- \uparrow\uparrow\uparrow$ $\rightarrow$ ground state

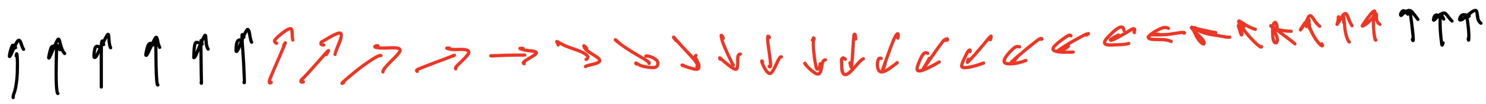
$\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow -- \downarrow\downarrow\uparrow\uparrow\uparrow$ $\rightarrow$ excited state

$\Delta E = 4J$

$S = k_B \ln N$

$\rightarrow$ reduce the free energy at any finite T

For the xy model in 2d the low-energy excitations that destroy the ferromagnetic order are the "spin waves"



Low-T expansion

1. $$H = -J \sum_{\langle \vec r, \vec r' \rangle} \cos(\theta_{\vec r} - \theta_{\vec r'})$$

Ground state $\longrightarrow$ $\theta_{\vec r} = \theta_0 \quad \forall \vec r$ $\qquad E = -2JN$

(infinitely degenerate)

2. Low-T $\longrightarrow$ "close" to the ground state

$$\theta_{\vec r} \simeq \theta_0 + \delta\theta_{\vec r}$$

$$\cos(\theta_{\vec r} - \theta_{\vec r'}) \simeq 1 - \frac{1}{2}(\theta_{\vec r} - \theta_{\vec r'})^2 + \dots$$

$$H_{sw} = -2JN + \frac{J}{2} \sum_{\langle r, r' \rangle} (\theta_{\vec r} - \theta_{\vec r'})^2$$

3. $$\frac{df}{dx} = \frac{f(x+a/2) - f(x-a/2)}{a}$$

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial f}{\partial x}(x+a/2, y) - \frac{\partial f}{\partial x}(x-a/2, y) \right) / a$$

$$= \frac{1}{a}\left(\frac{f(x+a,y) - f(x,y)}{a} - \frac{f(x,y) - f(x-a)}{a} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f(x+a,y) + f(x-a,y) - 2f(x,y)}{a^2}$$

$$\nabla^2 f = \frac{f(x+a,y) + f(x-a,y) + f(x,y+a) + f(x,y-a) - 4f(x,y)}{a^2}$$

4. $$-a\nabla^2 G_r = \delta_{\vec{r},\vec{0}}$$

Green's function of the Laplacian on the square lattice

$$\hat{G}_{\vec{q}} = \sum_{\vec{r}} e^{i\vec{q}\cdot\vec{r}} G_{\vec{r}}$$

$$\vec{q} = \frac{2\pi}{L}(n_x, n_y)$$

$$G_{\vec{r}} = \frac{1}{N} \sum_{\vec{q}\neq\vec{0}} e^{-i\vec{q}\cdot\vec{r}} G_{\vec{q}}$$

$$-\frac{L}{2a} \leq n_{x,y} \leq \frac{L}{2a}$$

$$-a^2\nabla^2 G_{\vec{r}} = 4G(x,y) - G(x+a,y) - G(x-a,y) - G(x,y+a) - G(x,y-a)$$

$$G(x+a,y) = \frac{1}{N} \sum_{\vec{q}\neq\vec{0}} e^{-i\vec{q}\cdot\vec{r} - iq_x a} G_{\vec{q}}$$

$$G(x-a,y) = \frac{1}{N} \sum_{\vec{q}\neq\vec{0}} e^{-i\vec{q}\cdot\vec{r} + iq_x a} G_{\vec{q}}$$

$$-a^2\nabla^2 G_{\vec{r}} = \frac{1}{N} \sum_{\vec{q}\neq\vec{0}} e^{-i\vec{q}\cdot\vec{r}} G_{\vec{q}} \left(4 - 2\cos(q_x a) - 2\cos(q_y a)\right) = \delta_{\vec{r},\vec{0}}$$

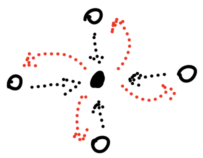
$$G_{\vec{q}} = \frac{1}{4 - 2\cos(q_x a) - 2\cos(q_y a)}$$

$$G_{\vec{0}} \simeq \frac{1}{2\pi} \ln \frac{L}{a}$$

$$G_{|\vec{r}|\gg a} - G_0 \simeq -\frac{1}{2\pi} \ln \frac{|r|}{a} - c + o(1) \qquad c = \frac{1}{2\pi}\left(\gamma + \frac{3}{2}\ln 2\right)$$

$$Z_{SW} = \int \prod_{\vec{r}} d\theta_{\vec{r}}\, e^{\beta H_{SW}} = e^{2\beta J N} \int \prod_{\vec{r}} \theta_{\vec{r}}\, e^{-\frac{K}{2}\sum_{\langle r,r'\rangle}(\theta_r-\theta_{r'})^2}$$

$$\sum_{\langle r,r'\rangle} (\theta_r - \theta_{r'})^2 = \frac{1}{2}\sum_r \Big[(\theta_r - \theta_{r+ae_x})^2 + (\theta_r - \theta_{r-ae_x})^2$$



$$+ (\theta_r - \theta_{r+ae_y})^2 + (\theta_r - \theta_{r-ae_y})^2 \Big]$$

$$= \frac{1}{2}\sum_r \Big[\theta_r^2 + \theta_{r+ae_x}^2 - 2\theta_r\theta_{r+ae_x} + \theta_r^2 + \theta_{r-ae_x}^2 - 2\theta_r\theta_{r-ae_x}$$

$$+ \theta_r^2 + \theta_{r+ae_y}^2 - 2\theta_r\theta_{r-ae_y} + \theta_r^2 + \theta_{r-ae_y}^2 - 2\theta_r\theta_{r-ae_y} \Big]$$

$$= \sum_r \Big[\underbrace{4\theta_r^2 - \theta_r\theta_{r+ae_x} - \theta_r\theta_{r-ae_x} - \theta_r\theta_{r+ae_y} - \theta_r\theta_{r-ae_y}}_{\displaystyle \parallel \atop \displaystyle \theta_r(-a\nabla^2\theta_r)} \Big]$$

$$\boxed{Z_{SW} \propto \int \mathcal{D}\theta\, e^{-\frac{K}{2}\sum_r \theta_r(-a\nabla^2\theta_r)}}$$

Alternative route: (continuum limit)

$$\sum_{\langle r,r'\rangle} (\theta_r - \theta_{r'})^2 = \frac{1}{2} \sum_{\vec{r}} \sum_{\vec{r}' \in \partial \vec{r}} (\theta_r - \theta_{r'})^2$$

$$= \frac{1}{2} \sum_{\vec{r}} \Big[(\theta_{\vec{r}} - \theta_{\vec{r}-a\vec{e}_x})^2 + (\theta_r - \theta_{r+ae_x})^2 + (\theta_r - \theta_{r-ae_y})^2 + (\theta_r - \theta_{r+ae_y})^2 \Big]$$

$$\theta_r - \theta_{r-ae_x} \simeq a \frac{\partial \theta_r}{\partial x}$$

$$= \frac{1}{2} \sum_r \left(2\left(a \frac{\partial \theta_r}{\partial x}\right)^2 + 2\left(a \frac{\partial \theta_r}{\partial y}\right)^2 \right)$$

$$= \frac{1}{2} \cdot 2\, a^2 \sum_r (\nabla \theta_r)^2$$

$$Z_{sw} = e^{2\beta J N} \int D\theta \; e^{-\frac{K}{2} \sum_r a^2 (\nabla \theta_r)^2}$$

$$Z_{sw} \simeq e^{2\beta J N} \int D\theta_r \; e^{-\frac{K}{2} \int d^2\vec{r} \, (\nabla \theta_r)^2}$$

$$\int d^2\vec{r} \, (\vec{\nabla} \theta_r)(\vec{\nabla} \theta_r) \simeq - \int d^2\vec{r} \; \theta_r \nabla^2 \theta_r = \int d^2\vec{q} \; q^2 |\theta_q|^2$$

$$Z_{sw} \propto \int D\theta_q \; e^{-\frac{K}{2} \int d^2\vec{q} \; q^2 |\theta_q|^2} \Rightarrow \langle \theta_q \theta_{q'} \rangle = \frac{1}{K q^2} \delta_{q+q',0}$$

$$\langle \theta_r \theta_{r'} \rangle = \frac{1}{Z_{SW}} \int \mathcal{D}\theta \, \theta_r \theta_{r'} \, e^{-\frac{K}{2}\sum_{i,j} \theta_i (-a\nabla^2) \theta_j}$$

Gaussian integral $\quad \langle \theta_r \theta_{r'} \rangle = K \left(-a\nabla^2\right)^{-1}_{r,r'}$

$$-a\nabla^2 = \begin{pmatrix} 4 & 0\,0\,0\,0 & -1 & 0\,0\,0 & -1 & 0\,0 & -1 & 0 & -1 & 0\,0\,0 \\ & \ddots & & & & & & & & \\ & & & & & & & & & 4 \end{pmatrix}$$

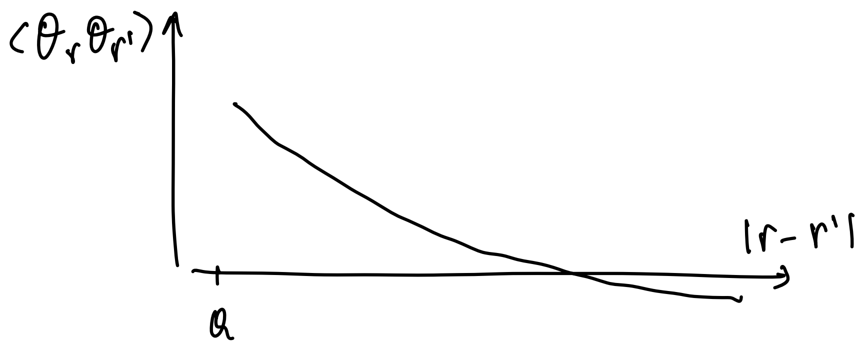
nearest neighbors on a square lattice

$$-a\nabla^2 \cdot G_{\vec{r}} = \delta_{\vec{r},\vec{0}} \qquad -a\nabla^2 G_{|r-r'|} = \delta_{\vec{r},\vec{r}'} = \mathbb{1}$$

$$\left(-a\nabla^2\right)^{-1}_{r,r'} = G_{r,r'} = G_{|r-r'|}$$

$$\langle \theta_r \theta_{r'} \rangle = K^{-1} G_{|r-r'|} \simeq K^{-1}\left(G_0 - \frac{1}{2\pi} \ln \frac{|r-r'|}{a} - c + o(1)\right)$$

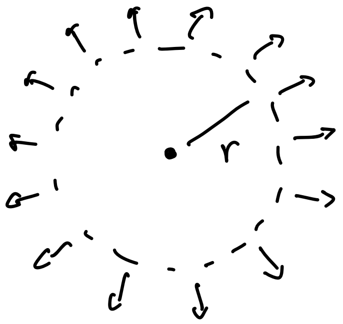
$$\langle \theta_r \theta_{r'} \rangle \simeq \frac{1}{2\pi K} \ln \frac{L}{a} - \frac{1}{2\pi K} \ln \frac{|r-r'|}{a} - \frac{c}{K}$$



Poisson equation (electrostatic in 2d)

$$\nabla^2 V = \rho / \epsilon_0 \qquad \rho(\vec r) = \delta(\vec r)$$

Gauss theorem



$$2\pi r\, |E(r)| = \frac{1}{\epsilon_0}$$

$$\vec E = \frac{1}{2\pi\epsilon_0 r}\vec e_r = -\vec\nabla V = -\frac{dV}{dr}\vec e_r$$

$$V = -\frac{1}{2\pi\epsilon_0}\ln r$$

5. $\langle \theta_{\vec r} \rangle = 0$

$$\langle \vec S_{\vec r} \rangle = \langle \cos\theta_{\vec r} \rangle \vec e_x + \langle \sin\theta_{\vec r} \rangle \vec e_y$$

$$\langle \cos\theta_{\vec r} \rangle = \langle \mathrm{Re}\, e^{i\theta_{\vec r}} \rangle = \mathrm{Re}\langle e^{i\theta_{\vec r}} \rangle$$

$$\vec J = (0,0,\dots 0, \underset{\substack{\uparrow \\ \text{r-th position}}}{i}, 0,0,\dots 0) \qquad J_{\vec r'} = i\,\delta_{r',r}$$

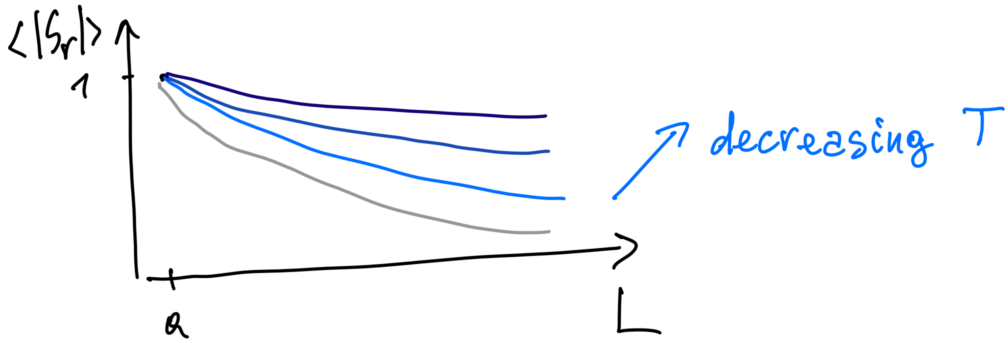
$$e^{i\theta_{\vec r}} = e^{\sum_{r'} J_{r'}\cdot\theta_{r'}} \qquad \langle \cos\theta_r \rangle = \mathrm{Re}\langle e^{J\cdot\theta} \rangle$$

$$\langle e^{J\cdot\theta} \rangle = \frac{1}{Z_{SW}} \int D\theta\, e^{-\frac{K}{2}\theta\cdot(-a\nabla^2)\cdot\theta + J\theta}$$

$$= e^{\frac{1}{2K} J\cdot(-a\nabla^2)^{-1}\cdot J} = e^{\frac{1}{2K} J\cdot G\cdot J}$$

$$\langle \cos\theta_r \rangle = \mathrm{Re}\; e^{\frac{1}{2K}\sum_{r,r'} J_r G_{r,r'} J_{r'}} = \mathrm{Re}\; e^{-\frac{1}{2K} G_0} \simeq e^{-\frac{1}{2K}\cdot\frac{1}{2\pi}\ln\frac{L}{a}}$$

$$\langle \cos\theta_r \rangle \simeq e^{-\frac{1}{4\pi K}\ln\left(\frac{L}{a}\right)} = e^{\frac{1}{4\pi K}\ln\left(\frac{a}{L}\right)} = \left(\frac{a}{L}\right)^{\frac{k_B T}{4\pi J}}$$



No finite magnetization in the thermodynamic limit but there is a "residual" magnetization for L finite

6. $\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \langle \cos(\theta_r - \theta_{r'}) \rangle = \mathrm{Re}\,\langle e^{i(\theta_r - \theta_{r'})} \rangle$

$$\vec{J} = (000 \ldots 0\; \underset{r}{\uparrow 1}\; 00000\; \underset{r'}{\uparrow -i}\; 00 \ldots 00)$$

$$\vec{J}(w) = i\left(\delta_{\vec{r},\vec{w}} - \delta_{\vec{r}',\vec{w}}\right)$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \mathrm{Re}\,\langle e^{\vec{J}\cdot\vec{\theta}} \rangle = \frac{1}{Z_{SW}} \int D\theta\, e^{-\frac{K}{2}\theta\cdot(-a\nabla^2)\theta + J\theta}$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \mathrm{Re}\; e^{\frac{1}{2K} J\cdot G\cdot J} = e^{\frac{1}{2K}\sum_{a,b} J_a G_{a,b} J_b}$$

$$\sum_{a,b} J_a G_{a,b} J_b = -\sum_{a,b} \left(\delta_{ar} - \delta_{ar'}\right) G_{a,b} \left(\delta_{br} - \delta_{br'}\right)$$

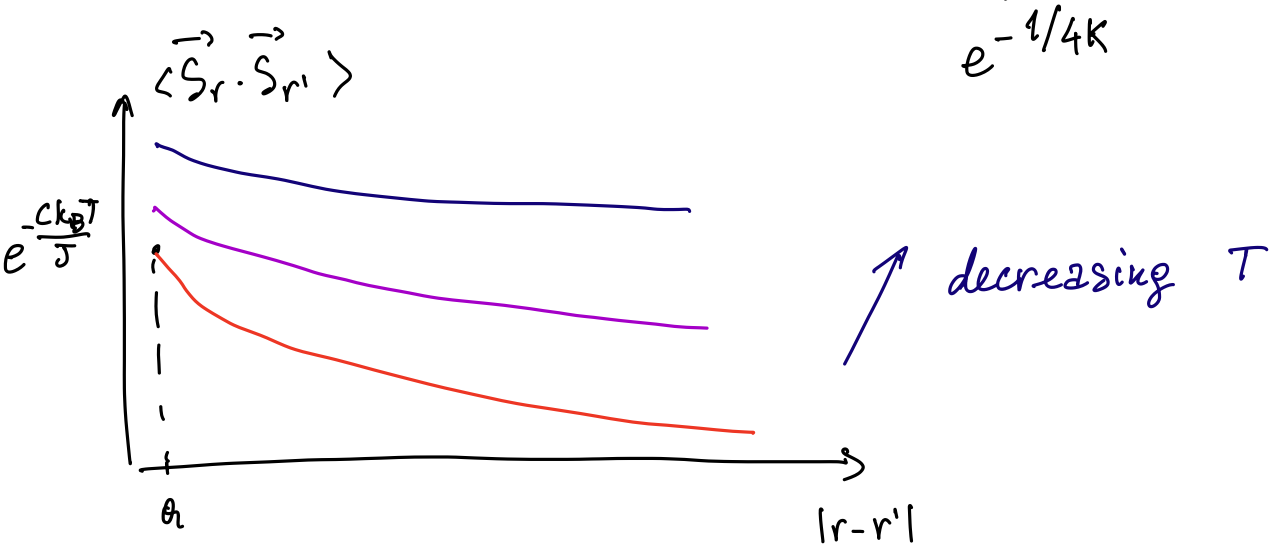
$$= -\sum_{a,b} \left(\delta_{ar}\delta_{br} + \delta_{ar'}\delta_{br'} - \delta_{ar}\delta_{br'} - \delta_{ar'}\delta_{br} \right) G_{ab}$$

$$= -\left(G_{rr} + G_{r'r'} - G_{rr'} - G_{r'r}\right)$$

$$= -2\left(G_0 - G_{|r-r'|}\right) = 2\left(G_{|r-r'|} - G_0\right)$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = e^{\frac{1}{K}\left(G_{|r-r'|} - G_0\right)} \simeq e^{-\frac{1}{2\pi K} \ln \frac{|r-r'|}{a} - \frac{c}{K}}$$

$$= e^{-\frac{c}{K}} \, e^{\frac{1}{2\pi K} \ln \frac{a}{|r-r'|}} = \underbrace{e^{-c/K}}_{\simeq\, e^{-1/4K}} \left(\frac{a}{|r-r'|}\right)^{\frac{k_B T}{2\pi J}}$$



power-law correlations

7. $\xi = \infty$

8. $\chi = \left| \dfrac{d\vec{m}}{d\vec{h}} \right|$

$$\mathcal{H} = -J \sum_{\langle r r' \rangle} S_r \cdot S_{r'} - \vec{h} \cdot \sum_{\vec{r}} \vec{S}_r \qquad \vec{h} = h \vec{e}_x$$

$$\mathcal{H}_{sw} \simeq -2JN + \frac{J}{2} \sum_{\langle r r' \rangle} (\theta_r - \theta_{r'})^2 - h \sum_{\vec{r}} \cos(\theta_r)$$

$$\frac{\partial \ln Z}{\partial (\beta h)} = \left\langle \sum_{\vec{r}} \vec{S}_r \right\rangle = \vec{M}$$

$$\frac{\partial^2 \ln Z}{\partial (\beta h)} = \begin{cases} \dfrac{\partial}{\partial (\beta h)} \vec{M} = k_B T \chi \\ \left\langle \left(\sum_r S_r \right)^2 \right\rangle - \left(\left\langle \sum_r S_r \right\rangle \right)^2 \end{cases}$$

$$k_B T \chi = \sum_{r, r'} \langle S_r S_{r'} \rangle - N^2 \langle S_r \rangle^2$$

$$k_B T \chi = \underbrace{N}_{\left(\frac{L}{a}\right)^2} \sum_r \langle S_0 S_r \rangle - N^2 \langle S_r \rangle^2$$

$$\| $$

$$\int \frac{d^2 r}{a^2} e^{-c/k} \left(\frac{a}{r} \right)^{k_B T / 2\pi J} \simeq 2\pi e^{-c/k} \int_a^L \left(\frac{a}{r} \right)^{\frac{k_B T}{2\pi J}} \overbrace{\frac{r}{a}}^{= x} \frac{dr}{a}$$

$$= 2\pi e^{-c/k} \int_a^{L/a} x^{1 - k_B T / 2\pi J} \, dx$$

$$= 2\pi e^{-c/k} \left. \frac{x^{2 - k_B T / 2\pi J}}{2 - \frac{k_B T}{2\pi J}} \right|_a^{L/a}$$

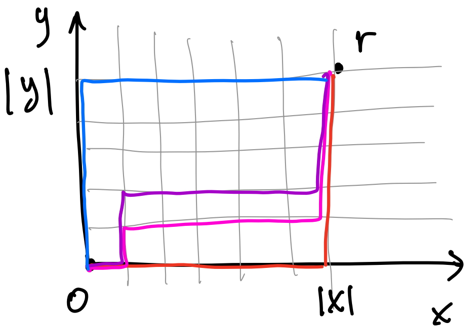
$$\simeq \frac{2\pi e^{-c/k}}{2 - \frac{k_B T}{2\pi J}} \left(\frac{L}{a} \right)^{2 - \frac{k_B T}{2\pi J}}$$

$$\frac{k_B T \chi}{N} \simeq \frac{2\pi e^{-c/k}}{2 - \frac{k_B T}{2\pi J}} \left(\frac{L}{a}\right)^{2-\frac{k_BT}{2\pi J}} - \left(\frac{L}{a}\right)^{2-\frac{k_BT}{2\pi J}} \propto \left(\frac{L}{a}\right)^{2-\frac{k_BT}{2\pi J}}$$

Diverges at low enough T in the thermodynamic limit ($T < 4\pi J$)

High-T expansion

1.



|x| steps right
|y| steps up
with any order

$$\mathcal{N}(\vec{r}) = \binom{|x+y|}{|x|} = \binom{|x|+|y|}{|y|} = \frac{(|x|+|y|)!}{|x|!\,|y|!}$$

$|y| = 0$

$\mathcal{N}(r) = 1$

$\mathcal{N}(r)$ is maximal when $|x| = |y|$

$$\mathcal{N}(r) = \frac{(2|x|)!}{(|x|!)^2} = \frac{(2x)^{2x}\,\cancel{e^{-2x}}\,\sqrt{4\pi x}}{\left(x^x\,\cancel{e^{-x}}\,\sqrt{2\pi x}\right)^2} = \left(\frac{2x}{x}\right)^{2x} \cancel{\frac{1}{\sqrt{\pi x}}}$$

$$\mathcal{N}(r) \simeq 2^{2|x|} = 2^{\|r\|_1}$$

In general $\mathcal{N}(r) \leq 2^{\|r\|_1}$

2.

$$\int_0^{2\pi} \underbrace{\cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3)}_{} d\theta_2 = \pi \cos(\theta_1-\theta_3)$$

$$(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)(\cos\theta_2 \cos\theta_3 + \sin\theta_2 \sin\theta_3)$$

$$\int_0^{2\pi} \cos^2\theta_2 d\theta_2 = \int_0^{2\pi} \sin^2\theta_2 \, d\theta_2 = \pi$$

$$\int_0^{2\pi} \cos\theta_2 \sin\theta_2 \, d\theta_2 = \int_0^{2\pi} \frac{\sin(2\theta_2)}{2} d\theta_2 = -\frac{\cos(2\theta_2)}{4}\Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3)\, d\theta_2 = \pi \cos\theta_1 \cos\theta_3 + \pi \sin\theta_1 \sin\theta_3$$

$$= \pi \cos(\theta_1-\theta_3)$$

Alternative method

$$\theta = \theta_1 - \theta_3 \implies \theta_3 = \theta_1 - \theta$$

$$\int_0^{2\pi} \cos(\theta_1-\theta_2)\cos(\theta_2-\theta_1+\theta)\, d\theta_2 = \pi\cos(\theta)$$

$$\theta' = \theta_2 - \theta_1$$

$$\int_0^{2\pi} \cos\theta' \cos(\theta'+\theta)\, d\theta' = \pi\cos(\theta)$$

$$\int_0^{2\pi} \cos(\theta')\cos(\theta'+\theta)\, d\theta' \equiv f(\theta)$$

$$f(0) = \int_0^{2\pi} \cos^2(\theta')\, d\theta' = \pi$$

$$f'(\theta) = -\int_0^{2\pi} \cos(\theta')\sin(\theta'+\theta)\, d\theta' \qquad f'(0) = 0$$

$$f''(\theta) = -\int_0^{2\pi} \cos(\theta')\cos(\theta'+\theta)\, d\theta' = -f(\theta)$$

$$f''(\theta) + f(\theta) = 0 \implies f(\theta) = A\cos\theta + B\sin\theta$$

$$f'(0) = 0 \implies B = 0$$

$$f(0) = \pi \implies A = \pi$$

3. $T \gg J$ $\qquad \beta J \ll 1$

$$Z = \int D\theta_r \, e^{\beta J \sum_{\langle rr' \rangle} \cos(\theta_r - \theta_{r'})} = \int D\theta_r \prod_{\langle rr' \rangle} e^{\beta J \cos(\theta_r - \theta_{r'})}$$

$$Z \simeq \int D\theta_r \prod_{\langle rr' \rangle} \left(1 + \beta J \cos(\theta_r - \theta_{r'}) + \ldots \right)$$

$$\simeq \int D\theta_r \left(1 + \sum_{\langle rr' \rangle} \beta J \cos(\theta_r - \theta_{r'}) + \ldots \right)$$

$$= (2\pi)^N + \beta J \sum_{\langle rr' \rangle} \cancel{\int d\theta_r d\theta_{r'} \cos(\theta_r - \theta_{r'})}$$

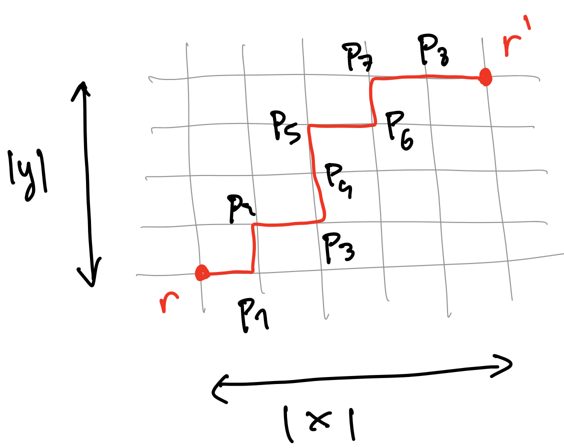
$$+ (\beta J)^2 \sum_{\substack{\langle rr' \rangle \\ \langle pp' \rangle}} \int d\theta_r d\theta_{r'} d\theta_p d\theta_{p'} \cos(\theta_r - \theta_{r'}) \cos(\theta_p - \theta_{p'})$$

only the term with $\langle rr' \rangle = \langle pp' \rangle$ survives

$$+ \ldots$$

$$\langle \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'} \rangle \equiv C(|\vec{r} - \vec{r}'|) = \frac{1}{Z} \int D\theta_r \cos(\theta_r - \theta_{r'}) \prod_{\langle pp' \rangle} e^{K \cos(\theta_p - \theta_{p'})}$$

$$C(|r - r'|) = \frac{1}{Z} \int D\theta_r \cos(\theta_r - \theta_{r'}) \prod_{\langle pp' \rangle} \left(1 + K \cos(\theta_p - \theta_{p'})\right)$$



the only terms that survive (at the lowest order in K) are of the form

$$K^{|r-r'|_1} \int D\theta_r \cos(\theta_r - \theta_{r'}) \cos(\theta_r - \theta_{p_1}) \cos(\theta_{p_1} - \theta_{p_2})$$

$$\times \ldots \times \cos(\theta_{p_7} - \theta_{p_8}) \cos\theta(p_8 - r')$$

$$\int \cos(\theta_r - \theta_{p_1}) \cos(\theta_{p_1} - \theta_{p_2}) d\theta_{p_1} = \pi \cos(\theta_r - \theta_{p_2})$$

$$\int \cos(\theta_r - \theta_{p_2}) \cos(\theta_{p_2} - \theta_{p_3}) d\theta_{p_2} = \pi \cos(\theta_r - \theta_{p_3})$$

$$\vdots$$

$$\int \cos(\theta_r - \theta_{p_8}) \cos(\theta_{p_8} - \theta_{r'}) d\theta_{p_8} = \pi \cos(\theta_r - \theta_{r'})$$

$$\int d\theta_r \, d\theta_{r'} \cos^2(\theta_r - \theta_{r'}) = 2\pi^2$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \frac{1}{(2\pi)^N} (2\pi)^{N - \|r - r'\|_1} \pi^{\|r - r'\|_1} K^{\|r - r'\|_1} \mathcal{N}(|r - r'|)$$

$$C(|r - r'|) \simeq \left(\frac{K}{2}\right)^{\|r - r'\|_1} \mathcal{N}(|r - r'|)$$

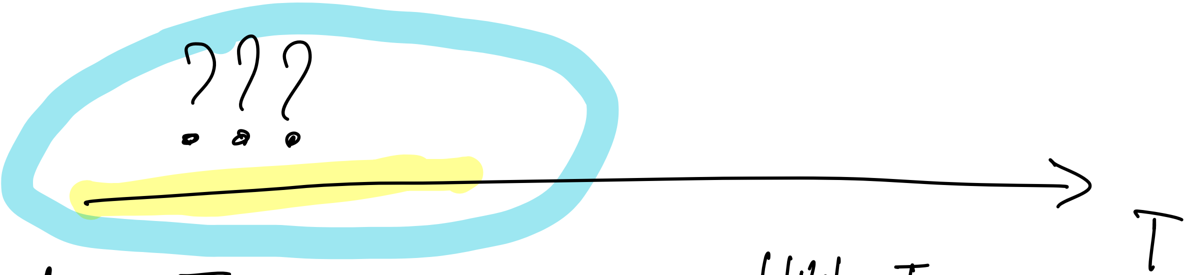
$$C(|r - r'|) \leq K^{\|r - r'\|_1} = K^{|x| + |y|}$$

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y| \geq |x|^2 + |y|^2 = r^2$$

$$K << 1 \qquad K^{|x| + |y|} \leq |K|^r$$

$$C(|r - r'|) \leq K^r = e^{r \ln K} = e^{-r/\xi}$$

$$\xi \simeq -\frac{1}{\ln K} = -\frac{1}{\ln J/T}$$



Low-T

power law correlations

$\xi = \infty \quad \chi = \infty$

critical phase

$$\langle m \rangle \propto \left(\frac{1}{L}\right)^{\frac{k_B T}{4\pi J}} \xrightarrow[L \to +\infty]{} 0$$

High-T

exponential correlations

"standard" paramagnetic phase