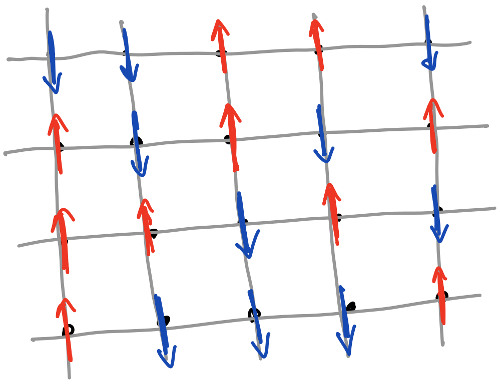
TD — The Mean-field Approach

1 — Mean-field approximation(s) for the Ising model



$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$$

$s_i = \pm 1$ $\qquad J > 0$

1.1 Preliminary questions

a) Ground states

- $h \neq 0$ $\rightarrow$ All the spins aligned to the direction of the external magnetic field : $\underline{s_i = \text{sign}(h)}$

$h > 0 \quad \rightarrow$ All spins up

$h < 0 \quad \rightarrow$ All spins down

$$\boxed{E_{GS} = -\frac{JzN}{2} - hN}$$

z = "degree" or "connectivity" of the lattice

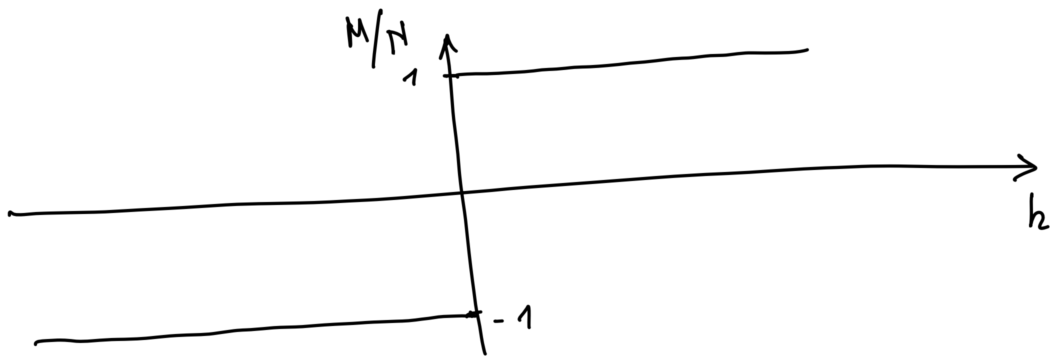
\# of neighbors of each node

For an euclidean hypercubic lattice (square, cubic, ...) $z = 2d$

• $h=0$ $\longrightarrow$ two degenerate ground states with all the spins aligned $s_i=+1$ or $s_i=-1$ $\forall i$

$$E_{GS} = -\frac{JzN}{2}$$

Magnetization vs h at $T=0$ $\qquad M = \sum_i^N s_i$



b) $h=0$ $\qquad H = -J \sum_{\langle ij \rangle} s_i s_j$

Reflexion with respect to the Oxy plane $\longrightarrow$ $\mathbb{Z}_2$ symmetry
(Inversion of the z-axis)

$s_i \longrightarrow -s_i$

$H[\{-s_i\}] = H[\{s_i\}]$ $\qquad H$ is INVARIANT

$$\langle O \rangle = \frac{\mathrm{Tr}\; O\, e^{-\beta H}}{\mathrm{Tr}\; e^{-\beta H}} = \frac{\sum_{\{s_i\}} O[\{s_i\}]\, e^{-\beta H[\{s_i\}]}}{\underbrace{\sum_{\{s_i\}} e^{-\beta H[\{s_i\}]}}_{\equiv Z}}$$

$$\langle M \rangle = \frac{\sum_{\{s_i\}} \left(\sum_{i=1}^N s_i\right) e^{\beta J \sum_{\langle ij \rangle} s_i s_j}}{\sum_{\{s_i\}} e^{\beta J \sum_{\langle ij \rangle} s_i s_j}} \qquad \tilde{s}_i = -s_i$$

$$\langle M \rangle = \frac{\sum_{\{\tilde{S}_i\}} \left(\sum_i -\tilde{S}_i\right) e^{\beta J \sum_{\langle ij \rangle} (-\tilde{S}_i)(-\tilde{S}_j)}}{\sum_{\{\tilde{S}_i\}} e^{\beta J \sum_{\langle ij \rangle} (-\tilde{S}_i)(-\tilde{S}_j)}} = -\langle M \rangle$$

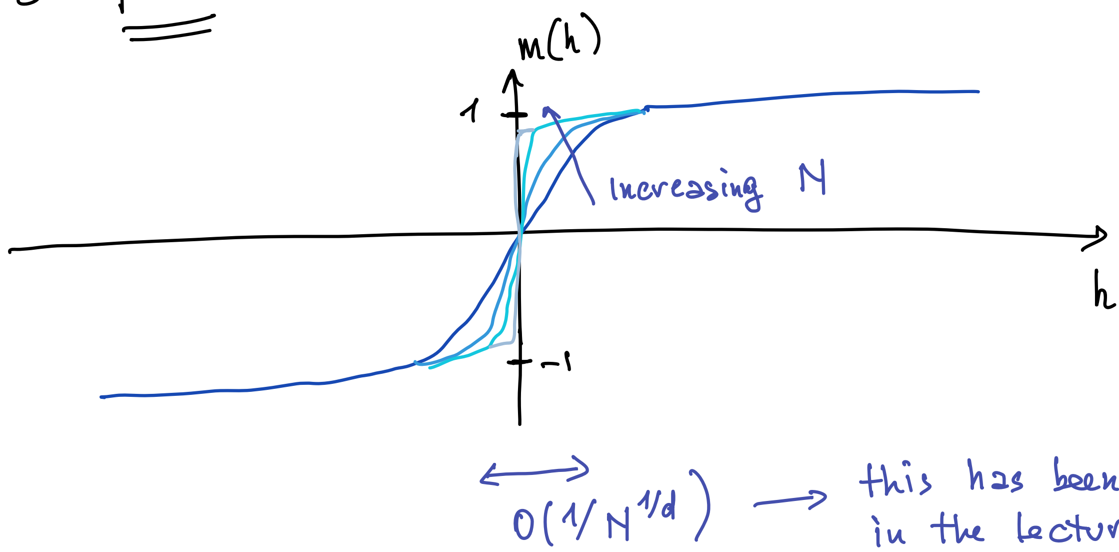
$$\langle M \rangle = -\langle M \rangle \implies \langle M \rangle = 0$$

c) Spontaneous symmetry of the $\mathbb{Z}_2$ invariance

this can only happen for $N \to \infty$

$m = \frac{\langle M \rangle}{N}$ INTENSIVE MAGNETIZATION (magnetization per spin)

$m(h)$ at finite N at low temperature $(T \lesssim T_c)$



this has been explained in the lectures, see Peierls criterion

for $T < T_c$:

$$\lim_{h \to 0^+} \lim_{N \to +\infty} m(h,N) \neq \lim_{N \to \infty} \lim_{h \to 0^+} m(h,N)$$

$$\vee \qquad\qquad\qquad\qquad \parallel$$

$$0 \qquad\qquad\qquad\qquad 0$$

The order of the limits does not commute

$m(h, N)$ becomes discontinuous for $N \to \infty$

$m = \frac{\partial \ln Z}{\partial \beta h}$ $\Rightarrow$ The free-energy becomes non analytic for $N \to \infty$

Z is a sum of analytic functions

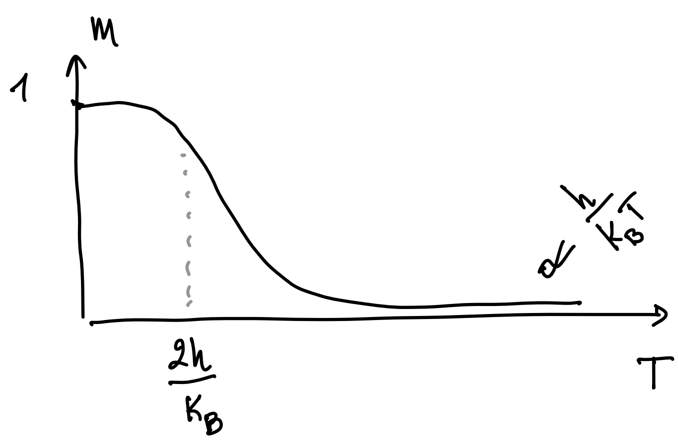
F can be non-analytic only if the number of terms in the sum becomes infinite

d) $J = 0$ $\qquad H = -h \sum_i S_i$ $\qquad$ independent spins

$$Z = \sum_{\{S_i = \pm 1\}} e^{\beta h \sum_i S_i} = \prod_i \sum_{S_i = \pm 1} e^{\beta h S_i} = \left(2 \operatorname{ch}(\beta h)\right)^N$$

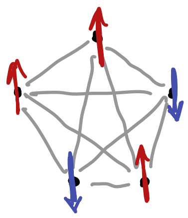
$$F = -K_B T \ln Z = -\frac{N}{\beta} \ln\left(2 \operatorname{ch}(\beta h)\right)$$

$$M = \frac{\partial \ln Z}{\partial (\beta h)} = \frac{\partial}{\partial \beta h} N \ln\left(2 \operatorname{ch}(\beta h)\right) = N \operatorname{th}(\beta h)$$



$\longrightarrow$ energy cost to flip one spin

1.2 The "fully-connected" approximation



$$H = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - h\sum_i s_i$$

a) $$\sum_{i\neq j} s_i s_j = \sum_i s_i \cdot \sum_j s_j - \sum_i (s_i)^2 = \Big(\sum_i s_i\Big)^2 - N$$

$$H = \underbrace{-\frac{J}{2N}\Big(\sum_i s_i\Big)^2}_{O(N)} + \underbrace{\cancel{\frac{J}{2}}}_{O(1)} \underbrace{- h\sum_i s_i}_{O(N)}$$

$$H[\{s_i\}] = H[M] = -\frac{J}{2N}M^2 - hM + o(1)$$

b) $$Z = \sum_{\{s_i=\pm 1\}} e^{\frac{\beta J}{2N}M^2 + \beta h M}$$

Hubbard-Stratonovich transformation to decouple the quadratic term

$$e^{\frac{\beta J}{2N}M^2} = e^{\overset{\sigma^{-2}}{\frac{\beta J N}{2}}\big(\overset{x}{\frac{M}{N}}\big)^2} = \int \frac{dz}{\sqrt{\frac{2\pi}{\beta J N}}}\, e^{-\frac{\beta J N}{2}z^2 + \beta J \cancel{N} z \frac{M}{\cancel{N}} + \beta h M}$$

$$Z = \sqrt{\frac{\beta J N}{2\pi}} \int dz \; e^{-\frac{\beta J N}{2} z^2} \sum_{\{s_i = \pm 1\}} e^{\beta (Jz + h) M[\{s_i\}]}$$

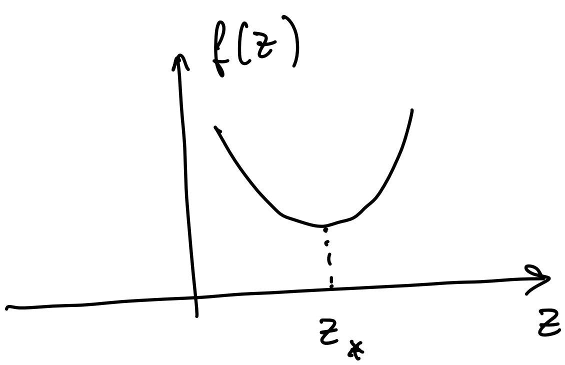
$$\sum_{\{s_i = \pm 1\}} e^{\beta (Jz+h) \sum_i s_i} = \Big(2\,\mathrm{ch}\big(\beta(Jz+h)\big)\Big)^N = e^{N \ln\left[2\,\mathrm{ch}\left(\beta(Jz+h)\right)\right]}$$

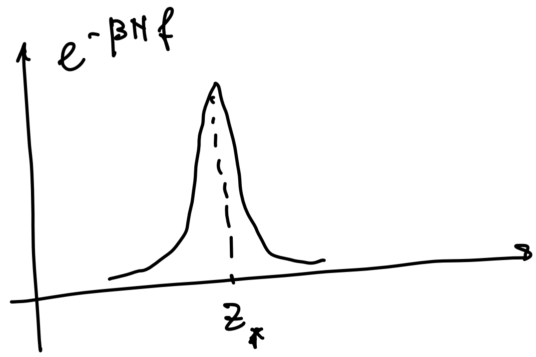
$$\boxed{Z = \sqrt{\frac{\beta J N}{2\pi}} \int dz \; e^{-N\beta\left[\frac{Jz^2}{2} - \frac{1}{\beta} \ln\left(2\,\mathrm{ch}\left(\beta(Jz+h)\right)\right)\right]}}$$

$$Z = \int dz \; e^{-N\beta f(z) + \cancel{\frac{1}{2} \ln \frac{\beta J N}{2\pi}}} \rightsquigarrow O(\ln N)$$

sub-leading

c) In the $N \to \infty$ limit the integral over z is dominated by the minimum of f (plus subleading logarithmic corrections)





$$f(z) \simeq f(z_*) + \frac{1}{2} f''(z_*) (z - z_*)^2 + \dots \qquad f''(z_*) > 0$$

$$\int dz \; e^{-\beta N f(z)} \simeq e^{-\beta N f(z_*)} \int dz \; e^{-\frac{\beta N f''(z_*)}{2} (z - z_*)^2}$$

$$Z \simeq e^{-\beta N f(z_*)} \sqrt{\frac{2\pi}{\beta N f''(z_*)}} = e^{-\beta N f(z_*) + \cancel{\frac{1}{2} \ln \frac{2\pi}{\beta N f''(z_*)}}}$$

$\hookrightarrow O(\ln N)$

$$Z \underset{N \to \infty}{\simeq} e^{-\beta N f(z_*)}$$

$z_* \to$ absolute minimum of $f(z)$

Free - energy : $F \simeq N f(z_*)$

$$f(z) = J\frac{z^2}{2} - \frac{1}{\beta} \ln 2 \operatorname{ch}(\beta(Jz+h))$$

$$\frac{\partial f}{\partial z} = Jz - \frac{1}{\cancel{\beta}} \operatorname{th}(\beta(Jz+h)) \cdot \cancel{\beta} J$$

Self-consistent equation : $\boxed{z_* = \operatorname{th}[\beta(Jz_* + h)]}$

$$\langle M \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial h} = -\frac{1}{\beta} \frac{\partial}{\partial h} \beta N f(z_*)$$

$$m = \frac{\langle M \rangle}{N} = -\frac{1}{\cancel{\beta}} \frac{\partial}{\partial h} \cancel{\beta} f(z_*) = -\frac{\partial}{\partial h}\left[\frac{Jz_*^2}{2} - \frac{1}{\beta} \ln 2\operatorname{ch}(\beta(Jz_*+h))\right]$$

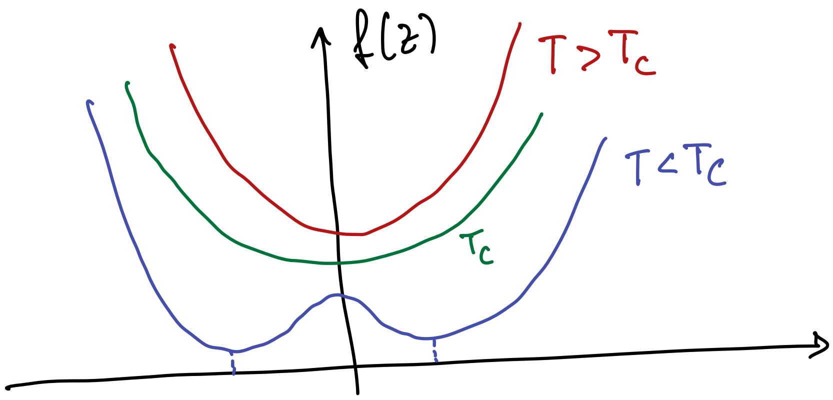
$$m = -\left[J z_* \cdot \frac{\partial z_*}{\partial h} - \frac{1}{\cancel{\beta}} \operatorname{th}(\beta(Jz_*+h)) \frac{\partial \cancel{\beta}(h + Jz_*)}{\partial h}\right]$$

$$m = \underbrace{\text{th}\left(\beta(Jz_* + h)\right)}_{= z_*} - J\,\underbrace{\left[z_* - \text{th}\left(\beta(Jz_* + h)\right)\right]}_{= 0}\,\frac{\partial z_*}{\partial h}$$

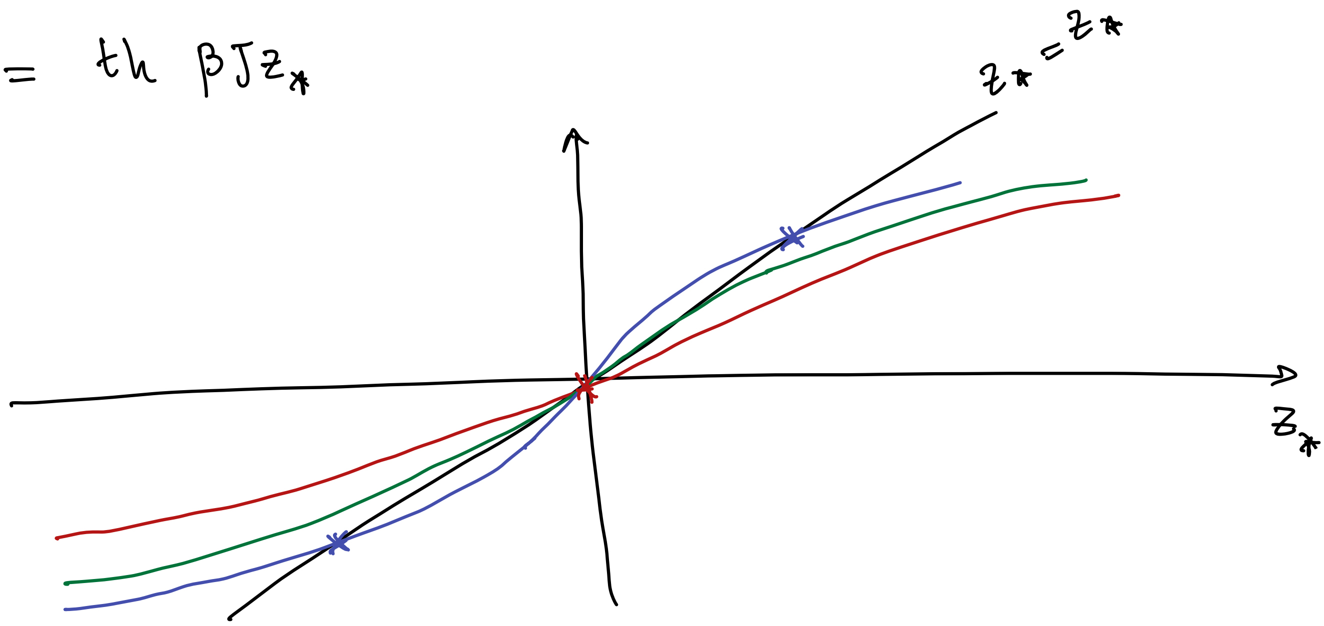
(self-consistent equation)

$$\boxed{m = z_*}$$

d) $h = 0$ $\qquad f(z) = \frac{Jz^2}{2} - \frac{1}{\beta}\ln\left[2\,\text{ch}(\beta J z)\right]$



$$z_* = \text{th}\ \beta J z_*$$



$T \geq T_c \longrightarrow$ one solution at $z_* = 0$

$T < T_c \longrightarrow$ three solutions, $z_* = 0$ et $\pm z_*$

At T_c :

$$z_* = \beta_c J z_* + \dots$$

$$\boxed{\beta_c = \frac{1}{J}}$$

$T \lesssim T_c$

$$\mathrm{th}(x) \simeq x - \frac{x^3}{3} + \dots$$

$$z_* \simeq \beta J z_* - \frac{(\beta J z_*)^3}{3}$$

$$(\beta J - 1) z_* = \frac{(\beta J)^3}{3} z_*^3$$

$z_* = 0$ ou $z_* = \left(3 \, \frac{\beta J - 1}{(\beta J)^3}\right)^{1/2}$

$$\boxed{T = T_c (1 - \epsilon)}$$

$$\beta = \frac{1}{T_c(1-\epsilon)}$$

$[k_B \equiv 1]$

$$\boxed{\epsilon = \frac{T_c - T}{T_c} = \frac{\beta - \beta_c}{\beta} \simeq \frac{\beta - \beta_c}{\beta_c}}$$

$$\beta \simeq \beta_c (1 + \epsilon)$$

$$m \simeq \left(3 \, \frac{\cancel{\beta_c J}(1+\epsilon) - 1}{[\cancel{\beta_c J}(1+\epsilon)]^3}\right)^{1/2}$$

$\beta_c J = 1$

$$m \simeq \left[3\left(\epsilon + o(\epsilon^2)\right)\right]^{1/2} \simeq \sqrt{3\epsilon} = \sqrt{3\,\frac{T_c - T}{T_c}} \qquad \boxed{\beta = 1/2}$$

f) $$\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0} = \left.\frac{\partial z_*}{\partial h}\right|_{h=0}$$

$$f(m) = \frac{Jm^2}{2} - \frac{1}{\beta} \ln 2\,\mathrm{ch}(\beta(Jm+h))$$

$\beta(Jm+h) \ll 1$

$$\mathrm{ch}(x) \simeq 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$\ln(1+\epsilon) \simeq \epsilon - \frac{\epsilon^2}{2} + \dots$$

$$\ln[2\,\mathrm{ch}(x)] \simeq \ln 2 + \ln\left(1 + \frac{x^2}{2} + \frac{x^4}{24}\right) \simeq \ln 2 + \frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\frac{x^4}{4}$$

$$= \ln 2 + \frac{x^2}{2} - \frac{x^4}{12}$$

$$f(m) \simeq \frac{Jm^2}{2} - \frac{1}{\beta}\ln 2 - \frac{1}{\beta}\frac{\beta^2(Jm+h)^2}{2} + \frac{\beta^3(Jm+h)^4}{12}$$

$(Jm+h)^2$, $(Jm+h)^4$ $\longrightarrow$ Keep only linear terms in h

$\chi \rightarrow$ linear response

$$(Jm+h)^2 \simeq (Jm)^2 + 2Jmh + o(h^2)$$

$$(Jm+h)^4 \simeq (Jm)^4 + 4(Jm)^3 h + o(h^2)$$

$$f(m) \simeq \left(\frac{J}{2} - \frac{\beta J^2}{2}\right) m^2 - \frac{1}{\beta}\ln 2 + \frac{\beta^3 J^4 m^4}{12} - \beta J m h + \dots$$

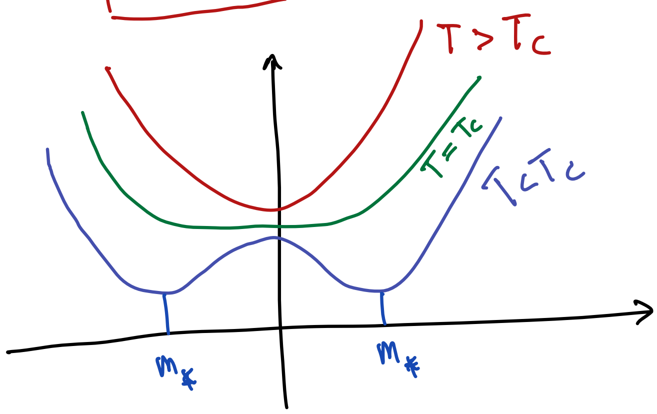
$$f(m) \simeq -\frac{1}{\beta}\ln 2 + \frac{J}{2}\left(1-\frac{\beta}{\beta_c}\right)m^2 + \frac{\beta^3 J^4}{12}m^4 - \beta J m h + \dots$$

$$\beta > \beta_c \quad (T < T_c) \longrightarrow 1 - \frac{\beta}{\beta_c} < 0$$

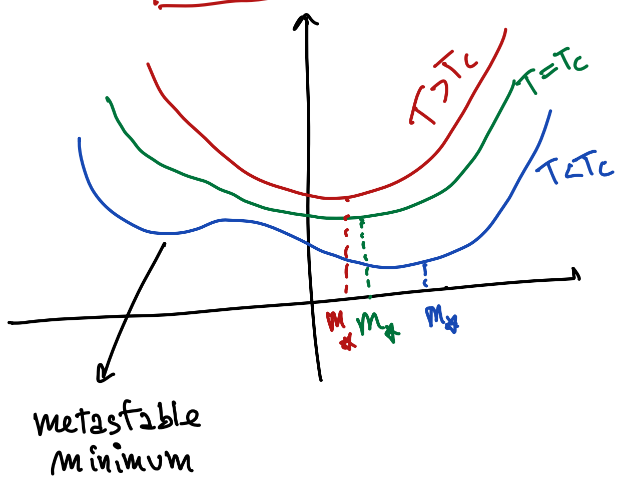
$$\beta < \beta_c \quad (T > T_c) \longrightarrow 1 - \frac{\beta}{\beta_c} > 0$$

The quadratic term of the GL free-energy changes sign at T_c

$\boxed{h = 0}$



$\boxed{h > 0}$



$$\frac{\partial f}{\partial m} = J\left(1-\frac{\beta}{\beta_c}\right)m + \frac{\beta^3 J^4}{3}m^3 - \beta J h$$

- $T > T_c \longrightarrow m_* \simeq \dfrac{\beta J h}{J\left(1-\frac{\beta}{\beta_c}\right)} = \dfrac{\beta h}{1-\frac{T_c}{T}} = \dfrac{h}{T-T_c}$

- $T < T_c \longrightarrow \left(1-\dfrac{\beta}{\beta_c}\right)m_* + \dfrac{\beta^3 J^3}{3}m_*^3 = \beta h$

$$\left[\left(1-\frac{\beta}{\beta_c}\right) + \beta^3 J^3 m_*^2\right]\frac{\partial m_*}{\partial h} = \beta$$

$$\chi = \left.\frac{\beta}{\left(1-\frac{\beta}{\beta_c}\right) + \beta^3 J^3 m_*^2}\right|_{h=0}$$

$T > T_c \qquad m_*\big|_{h=0} = 0$

$$\chi = \frac{\beta}{1 - \frac{\beta}{\beta_c}} \simeq \frac{1}{T - T_c}$$

we recover the previous result

$T < T_c \qquad m_*\big|_{h=0} \simeq \pm \sqrt{3 \frac{T_c - T}{T_c}} = \pm \sqrt{3\left(\frac{1}{\beta_c} - \frac{1}{\beta}\right)\beta_c}$

$$= \pm \sqrt{3 \frac{\beta - \beta_c}{\beta}} \simeq \pm \sqrt{3 \frac{\beta - \beta_c}{\beta_c}}$$

$$\chi = \frac{\beta}{1 - \frac{\beta}{\beta_c} + \beta^3 J^3 \left(3 \frac{\beta - \beta_c}{\beta}\right)} = \frac{\beta}{1 - \frac{\beta}{\beta_c} + 3\left(\frac{\beta}{\beta_c}\right)^3 \left(1 - \frac{\beta_c}{\beta}\right)}$$

$$\beta = \beta_c (1 + \epsilon)$$

$$\chi = \frac{\beta}{-\epsilon + 3(1+\epsilon)^3 \epsilon} \simeq \frac{\beta}{-\epsilon + 3\epsilon} = \frac{\beta}{2\epsilon}$$

$$\chi \simeq \begin{cases} \dfrac{1}{T - T_c} & T > T_c \\[2ex] \dfrac{1}{2(T_c - T)} & T < T_c \end{cases}$$

$$\boxed{\gamma = 1}$$

Alternative strategy to compute χ

$$m_* = \text{th}\left(\beta(Jm_* + h)\right)$$

$$\frac{\partial m_*}{\partial h} = \left[1 - \text{th}^2\left(\beta(Jm_* + h)\right)\right] \beta \left(J \frac{\partial m_*}{\partial h} + 1\right)$$

$$\text{th}^2\left(\beta(Jm_* + h)\right) = m_*^2$$

$$\frac{\partial m_*}{\partial h} = \left(1 - m_*^2\right) \beta \left(J \frac{\partial m_*}{\partial h} + 1\right)$$

$$\frac{\partial m_*}{\partial h}\left(1 - \beta J\left(1 - m_*^2\right)\right) = \beta\left(1 - m_*^2\right)$$

$$\chi = \left.\frac{\beta\left(1 - m_*^2\right)}{1 - \beta J\left(1 - m_*^2\right)}\right|_{h=0}$$

$$T > T_c \longrightarrow m_* = 0 \longrightarrow \chi = \frac{\beta}{1 - \beta J} = \frac{\beta}{1 - \frac{\beta}{\beta_c}} = \frac{1}{\cancel{T}} \frac{1}{\frac{T - T_c}{\cancel{T}}}$$

$$T < T_c \longrightarrow m_*^2 = \sqrt{3\epsilon}$$

$$\chi = \frac{\beta(1 - 3\epsilon)}{1 - \frac{\beta}{\beta_c}(1 - 3\epsilon)} \qquad \beta \simeq \beta_c(1 + \epsilon)$$

$$\chi \simeq \frac{\beta_c(1+\epsilon)(1-3\epsilon)}{1 - (1+\epsilon)(1-3\epsilon)} \simeq \frac{\beta_c(1 - 2\epsilon)}{1 - (1 - 2\epsilon)} \simeq \frac{\beta_c}{2\epsilon} = \frac{1}{2(T_c - T)}$$

* Why is it possible to neglect the term $\propto hm^3$ in the free-energy?

$$f(m) \simeq \left(\frac{J}{2} - \frac{\beta J^2}{2}\right) m^2 - \frac{1}{\beta}\ln 2 + \frac{\beta^3 J^4 m^4}{12} - \beta J m h + \frac{\beta^3}{3}(Jm)^3 h + \dots$$

$$f(m) \simeq \frac{J}{2}\left(1 - \frac{\beta}{\beta_c}\right) m^2 - \frac{1}{\beta}\ln 2 + J\left(\frac{\beta}{\beta_c}\right)^3 \frac{m^4}{12} - \beta J m h + \left(\frac{\beta}{\beta_c}\right)^3 \frac{m^3}{3} h$$

$$\frac{df}{dm} = J\left(1 - \frac{\beta}{\beta_c}\right) m + J\left(\frac{\beta}{\beta_c}\right)^3 \frac{m^3}{3} - \beta J h + \left(\frac{\beta}{\beta_c}\right)^3 m^2 h$$

$$\left(1 - \frac{\beta}{\beta_c}\right) m = \beta h - \left(\frac{\beta}{\beta_c}\right)^3 \frac{m^3}{3} - \left(\frac{\beta}{\beta_c}\right)^3 m^2 h$$

- Take the derivative of both sides of the equation wrt h

$$\left(1 - \frac{\beta}{\beta_c}\right) \frac{dm}{dh} = \beta - \left(\frac{\beta}{\beta_c}\right)^3 m^2 \frac{dm}{dh} - \left(\frac{\beta}{\beta_c}\right)^3 m^2 - \left(\frac{\beta}{\beta_c}\right)^3 2m \frac{dm}{dh} \cdot h$$

- Take the limit $h \to 0$ $\qquad m \simeq \sqrt{3\epsilon} \qquad \left.\frac{dm}{dh}\right|_{h=0} = \chi$

$$\underbrace{\left(1 - \frac{\beta}{\beta_c}\right)}_{= -\epsilon} \chi = \beta - \underbrace{\left(\frac{\beta}{\beta_c}\right)}_{= 1} 3\epsilon\, \chi - \underbrace{\left(\frac{\beta}{\beta_c}\right)^3}_{\simeq 1} 3\epsilon$$

$$(-\epsilon + 3\epsilon)\chi \simeq \beta - 3\epsilon$$

$$\chi \simeq \frac{\beta - 3\epsilon}{2\epsilon} = \frac{\beta - 3\epsilon}{2\frac{T_c - T}{T_c}} \simeq \frac{1 - 3(T_c - T)}{2(T_c - T)} = \frac{1}{2(T_c - T)} - \frac{3}{2}$$

This term only gives a sub-leading contribution

g) $C = \frac{\partial \langle E \rangle}{\partial T}$ $\qquad \langle E \rangle = - \frac{\partial \ln Z}{\partial \beta}$

$$\langle E \rangle = - \frac{\partial}{\partial \beta} \left(- N \beta f(m_*) \right) = N \frac{\partial}{\partial \beta} \left(\beta f(m_*) \right)$$

$$h = 0 \longrightarrow f(m) \simeq - \frac{1}{\beta} \ln 2 + \frac{J}{2} \left(1 - \frac{\beta}{\beta_c} \right) m^2 + \frac{\beta^3 J^4}{12} m^4$$

$$T < T_c \rightarrow m_* = 0 \longrightarrow f(m_*) = - \frac{1}{\beta} \ln 2$$

$$T < T_c \longrightarrow m_* \simeq \sqrt{3\epsilon} \longrightarrow f(m_*) = - \frac{1}{\beta} \ln 2 + \frac{J}{2} \left(1 - \frac{\beta}{\beta_c} \right) 3\epsilon + \frac{\beta^3 J^4}{12} \cdot 9\epsilon^2$$

$$f(m_*) \simeq - \frac{1}{\beta} \ln 2 + \frac{J}{2} \left(1 - (1+\epsilon) \right) \cdot 3\epsilon + \frac{3}{4} \beta^3 J^4 \epsilon^2 + o(\epsilon^3)$$

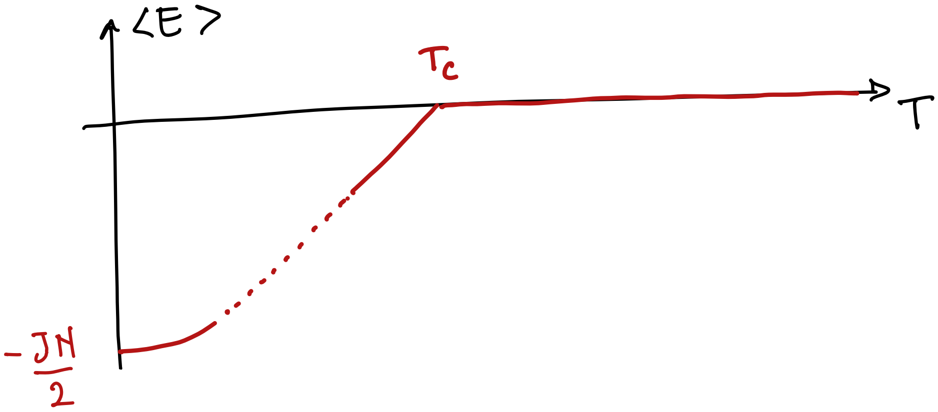
$$= - \frac{1}{\beta} \ln 2 + \frac{3J}{2} \left(\frac{\beta^3 J^3}{2} - 1 \right) \epsilon^2 + o(\epsilon^3)$$

$$f(m_*) \simeq \begin{cases} - \frac{1}{\beta} \ln 2 & T > T_c \\ - \frac{1}{\beta} \ln 2 - \frac{3J}{4} \epsilon^2 & T < T_c \end{cases}$$

$$\langle E \rangle = N \frac{\partial}{\partial \beta} \left(\beta f(m_*) \right) = N \frac{\partial}{\partial \epsilon} \left(\beta f(m_*) \right) \frac{\partial \epsilon}{\partial \beta}$$

$$\frac{\partial \epsilon}{\partial \beta} \simeq \frac{1}{\beta_c}$$

$$\langle E \rangle = \frac{N}{\beta_c} \frac{\partial \beta f(m_*)}{\partial \epsilon} \simeq \begin{cases} 0 & T > T_c \\ -\dfrac{3\epsilon}{2} & T < T_c \end{cases}$$



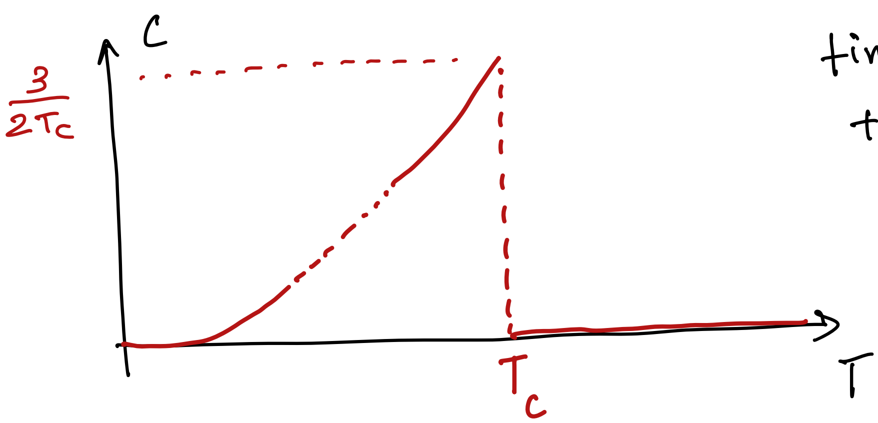
$$C = \frac{d\langle E \rangle}{dT} = \frac{d\langle E \rangle}{d\epsilon} \frac{d\epsilon}{dT}$$

$$\frac{d\epsilon}{dT} = -\frac{1}{T_c}$$

$$C \simeq \begin{cases} 0 & T > T_c \\ \dfrac{3}{2T_c} & T < T_c \end{cases}$$

$C \propto |T - T_c|^{\alpha}$ $\alpha = 0$

finite jump at the transition

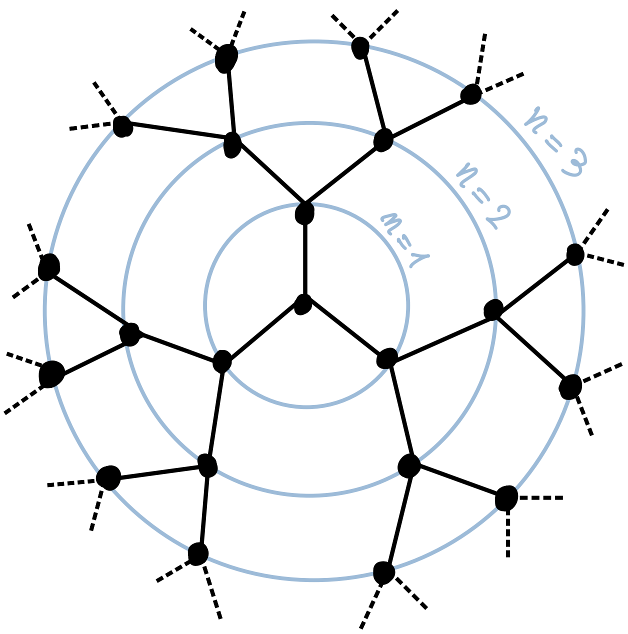


$$\boxed{\alpha + 2\beta + \gamma = 2}$$

SCALING RELATION

(See lectures' notes)

1.3 The Bethe Approximation



a)

n	$S(n)$	N
0	1	1
1	$k+1$	$1+k+1$
2	$k(k+1)$	$1+(k+1)(1+k)$
3	$k^2(k+1)$	$1+(k+1)(1+k+k^2)$
$\vdots$	$\vdots$	
n	$k^{n-1}(k+1)$	$1+(k+1)(1+\ldots+k^{n-1})$

b) $$N = \sum_{n=0}^{R} S(n) = 1 + (k+1)\sum_{m=0}^{R-1} k^m = 1 + (k+1)\,\frac{1-k^R}{1-k}$$

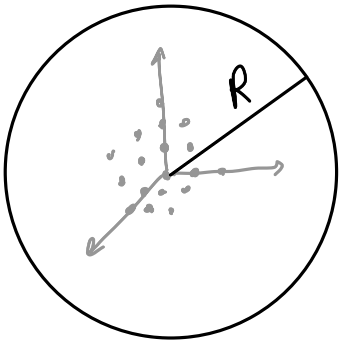
$$N = 1 + (k+1)\,\frac{k^R-1}{k-1} = 1 + \frac{k^{R+1}-k+k^R-1}{k-1}$$

$$= \frac{\cancel{k}-1+k^{R+1}-\cancel{k}+k^R-1}{k-1}$$

$$= \frac{k^{R+1}+k^R-2}{k-1}$$

$R=3,\ k=2 \longrightarrow N=22$ (as in the figure)

c) Euclidean lattices



$$N(R) \propto R^d$$

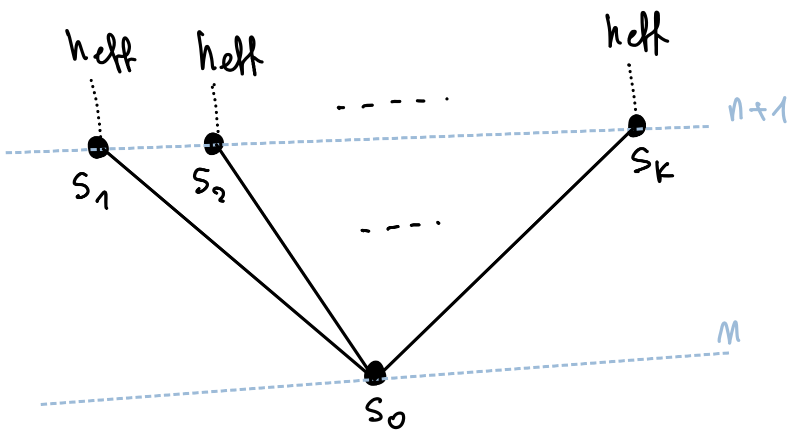
$$\lim_{R\to\infty} \frac{\ln N}{\ln R} = d$$

Bethe lattice : $\lim_{R\to\infty} \frac{\ln N}{\ln R} = \lim_{R\to\infty} \frac{R \ln K}{\ln R} \to +\infty \quad (d \to \infty)$

Effectively ∞-d lattice

d) $$P(s=+1) = \frac{e^{\beta h_{eff}}}{e^{\beta h_{eff}} + e^{-\beta h_{eff}}} = \frac{e^{\beta h_{eff}}}{2\,\mathrm{ch}(\beta h_{eff})}$$

$$\langle m \rangle = P(s=+1) - P(s=-1) = \mathrm{th}(\beta h_{eff})$$



$$H = -J s_0 \sum_{i=1}^{k} s_i - h s_0 - h_{eff} \sum_{i=1}^{k} s_i$$

e) $$Z = \sum_{s_0, s_1, \dots, s_K} e^{\beta J s_0 \sum_i^k s_i + \beta h s_0 + \beta h_{eff} \sum_i^k s_i}$$

$$Z = \sum_{s_0, s_1, \dots, s_K} e^{\beta h s_0} \prod_i^K e^{\beta (J s_0 + h_{eff}) s_i}$$

$$Z = \sum_{s_0 = \pm 1} e^{\beta h s_0} \prod_i^K \sum_{s_i = \pm 1} e^{\beta (J s_0 + h_{eff}) s_i}$$

f) $$\sum_{s = \pm 1} e^{\beta (J s_0 + h_{eff}) s} = c\, e^{\beta u s_0}$$

$s_0 = +1$ $$e^{\beta(J + h_{eff})} + e^{-\beta(J + h_{eff})} = c\, e^{\beta u}$$

$s_0 = -1$ $$e^{\beta(-J + h_{eff})} + e^{-\beta(-J + h_{eff})} = c\, e^{-\beta u}$$

- Take the sum and the difference of the two equations

$$* \quad c\left(e^{\beta u} + e^{-\beta u}\right) = e^{\beta(J + h_{eff})} + e^{-\beta(J + h_{eff})} + e^{\beta(-J + h_{eff})} + e^{-\beta(-J + h_{eff})}$$
$$= \left(e^{\beta J} + e^{-\beta J}\right)\left(e^{\beta h_{eff}} + e^{-\beta h_{eff}}\right)$$

$$* \quad c\left(e^{\beta u} - e^{-\beta u}\right) = e^{\beta(J + h_{eff})} + e^{-\beta(J + h_{eff})} - e^{\beta(-J + h_{eff})} - e^{-\beta(-J + h_{eff})}$$
$$= \left(e^{\beta J} - e^{-\beta J}\right)\left(e^{\beta h_{eff}} - e^{-\beta h_{eff}}\right)$$

take the ratio $\longrightarrow$ $$\text{th}(\beta u) = \text{th}(\beta J)\, \text{th}(\beta h_{eff})$$

$$\boxed{\begin{aligned} u &= \frac{1}{\beta}\,\mathrm{atanh}\left[\mathrm{th}(\beta J)\,\mathrm{th}(\beta h_{eff})\right] \\ c &= \frac{\mathrm{ch}(\beta J)\,\mathrm{ch}(\beta h_{eff})}{2\,\mathrm{ch}(\beta u)} \end{aligned}}$$

g)
$$Z = \sum_{S_0=\pm 1} e^{\beta h s_0} \prod_i \underbrace{\sum_{S_i=\pm 1} e^{\beta(J s_0 + h_{eff}) s_i}}_{= \, c\, e^{\beta u s_0}}$$

$$Z = \underbrace{c^k}_{\downarrow\, C} \sum_{S_0=\pm 1} e^{\beta \overbrace{(h + k u)}^{= \, h_{eff}} s_0}$$

$$C = c^k$$

$$\boxed{h_{eff} = h + \frac{k}{\beta}\,\mathrm{atanh}\left[\mathrm{th}(\beta J)\,\mathrm{th}(\beta h_{eff})\right]}$$

SELF-CONSISTENT EQUATION FOR h_{eff}

h) $\langle m \rangle = \mathrm{th}(\beta h_{eff})$

$$\boxed{\langle m \rangle = \mathrm{th}\left[\beta h + k\,\mathrm{atanh}\left[\mathrm{th}(\beta J)\langle m \rangle\right]\right]}$$

$\rightarrow$ Self-consistent equation for $\langle m \rangle$

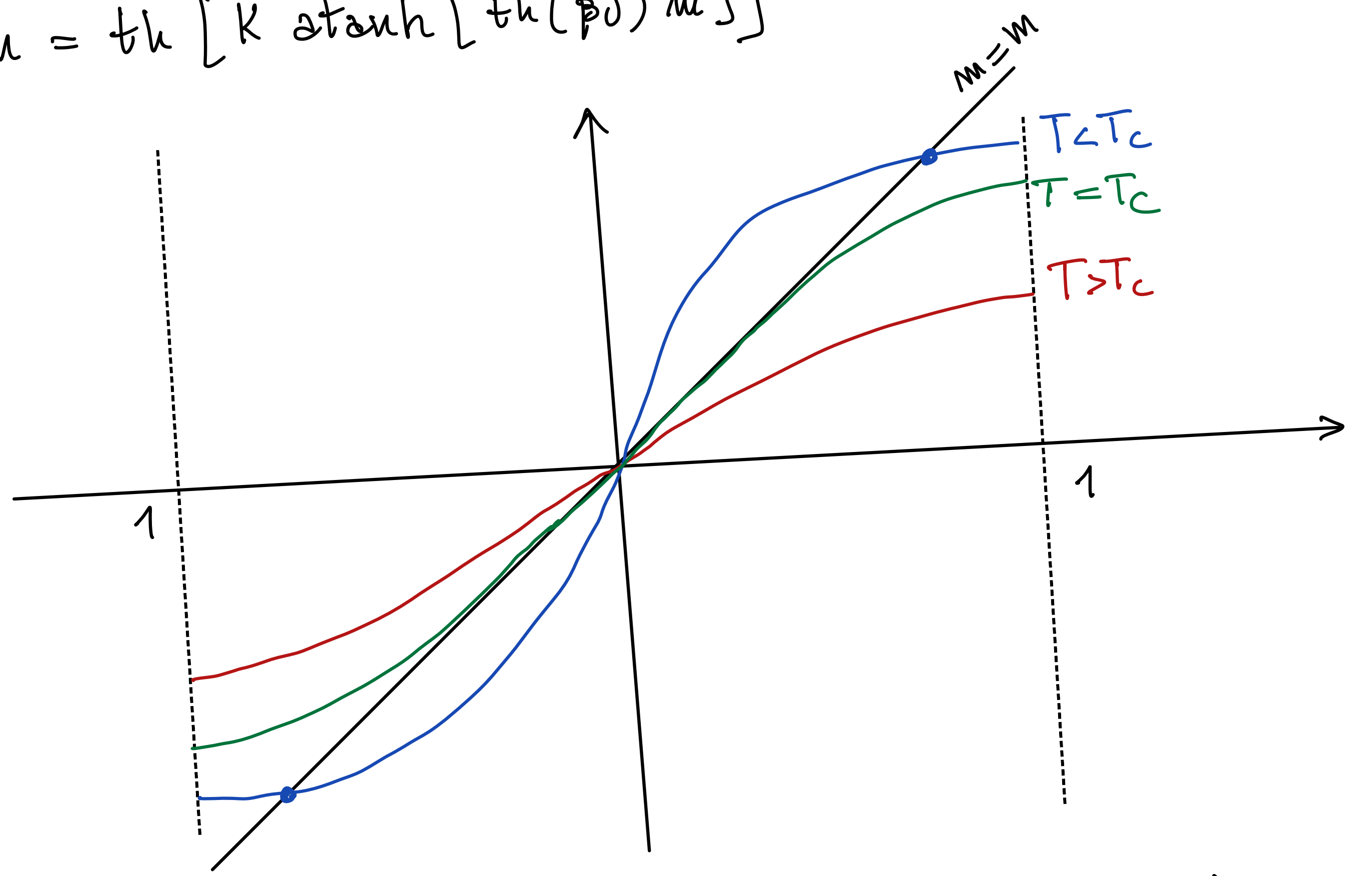
for $h = 0$:

$$\langle m \rangle = \mathrm{th}\left[k\,\mathrm{atanh}\left[\mathrm{th}(\beta J)\langle m \rangle\right]\right]$$

Similar but more complicated then Curie-Weiss an fully-connected approximations

i) $\boxed{h=0}$

$$m = \text{th}\left[K \,\text{atanh}\left[\text{th}(\beta J)\, m\right]\right]$$



Critical temperature : $\frac{d}{dm}\left(\text{th}\left[K \,\text{atanh}\left[\text{th}(\beta J)\, m\right]\right]\right)_{m=0} = 1$

expand around $m=0$

$$\text{atanh}\left[\text{th}(\beta J)\, m\right] \approx \text{th}(\beta J)\, m$$

$$\text{th}\left[K\, \text{th}(\beta J)\, m\right] \approx K\, \text{th}(\beta J)\, m$$

$$\text{th}(\beta_c J) = 1/K \qquad \beta_c J = \text{atanh}(1/K)$$

$$\boxed{T_c = \frac{J}{\text{atanh}(1/K)}}$$

$$\operatorname{atan}(1/k) > 1/k \qquad (k \geq 1)$$

$$T_c^{(\text{Bethe})} \leq \frac{J}{1/k} = kJ < (k+1)J = T_c^{(CW)}$$

J) $k+1 = 2d$

$k = 1$ for $d = 1$ $\longrightarrow$ $T_c^{(\text{Bethe})}(d=1) \longrightarrow 0$

No phase transition for $d = 1$

k) $\operatorname{atanh}(x) \simeq x + \frac{x^3}{3}$ $\qquad t \equiv \operatorname{th}(\beta J)$

$$\operatorname{th}\left[k \operatorname{atanh}[\operatorname{th}(\beta J) m]\right] \simeq \operatorname{th}\left[k\left(tm + \frac{(tm)^3}{3}\right)\right]$$

$$\operatorname{th}(x) \simeq x - \frac{x^3}{3}$$

$$\operatorname{th}\left[k\left(tm + \frac{(tm)^3}{3}\right)\right] \simeq k\left(tm + \frac{(tm)^3}{3}\right) - \frac{1}{3}k^3\left(tm + \frac{(tm)^3}{3}\right)^3$$

$$= ktm + \frac{kt^3}{3}m^3 - \frac{1}{3}k^3t^3m^3 + o(m^5)$$

$$= ktm + \frac{k^3t^3}{3}\left(\frac{1}{k^2} - 1\right)m^3 + o(m^5)$$

$$\operatorname{th}(\beta_c J) = 1/k \Rightarrow t_c = 1/k$$

$$m \simeq \frac{t}{t_c} \cdot m + \frac{1}{3}\left(\frac{t}{t_c}\right)^3 (t_c^2 - 1)\, m^3$$

$$T > T_c \Rightarrow \beta < \beta_c \Rightarrow \mathrm{th}(\beta J) < \mathrm{th}(\beta_c J) \Rightarrow t < t_c$$

$$T < T_c \Rightarrow \beta > \beta_c \Rightarrow \mathrm{th}(\beta J) > \mathrm{th}(\beta_c J) \Rightarrow t > t_c$$

$$m\left(\frac{t}{t_c} - 1\right) \simeq \frac{1}{3}\left(\frac{t}{t_c}\right)^3 (1 - t_c^2)\, m^2$$

$$m = \pm \sqrt{\frac{3\left(\frac{t}{t_c} - 1\right)}{\left(\frac{t}{t_c}\right)^3 (1 - t_c^2)}}$$

$$T = T_c (1 - \epsilon) \Rightarrow \epsilon = \frac{T_c - T}{T_c}$$

$$\beta = \beta_c (1 + \epsilon) \Rightarrow \epsilon = \frac{\beta - \beta_c}{\beta_c}$$

$$\begin{aligned} t = \mathrm{th}[\beta_c (1+\epsilon) J] &= \mathrm{th}[\beta_c J + \beta_c J \epsilon] \\ &= \mathrm{th}(\beta_c J) + (1 - \mathrm{th}^2(\beta_c J))\, \beta_c J \epsilon \\ &= t_c + (1 - t_c^2)\, \beta_c J \epsilon \end{aligned}$$

$$\frac{t - t_c}{t_c} = \frac{(1 - t_c^2)\, \beta_c J \epsilon}{t_c}$$

$$m = \pm \sqrt{\frac{3\, \frac{\cancel{(1 - t_c^2)}\, \beta_c J \epsilon}{t_c}}{\left[1 + \frac{(1 - t_c^2)}{t_c} \beta_c J \epsilon\right] \cancel{(1 - t_c^2)}}} \simeq \pm \sqrt{\frac{3 \beta_c J}{t_c}}\, \epsilon$$

$$m = \pm \sqrt{\frac{3\beta_c J}{th(\beta_c J)} \frac{T_c - T}{T_c}}$$

$$\boxed{\beta = 1/2}$$

e) $\langle m \rangle = th\left[\beta h + k \,\mathrm{atanh}\left[th(\beta J)\langle m \rangle\right]\right]$

$$\frac{dm}{dh} = \left(1 - th^2\left[\beta h + k\,\mathrm{atanh}\left[th(\beta J) m\right]\right]\right) \times$$

$$\times \left(\beta + \frac{k}{1-(th(\beta J)\, m)^2}\, th(\beta J) \frac{dm}{dh}\right)$$

$$\frac{dm}{dh} = \left(1 - th^2\left[\beta h + k\,\mathrm{atanh}(tm)\right]\right)\left(\beta + \frac{k}{1-(tm)^2}\, t \frac{dm}{dh}\right)$$

$h \to 0 \qquad m \to m_*(h=0)$

- $T > T_c \qquad m_*(h=0) = 0$

$$\left.\frac{dm}{dh}\right|_{h=0} = \beta + k t \left.\frac{dm}{dh}\right|_{h=0}$$

$$\chi(1 - kt) = \beta \implies \chi = \frac{\beta}{1 - \frac{t}{t_c}} = \frac{t_c}{T(t_c - t)}$$

$t_c - t \simeq -(1 - t_c^2)\beta_c J \epsilon \qquad$ for $T \gtrsim T_c$ $\left[\begin{array}{l} T = T_c(1-\epsilon) \\ \text{with } \epsilon < 0 \end{array}\right]$

$$\chi \simeq \frac{t_c}{T(1-t_c^2)\beta_c J|\epsilon|} \simeq \frac{t_c}{(1-t_c^2)\beta_c J(T-T_c)}$$

$$\boxed{\gamma = 1}$$

- $T < T_c$ $\qquad m_*(h=0) \simeq \pm\sqrt{\frac{3\beta_c J}{t_c}\epsilon}$

$$\left.\frac{dm}{dh}\right|_{h=0} \simeq \left(1 - th^2(Ktm)\right)\left(\beta + \frac{K}{1-(tm)^2}\, t \left.\frac{dm}{dh}\right|_{h=0}\right)$$

$$\chi \simeq \left(1 - (Ktm)^2\right)\left(\beta + \frac{tK}{1-(tm)^2}\chi\right)$$

$$\chi \simeq \left(1 - \left(\frac{t}{t_c}m\right)^2\right)\left(\beta + \frac{t/t_c}{1-(tm)^2}\chi\right)$$

$$m^2 \simeq \frac{3\beta_c J}{t_c}\epsilon \qquad\qquad t = t_c + (1-t_c)^2\beta_c J\epsilon$$

$$1 - \left(\frac{t}{t_c}m\right)^2 \simeq 1 - \frac{3\beta_c J}{t_c}\epsilon = 1 - m^2$$

$$1 - (tm)^2 \simeq 1 - t_c^2 m^2$$

$$\frac{t/t_c}{1-t_c^2m^2} \simeq \frac{t}{t_c}\left(1 + t_c^2 m^2\right) \simeq 1 + t_c^2 m^2 + \frac{(1-t_c)^2}{t_c}\beta_c J\epsilon$$

$$\simeq 1 + \left[3t_c\beta_c J + \frac{(1-t_c)^2}{t_c}\beta_c J\right]\epsilon$$

$$\chi \simeq \left(1 - \frac{3\beta_c J}{t_c}\epsilon\right)\left(\beta + (1+\alpha\epsilon)\chi\right)$$

$$\chi \simeq \beta + (1+\alpha\epsilon)\chi - \frac{3\beta_c J}{t_c}\beta\epsilon - \frac{3\beta_c J}{t_c}\epsilon\chi$$

$$\chi\left(\frac{3\beta_c J}{t_c} - \alpha\right)\epsilon \simeq \beta_c + O(\epsilon)$$