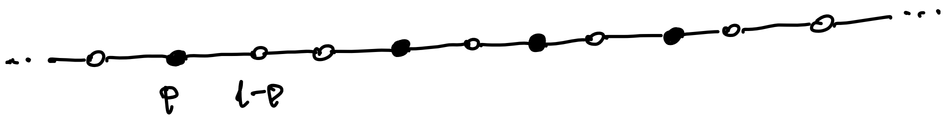
TD : Real space renormalization Group

I | Percolation transition

(A) 1D chain - Exact solution

1.



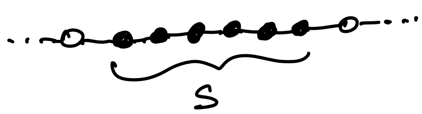
$$\sum_{s=1}^{+\infty} s N_s = N p \Rightarrow \sum_s s n_s = p$$

$\sum_{s=1}^{+\infty} s N_s$: # of nodes in a cluster of size s; Np: # of occupied nodes

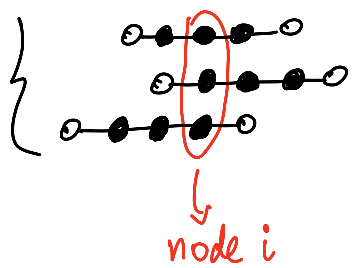
N_s = # of clusters of size s $\Longrightarrow$ $n_s = N_s / N$

P_s = probability that a node belongs to a cluster of size s

$s\, p^s (1-p)^2$



example : $s=3$ $\longrightarrow$ 3 configurations $\longrightarrow$



$p^3(1-p)^2$

the probability that node i belongs to a cluster of size 3 is $3p^3(1-p)^2$

$$\sum_{s=1}^{+\infty} P_s = \text{probability that a node is occupied} = p$$

$$Q = \sum_{s=0}^{+\infty} p^s \qquad \frac{dQ}{dp} = \sum_{s=1}^{+\infty} s p^{s-1} \qquad \sum_{s=1}^{+\infty} s p^s = p \frac{dQ}{dp}$$

$$\sum_{s=1}^{+\infty} P_s = (1-p)^2 \sum_{s=1}^{+\infty} s p^s = (1-p)^2 p \frac{d}{dp}\left(\frac{1}{1-p}\right) = p$$

# of nodes in a cluster of size s $\begin{cases} \nearrow s N_s \\ \searrow N P_s \end{cases}$ $\Longrightarrow N P_s = s N_s = N s n_s$

$$\boxed{n_s = P_s / s}$$

$\sum_s s N_s$ = total number of occupied nodes $= Np$

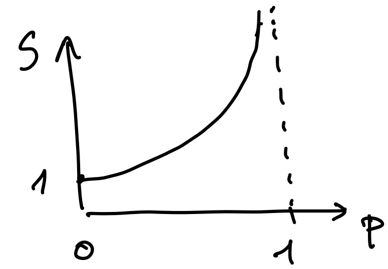
2. Average size of a cluster S

N_s = # of clusters of size s

$$S = \frac{\sum_{s=1}^{+\infty} s N_s}{\sum_{s=1}^{+\infty} N_s} = \frac{\sum_s s n_s}{\sum_s n_s}$$

$$S = \frac{\sum_s s\, n_s}{\sum_s n_s} = \frac{p}{\sum_s (1-p)^2 p^s} = \frac{p}{(1-p)^2\left(\frac{1}{1-p} - 1\right)}$$

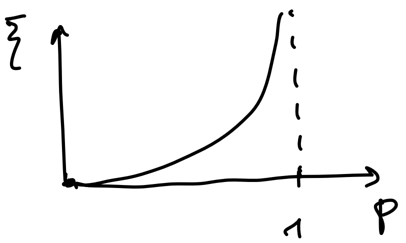
$$S = \frac{\cancel{p}}{(1-p)^{\cancel{2}} \frac{\cancel{p}}{\cancel{1-p}}} = \frac{1}{1-p}$$



3.

same cluster

$$g(r) = p^r = e^{r \ln p} = e^{-r/\xi} \qquad \xi = -\frac{1}{\ln p}$$



$$p = 1 - \epsilon$$

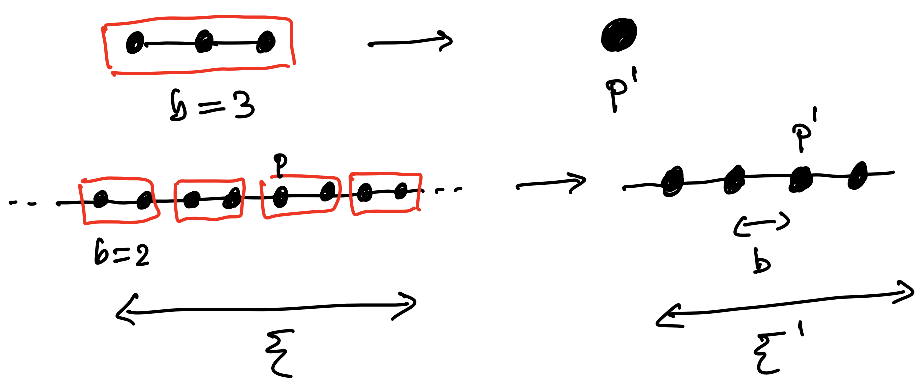
$$\ln(1-\epsilon) \approx -\epsilon$$

$$\xi \approx \frac{1}{\epsilon} = \frac{1}{1-p}$$

4.

$p_c = 1 \qquad S = \frac{1}{p_c - p} \qquad \xi \simeq \frac{1}{p_c - p} \qquad \nu = 1, \ \gamma = 1$

(B) 1D chain : RG approach



$$b\,\xi'(p') = \xi(p)$$

Fixed points: $\xi = 0$ or $\xi = +\infty$

1. $\xi \simeq A\,(p_c - p)^{-\nu}$

$\xi' \simeq A\,(p_c - p')^{-\nu}$

p close to the critical percolation threshold p_c

$p = p_c - \epsilon$

$p' = p'(p) = p'(p_c - \epsilon)$
$= p_c - \epsilon \left.\frac{dp'}{dp}\right|_{p_c}$

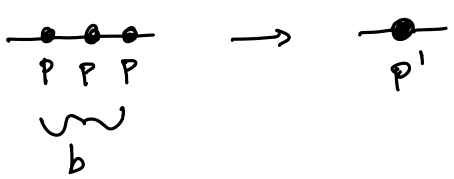
$$b\cancel{A}\,(p_c - p')^{-\nu} = \cancel{A}\,(p_c - p)^{-\nu}$$

$$b\,(p_c - p')^{-\nu} = (p_c - p)^{-\nu}$$

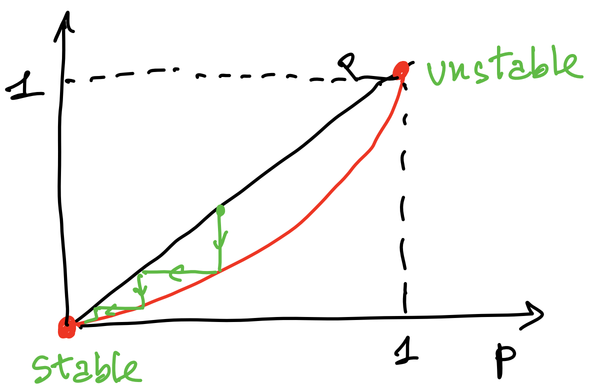
$$b\left(\cancel{p_c} - \cancel{p_c} + \epsilon \left.\frac{dp'}{dp}\right|_{p_c}\right)^{-\nu} = (p_c - p)^{-\nu} = \epsilon^{-\nu}$$

$$b\left(\cancel{\epsilon} \left.\frac{dp'}{dp}\right|_{p_c}\right)^{-\nu} = \cancel{\epsilon^{-\nu}}$$

$$\ln b - \nu \ln \left.\frac{dp'}{dp}\right|_{p_c} = 0 \implies \boxed{\nu^{-1} = \frac{\ln \left.\frac{dp'}{dp}\right|_{p_c}}{\ln b}}$$

2. 

$p' = p^b$

3. 

$p' = p^b$

$\begin{cases} p = 0 \\ p = 1 \end{cases}$ Fixed points ($\xi = 0$ and $\xi = \infty$)

4. $$\nu^{-1} = \frac{\ln \left.\frac{dp'}{dp}\right|_{p_c}}{\ln b}$$

$$\frac{dp'}{dp} = b\,p^{b-1}$$

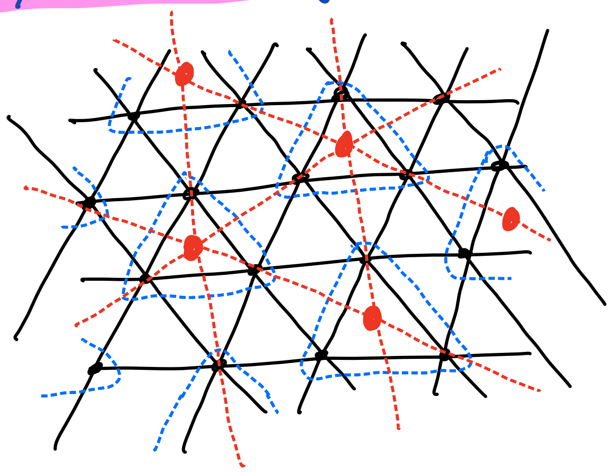
$b \geq 2$

$p = 0$ $$\left.\frac{dp'}{dp}\right|_{p=0} = 0$$

$p = 1 \qquad \left.\frac{dp'}{dp}\right|_{p=1} = b \qquad \boxed{\nu^{-1} = 1}$

(c) 2D triangular lattice : RG approach

1\.



$$p' = p^3 + 3p^2(1-p)$$

$$p' = p^2(p + 3 - 3p) = p^2(3-2p)$$

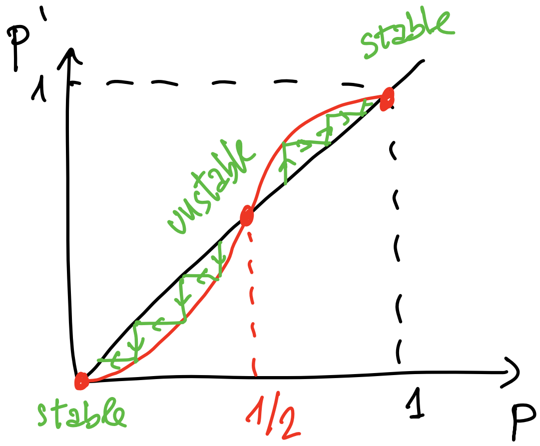
2\. Fixed points $\quad p'(p) = p$

$$\cancel{p} = p^{\cancel{2}}(3-2p) \qquad p = 0$$

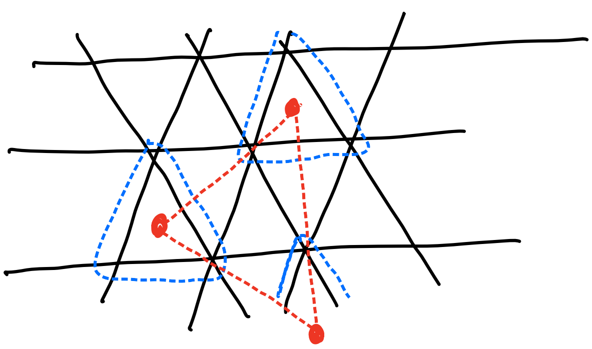
$$1 = 3p - 2p^2$$

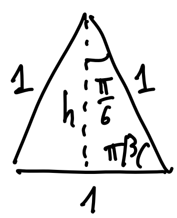
$$2p^2 - 3p + 1 = 0$$

$$p = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} \to 1, \ 1/2$$



3\.





$b = 2h$

$$h = \sin(\pi/3) = \frac{\sqrt{3}}{2}$$

$$b = \sqrt{3}$$

$$\frac{dp'}{dp} = \frac{d}{dp}\left(3p^2 - 2p^3\right) = 6p - 6p^2 = 6p(1-p)$$

$$\left.\frac{dp'}{dp}\right|_{p=0} = 0 \qquad \left.\frac{dp'}{dp}\right|_{p=1} = 0$$

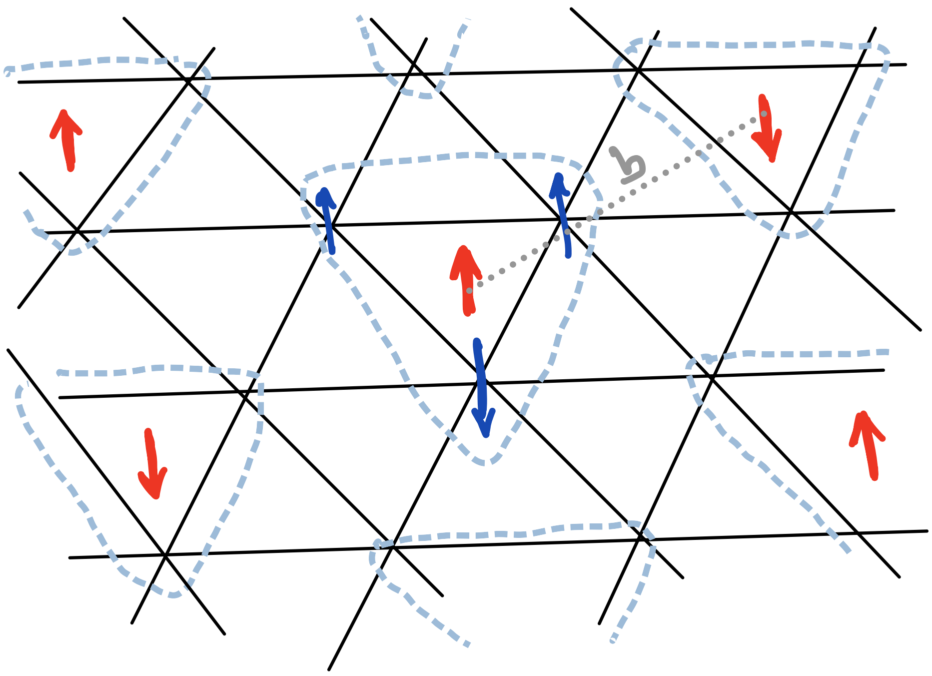
$$\left.\frac{dp'}{dp}\right|_{p=1/2} = \frac{6}{4} = \frac{3}{2}$$

$$\nu^{-1} = \frac{\ln \left. dp'/dp \right|_{p=1/2}}{\ln b} = \frac{\ln 3/2}{\ln \sqrt{3}} = \frac{\ln 3 - \ln 2}{(1/2)\ln 3} \simeq 0{,}738$$

$$\nu \simeq 1{,}355 \qquad \nu^{2d}_{exact} = \frac{4}{3} \simeq 1{,}333\ldots$$

II | Real-space RG for the Ising model on the triangular lattice (variational approximation)

$$H[\underline{s}] = -J \sum_{\langle ij \rangle} s_i s_j \qquad s_i = \pm 1$$



Majority rule : $\hat{s}_\alpha = \text{sign}\left(s_{1(\alpha)} + s_{2(\alpha)} + s_{3(\alpha)}\right)$

(similar to percolation, see previous exercise)

1. $b = \sqrt{3}$

$$\hat{N} = N/3 = N/b^2$$

$C(\hat{\underline{s}})$ = # of configurations of $\underline{s}$ that lead to $\hat{\underline{s}}$

$\hat{S}_\alpha = 1 \longrightarrow 4$ configurations

$S_{\alpha(1)}$	$S_{\alpha(2)}$	$S_{\alpha(3)}$
$\uparrow$	$\uparrow$	$\uparrow$
$\uparrow$	$\uparrow$	$\downarrow$
$\uparrow$	$\downarrow$	$\uparrow$
$\downarrow$	$\uparrow$	$\uparrow$

$\hat{S}_\alpha = -1 \longrightarrow 4$ configurations

$S_{\alpha(1)}$	$S_{\alpha(2)}$	$S_{\alpha(3)}$
$\downarrow$	$\downarrow$	$\downarrow$
$\downarrow$	$\downarrow$	$\uparrow$
$\downarrow$	$\uparrow$	$\downarrow$
$\uparrow$	$\downarrow$	$\downarrow$

$$C(\hat{\underline{S}}) = 4^{\hat{N}} = 4^{N/3}$$

2. $$e^{-\beta \hat{H}[\hat{\underline{S}}]} = \sum_{\underline{S} \in C(\hat{\underline{S}})} e^{-\beta H[\underline{S}]}$$

$$\hat{Z} = \sum_{\{\hat{S}=\pm 1\}} e^{-\beta \hat{H}[\hat{\underline{S}}]} = \sum_{\{\hat{S}=\pm 1\}} \sum_{\underline{S} \in C(\hat{\underline{S}})} e^{-\beta H[\underline{S}]}$$

$$\sum_{\{\hat{S}=\pm 1\}} \sum_{\underline{S} \in C(\hat{\underline{S}})} \longrightarrow \sum_{\{\underline{S}=\pm 1\}}$$

Hence $\hat{Z} = Z$

$$\text{Prob}[\hat{\underline{s}}] = \frac{e^{-\beta \hat{H}[\hat{\underline{s}}]}}{\hat{Z}} = \frac{1}{Z} \sum_{\underline{s} \in C(\hat{\underline{s}})} e^{-\beta H[\underline{s}]}$$

$$\text{Prob}[\hat{\underline{s}}] = \sum_{\underline{s} \in C(\hat{\underline{s}})} \text{Prob}[\underline{s}]$$

Prob of $\hat{\underline{s}}$ in the decimated system is the sum of the probabilities of the configurations of the original spins that are decimated into $\hat{\underline{s}}$ (i.e. $C(\hat{\underline{s}})$)

$\hat{H}[\hat{\underline{s}}]$ might contain all possible interaction terms allowed by the symmetry

e.g. next-to-nearest neighbors couplings

multi-body interactions

3. Approximate Hamiltonian to compute averages

$$\langle \cdots \rangle_{0,\hat{\underline{s}}} = \frac{1}{Z_0[\hat{\underline{s}}]} \sum_{\underline{s} \in C(\hat{\underline{s}})} \cdots \, e^{-\beta H_0[\underline{s}]}$$

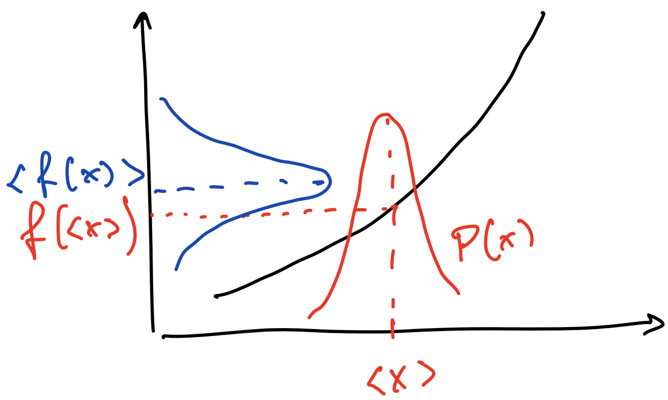
$$Z_0[\hat{\underline{s}}] = \sum_{\underline{s} \in C(\hat{\underline{s}})} e^{-\beta H_0[\underline{s}]}$$

$$\langle e^{-\beta(H[\underline{s}]-H_0[\underline{s}])}\rangle_{0,\hat{\underline{s}}} = \frac{1}{Z_0[\hat{\underline{s}}]} \underbrace{\sum_{\underline{s}\in C[\hat{\underline{s}}]} e^{-\beta(H[\underline{s}]-\cancel{H_0[\underline{s}]})}\, e^{-\beta \cancel{H_0[\underline{s}]}}}_{=\, e^{-\beta \hat{H}[\hat{\underline{s}}]}}$$

$$e^{-\beta \hat{H}[\hat{\underline{s}}]} = Z_0[\hat{\underline{s}}]\, \langle e^{-\beta(H[\underline{s}]-H_0[\underline{s}])}\rangle$$

Jensen inequality: for any convex function f

$\langle f(x)\rangle > f(\langle x\rangle)$ $\quad x \to$ random variable



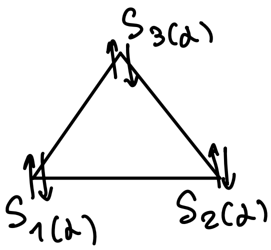
$$\langle e^{-\beta(H[\underline{s}]-H_0[\underline{s}])}\rangle_{0,\hat{\underline{s}}} \leq e^{-\beta(\langle H[\underline{s}]-H_0[\underline{s}]\rangle_{0,\hat{\underline{s}}})}$$

Taking the log of the expression above:

$$\boxed{H[\hat{\underline{s}}] \leq -\frac{1}{\beta}\ln Z_0[\hat{\underline{s}}] + \langle H[\underline{s}]-H_0[\underline{s}]\rangle_{0,\hat{\underline{s}}}}$$

4. $H_0[\underline{S}] = -J \sum_{\alpha=1}^{\hat{N}} \sum_{\langle ij \rangle \in \alpha} S_{i(\alpha)} S_{j(\alpha)}$

$\hat{N}$ independent triangular plaquettes



8 configurations

$S_{i(\alpha)} = \pm 1$

Sum over configurations compatible with $\hat{S}_\alpha$

$$\odot \quad Z_0[\hat{\underline{S}}] = \prod_{\alpha=1}^{\hat{N}} \sum_{\substack{\{S_{1(\alpha)}, S_{2(\alpha)}, S_{3(\alpha)}\} \\ \mathrm{sign}(S_{1(\alpha)}+S_{2(\alpha)}+S_{3(\alpha)}) = \hat{S}_\alpha}} e^{\beta J (S_{1\alpha}S_{2\alpha} + S_{2\alpha}S_{3\alpha} + S_{3\alpha}S_{1\alpha})}$$

• If $\hat{S}_\alpha = +1$:

4 configurations

$\uparrow\uparrow\uparrow \longrightarrow 3\beta J$

$\uparrow\uparrow\downarrow \quad \uparrow\downarrow\uparrow \quad \downarrow\uparrow\uparrow \longrightarrow -\beta J$

• If $\hat{S}_\alpha = -1$: 4 configurations

$\downarrow\downarrow\downarrow \longrightarrow 3\beta J$

$\downarrow\downarrow\uparrow \quad \downarrow\uparrow\downarrow \quad \uparrow\downarrow\downarrow \longrightarrow -\beta J$

$$\boxed{Z_0[\hat{\underline{S}}] = \left(e^{3\beta J} + 3e^{-\beta J}\right)^{\hat{N}}}$$

$\longrightarrow$ It is independent of $\hat{\underline{S}}$!

$$\odot \quad \langle S_{1\alpha} \rangle_{0,\hat{S}} = \frac{\displaystyle\sum_{\substack{\{S_{1(\alpha)}, S_{2(\alpha)}, S_{3(\alpha)}\} \\ \mathrm{sign}(S_{1\alpha}+S_{2\alpha}+S_{3\alpha}) = \hat{S}_\alpha}} S_{1\alpha} \, e^{+\beta J (S_{1\alpha}S_{2\alpha} + S_{2\alpha}S_{3\alpha} + S_{3\alpha}S_{1\alpha})}}{\displaystyle\sum_{\substack{\{S_{1(\alpha)}, S_{2(\alpha)}, S_{3(\alpha)}\} \\ \mathrm{sign}(S_{1\alpha}+S_{2\alpha}+S_{3\alpha}) = \hat{S}_\alpha}} e^{+\beta J (S_{1\alpha}S_{2\alpha} + S_{2\alpha}S_{3\alpha} + S_{3\alpha}S_{1\alpha})}}$$

- if $\hat{S}_\alpha = +1$

$\uparrow\uparrow\uparrow$		$e^{3\beta J}$
$\uparrow\uparrow\downarrow$	$\uparrow\downarrow\uparrow$	$2e^{-\beta J}$
$\downarrow\uparrow\uparrow$		$-e^{-\beta J}$

$$\langle S_{1\alpha} \rangle = \frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}}$$

- if $\hat{S}_\alpha = -1$

$\downarrow\downarrow\downarrow$		$-e^{3\beta J}$
$\downarrow\downarrow\uparrow$	$\downarrow\uparrow\downarrow$	$-2e^{-\beta J}$
$\uparrow\downarrow\downarrow$		$+e^{-\beta J}$

$$\langle S_{1\alpha} \rangle = -\frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}}$$

$$\boxed{\langle S_i \rangle = \frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \hat{S}_{\alpha(i)}}$$

Different blocks are uncorrelated using the Boltzmann measure associated with H_0

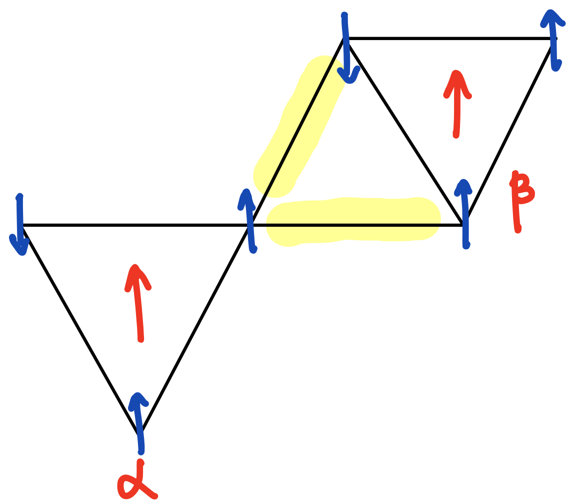
Hence $(\alpha \neq \beta)$

$$\langle S_{i(\alpha)} S_{J(\beta)} \rangle_{0,\hat{\underline{S}}} = \langle S_{i(\alpha)} \rangle_{0,\hat{\underline{S}}} \langle S_{J(\beta)} \rangle_{0,\hat{\underline{S}}}$$

$$= \left(\frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \right)^2 \hat{S}_{\alpha(i)} \hat{S}_{\beta(J)}$$

5. $$\langle H[\underline{S}] - H_0[\underline{S}] \rangle_{0,\hat{S}} = \left\langle -J \sum_{\langle i,j \rangle} s_i s_j \; - J \sum_{\langle i,j \rangle \in \alpha} s_i s_j \right\rangle_{0,\hat{\underline{S}}}$$

$$= \left\langle -J \sum_{\substack{\langle i,j \rangle \\ \alpha(i) \neq \beta(j)}} s_i s_j \right\rangle$$



For each pair of neighboring plaquettes in the decimated lattices there are two edges between the spins of the original lattice

$$\langle H[\underline{S}] - H_0[\underline{S}] \rangle_{0,\hat{S}} = -2J \sum_{\langle \alpha,\beta \rangle} \left(\frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \right)^2 \hat{S}_\alpha \hat{S}_\beta$$

$$\hat{H}[\hat{S}] \leq -\frac{1}{\beta} \hat{N} \ln\left(e^{3\beta J} + 3e^{-\beta J} \right) \; - J' \sum_{\langle \alpha\beta \rangle} \hat{S}_\alpha \hat{S}_\beta$$

$$\boxed{J' = 2 \left(\frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \right)^2 J}$$

6. $J = J' = 0$ is a fixed point (stable fixed point associated to $T=\infty$ paramagnetic phase)

$$J = 2J\left(\frac{e^{3\beta J}+e^{-\beta J}}{e^{3\beta J}+3e^{-\beta J}}\right)^2 \qquad e^{\beta J}\equiv x>0$$

$$2\left(\frac{x^3+1/x}{x^3+3/x}\right)^2 = 1$$

$$\frac{x^4+1}{x^4+3} = \frac{1}{\sqrt{2}} \implies x^4\left(1-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}-1$$

$$x^4 = \frac{3\sqrt{2}-2}{2-\sqrt{2}} = \frac{(3\sqrt{2}-2)(2+\sqrt{2})}{4-2} = \frac{6\sqrt{2}+6-4-2\sqrt{2}}{2}$$

$$x^4 = \frac{4\sqrt{2}+2}{2} = 2\sqrt{2}+1$$

$$\boxed{(\beta J)_* = \frac{1}{4}\ln\left(2\sqrt{2}+1\right)}$$

Stability: Introduce $K=\beta J$

$$K = 2K\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2$$

$$\frac{dk'}{dk} = 2\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right)^2$$

$$+ \; 4k\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right)\frac{\left(3e^{3k}-e^{-k}\right)\left(e^{3k}+3e^{-k}\right)-\left(e^{3k}+e^{-k}\right)\left(3e^{3k}-3e^{-k}\right)}{\left(e^{3k}+3e^{-k}\right)^2}$$

At $k = k_*$ $\quad \left(\frac{e^{3k_*}+e^{-k_*}}{e^{3k_*}+3e^{-k_*}}\right)^2 = \frac{1}{2}$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \frac{4k_*}{\sqrt{2}}\left(\cancel{3e^{6k_*}} + 9e^{2k_*} - e^{2k_*} - \cancel{3e^{-2k_*}} - \cancel{3e^{-6k_*}} \right.$$
$$\left. \cancel{+3e^{+2k_*}} - \cancel{3e^{2k_*}} \cancel{+3e^{-2k_*}}\right) \Big/ \left(e^{3k_*}+3e^{-k_*}\right)^2$$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \frac{4k_*}{\sqrt{2}} \; 8e^{2k_*} \Big/ \left(e^{3k_*}+3e^{-k_*}\right)^2$$

$x_* = e^{k_*}$ $\qquad \frac{8x_*^2}{\left(x_*^3+\frac{3}{x_*}\right)^2} = \frac{8x_*^4}{x_*^4+3}$

$x_*^4 = 2\sqrt{2}+1$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \frac{4k_*}{\sqrt{2}} \frac{16\sqrt{2}+8}{(2\sqrt{2}+1+3)^2} = 1 + \frac{4k_*}{\sqrt{2}} \frac{16\sqrt{2}+8}{(2\sqrt{2}+4)^2}$$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \frac{4k_*}{\sqrt{2}} \frac{8}{4} \frac{2\sqrt{2}+1}{(\sqrt{2}+2)^2} = 1 + \frac{8k_*}{\sqrt{2}} \frac{2\sqrt{2}+1}{6+4\sqrt{2}}$$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \frac{4k_*}{\sqrt{2}} \frac{2\sqrt{2}+1}{3+2\sqrt{2}} = 1 + 4k_* \frac{2\sqrt{2}+1}{3\sqrt{2}+4}$$

$$\left.\frac{dk'}{dk}\right|_{k_*} = 1 + \ln(1+2\sqrt{2}) \frac{2\sqrt{2}+1}{3\sqrt{2}+4} = 1 + \frac{8-5\sqrt{2}}{2} \ln(1+2\sqrt{2})$$

$$\frac{2\sqrt{2}+1}{3\sqrt{2}+4} \frac{3\sqrt{2}-4}{3\sqrt{2}-4} = \frac{12-8\sqrt{2}+3\sqrt{2}-4}{18-16} = \frac{8-5\sqrt{2}}{2}$$

7. $$\nu = \frac{\ln b}{\ln \left.\frac{dk'}{dk}\right|_{k_*}} = \frac{\ln\sqrt{3}}{\ln\left(1 + \frac{8-5\sqrt{2}}{2}\ln(1+2\sqrt{2})\right)} \simeq 1,13353\ldots$$

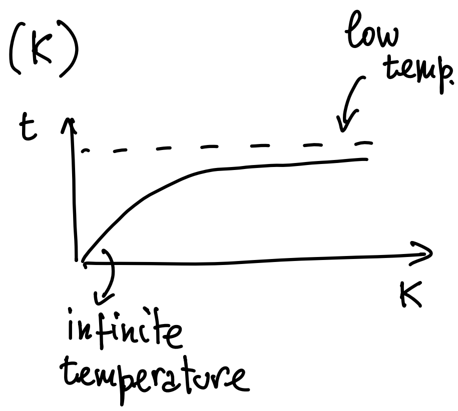
not too far from the exact value $\nu = 1$

The critical temperature is $\frac{T_c}{J} = \frac{4}{\ln(1+2\sqrt{2})} \simeq 2{,}97962\ldots$

The exact one is $T_c^{(\text{exact})} = 3{,}642\ldots$

II | Ising model - Migdal Kadanoff

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j \qquad K \equiv \beta J \qquad t = \mathrm{th}(K)$$



(A) Decimation in 1d

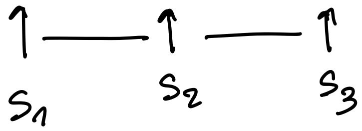
1. $$e^{\beta J S_i S_j} = \begin{cases} e^{\beta J} & \uparrow\uparrow \text{ or } \downarrow\downarrow \\ e^{-\beta J} & \uparrow\downarrow \text{ or } \downarrow\uparrow \end{cases}$$

$$A + B S_i S_j = \begin{cases} A+B & \uparrow\uparrow \text{ or } \downarrow\downarrow \\ A-B & \uparrow\downarrow \text{ or } \downarrow\uparrow \end{cases}$$

$$\begin{cases} A+B = e^{\beta J} \\ A-B = e^{-\beta J} \end{cases} \Rightarrow \begin{array}{l} A = \mathrm{ch}(\beta J) \\ B = \mathrm{sh}(\beta J) \end{array} \Rightarrow e^{K S_i S_j} = \mathrm{ch}(K) + S_i S_j\, \mathrm{sh}(K)$$

$$e^{K S_i S_j} = \mathrm{ch}(K)\left(1 + t S_i S_j\right)$$

2.

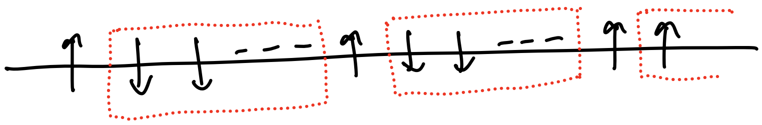


$$\sum_{s_2=\pm 1} e^{\beta J(s_1 s_2 + s_2 s_3)} = \sum_{s_2=\pm 1} ch^2(K)\,(1+t s_1 s_2)(1+t s_2 s_3)$$

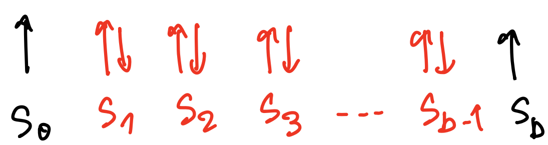
$$= ch^2(K) \sum_{s_2=\pm 1} \left(1 + \cancel{t s_1 s_2} + \cancel{t s_2 s_3} + t^2 s_1 s_3\right)$$

$$= 2\, ch^2(K)\,(1 + t^2 s_1 s_3)$$

3. $$Z = \sum_{\{s_i\}} e^{K \sum_i s_i s_{i+1}}$$



integrate out b-1 spins



$$\sum_{\left\{\substack{s_1=\pm 1\\ \vdots \\ s_{b-1}=\pm 1}\right\}} e^{K(s_0 s_1 + s_1 s_2 + \dots + s_{b-1} s_b)}$$

$$= ch^b(K) \sum_{\left\{\substack{s_1=\pm 1\\ \vdots \\ s_{b-1}=\pm 1}\right\}} (1+t s_0 s_1)(1+t s_1 s_2) \times \dots \times (1+t s_{b-1} s_b)$$

$$= 2\,\mathrm{ch}^b(K) \sum_{\{S_2=\pm 1, \dots, S_{b-1}=\pm 1\}} (1+t^2 S_0 S_2)(1+t S_2 S_3) \times \dots \times (1+t S_{b-1} S_b)$$

$$= \dots = 2^{b-1}\,\mathrm{ch}^b(K)\,(1+t^b S_0 S_b) = c\, e^{\beta \tilde{J} S_0 S_b}$$

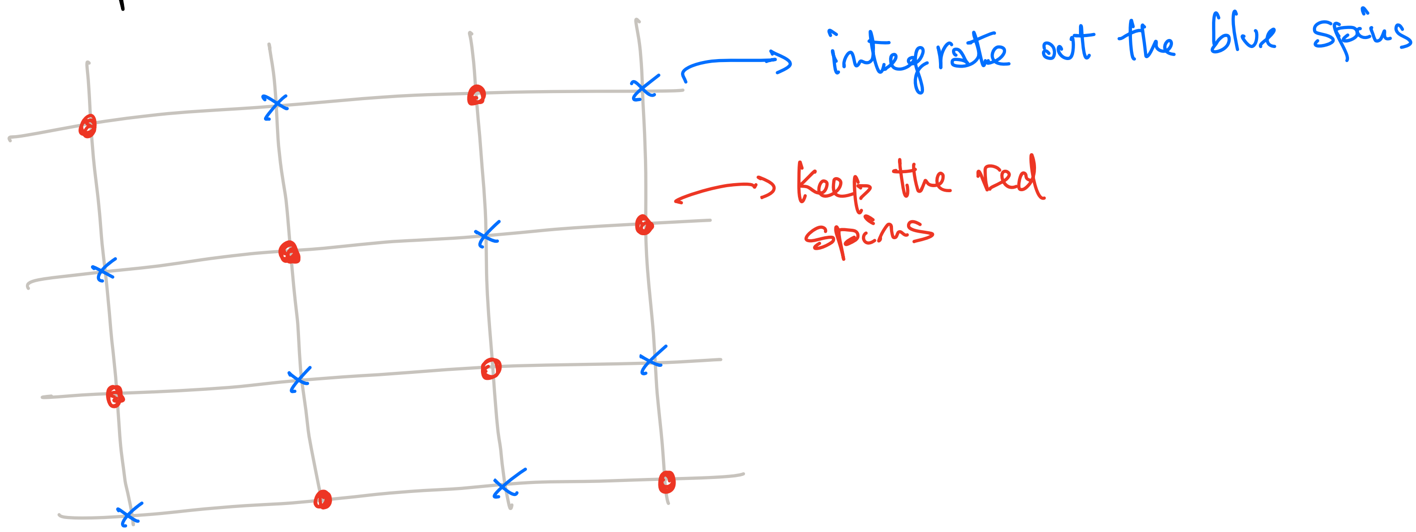
$$= c\,\mathrm{ch}(\tilde{K}) + c\,\mathrm{sh}(\tilde{K}) S_0 S_b$$

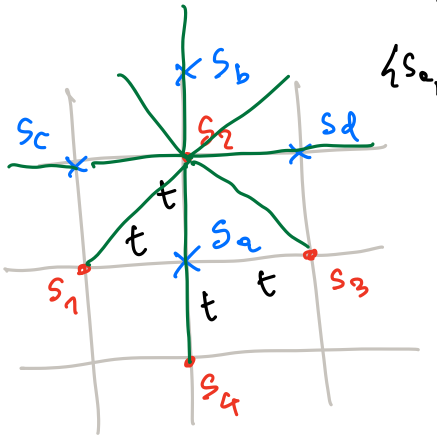
$$\begin{cases} c\,\mathrm{ch}(\tilde{K}) = 2^{b-1}\,\mathrm{ch}^b(K) \\ c\,\mathrm{sh}(\tilde{K}) = 2^{b-1}\,\mathrm{sh}^b(K) \end{cases} \Rightarrow \mathrm{th}(\tilde{K}) = (\mathrm{th}(K))^b$$

$$\boxed{\tilde{t} = t^b}$$

$d > 1$

Performing exact decimation is very hard due to the proliferation of coupling constants





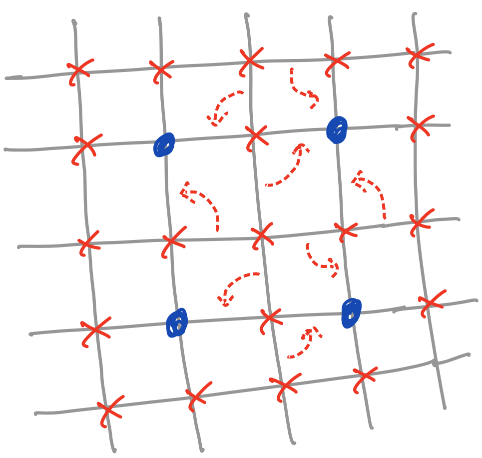
$$\sum_{\{s_a,s_b,s_c,s_d..\}} e^{\beta J s_a(s_1+s_2+s_3+s_4)}\, e^{\beta J s_b(s_2+\dots)}\, e^{\beta J s_c(s_2+\dots)}\, e^{\beta J s_d(s_2+\dots)}$$

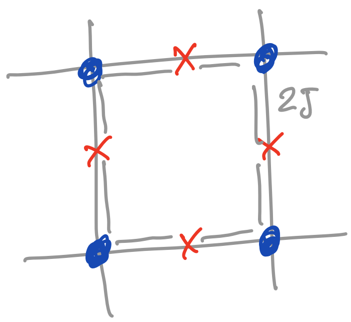
$$\propto (1+ts_as_2)(1+ts_bs_2)(1+ts_cs_2)(1+ts_ds_2)(1+ts_as_3)(1+ts_as_4)(1+ts_as_1)(1+ts_bs_5)\dots(1+ts_cs_8)\dots(1+ts_ds_{11})\dots$$

$$\propto 1+t^2\underline{s_2s_3}+t^2\underline{s_2s_4}+t^2\underline{s_2s_1}+t^2\underline{s_2s_5}+t^2\underline{s_2s_8}+\dots$$

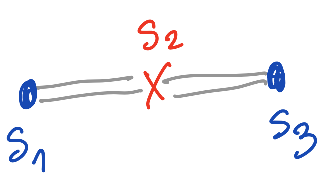
Second nearest-neighbors interactions + many-body terms

(B) 2d model



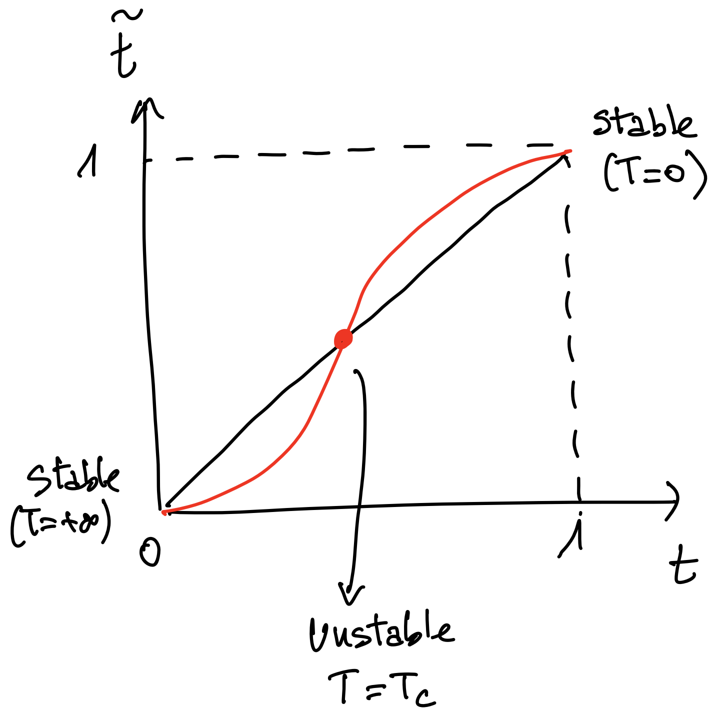


1.



$$\sum_{s_2=\pm1} e^{2K(s_1s_2+s_2s_3)} = 2\cdot ch^2(2K)\times (1+s_1s_3\, th^2(2K))$$

$$\tilde{t} = th^2(2K)$$



2. $$th(2x) = \frac{2\,th(x)}{1+th^2(x)}$$

$$\tilde{t} = \left(\frac{2t}{1+t^2}\right)^2 = \frac{4t^2}{(1+t^2)^2}$$

Fixed points

$$t = \frac{4t^2}{(1+t^2)^2}$$

$$t\left[(1+t^2)^2 - 4t\right] = 0$$

$$t\left[1+t^4+2t^2-4t\right] = 0$$

$$t\left[t^4+2t^2-4t+1\right] = 0$$

$$\underbrace{(t-1)}_{\text{t=1 is a fixed point}}(at^3+bt^2+ct+d) = at^4+bt^3+ct^2+dt - at^3-bt^2-ct-d$$

$a = 1$

$(b-a) = 0 \Rightarrow b = 1$

$c-b = 2 \Rightarrow c = 3$

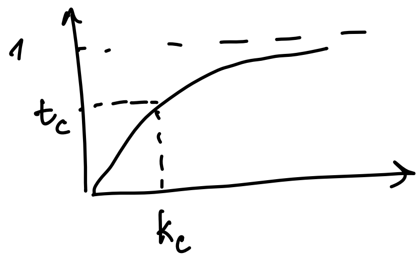
$d-c = -4 \Rightarrow d = -1$

$$t(t-1)(t^3+t^2+3t-1)=0$$

$$\boxed{\begin{array}{l} t_c \simeq 0{,}2958\ldots \\ K_c = 0{,}3047\ldots \end{array}}$$

$t_c \simeq 0{,}3$

$th(K_c) = 0{,}3 \sim K_c$

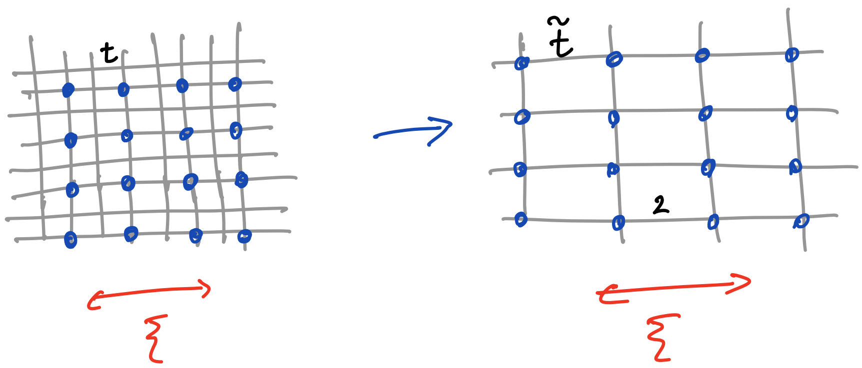


$$K_c^{exact} = \frac{\log(1+\sqrt{2})}{2} \simeq 0{,}44$$

$$K_c^{exact} > K_c^{MK} > K_c^{MF}$$

$$k_B T_c^{MF} = 4J \Rightarrow K_c^{MF} = 0{,}25$$

3.



$$\xi = \xi(\beta J) = \xi(K) = \xi(t)$$

b. $\xi(\tilde{t}) = \xi(t)$

$\tilde{t} = \tilde{t}(t)$

In the vicinity of the critical point:

$$\xi(t) \simeq A(t-t_c)^{-\nu} \quad \text{and} \quad \tilde{t}(t) \simeq t_c + \left.\frac{d\tilde{t}}{dt}\right|_{t_c}(t-t_c)$$

$$b \cdot \cancel{A}\left(\tilde{t}-t_c\right)^{-\nu} = \cancel{A}\left(t-t_c\right)^{-\nu}$$

$$b\left(\cancel{t_c}+\left.\frac{d\tilde{t}}{dt}\right|_{t_c}(t-t_c)-\cancel{t_c}\right)^{-\nu} = (t-t_c)^{-\nu}$$

$$b\left(\left.\frac{d\tilde{t}}{dt}\right|_{t_c}\right)^{-\nu}\cancel{(t-t_c)^{-\nu}} = \cancel{(t-t_c)^{-\nu}}$$

$$\ln b-\nu\ln\left.\frac{d\tilde{t}}{dt}\right|_{t_c} = 0 \Rightarrow \frac{1}{\nu} = \frac{\ln d\tilde{t}/dt|_{t_c}}{\ln b}$$

$$\frac{d\tilde{t}}{dt} = \frac{8t}{(1+t^2)^2}-2\,\frac{4t^2\cdot 2t}{(1+t^2)^3}$$

$$\frac{d\tilde{t}}{dt} = \frac{8t}{(1+t^2)^2}\left(1-\frac{2t^2}{1+t^2}\right) = \frac{8t}{(1+t^2)^2}\,\frac{1-t^2}{1+t^2}$$

$$t_c = \frac{4t_c^2}{(1+t_c^2)^2} \Rightarrow \frac{4t_c}{(1+t_c^2)^2} = 1$$

$$\left.\frac{d\tilde{t}}{dt}\right|_{t_c} = \frac{4t_c}{(1+t_c^2)^2}\,\frac{2(1-t_c^2)}{1+t_c^2} = \frac{2(1-t_c^2)}{1+t_c^2} \approx \frac{2\cdot 0{,}91}{1{,}09}$$

One can also compute $d\tilde{K}/dK|_{K_c}$

$$\tilde{t} = \frac{4t^2}{(1+t^2)^2} \qquad t = th(K)$$

$$\tilde{K} = \mathrm{atanh}\left(f(t)\right) \qquad f(t) = \frac{4t^2}{(1+t^2)^2}$$

$$\frac{d}{dx}\,\mathrm{atanh}\left(th(x)\right) = 1 = \frac{d\,\mathrm{atanh}(th(x))}{d\left(th(x)\right)}\left(1 - th^2(x)\right)$$

$$\frac{d}{dx}\,\mathrm{atanh}(x) = \frac{1}{1-x^2}$$

$$\frac{d\tilde{K}}{dK} = \frac{1}{1-f^2(t)}\,\frac{df}{dt}\,\frac{dt}{dK}$$

$$\frac{df}{dt} = \frac{8t}{(1+t^2)^2} - 2\,\frac{4t^2\cdot 2t}{(1+t^2)^3} = \frac{8t}{(1+t^2)^2}\left(1 - \frac{2t^2}{1+t^2}\right) = \frac{8t}{(1+t^2)^2}\,\frac{1-t^2}{1+t^2}$$

$$\left.\frac{d\tilde{K}}{dK}\right|_{K_c} = \cancel{\frac{1}{1-f^2(t_c)}}\,\left.\frac{df}{dt}\right|_{t_c}\cancel{\left(1-t_c^2\right)} \qquad f(t_c) = t_c$$

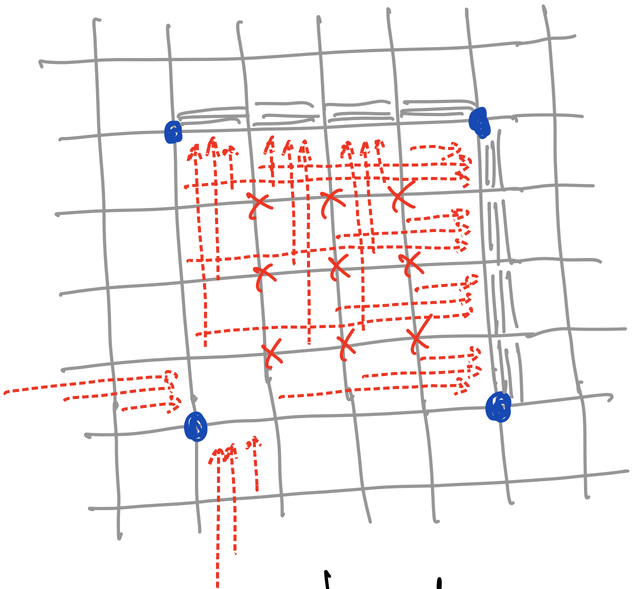
$$\left.\frac{d\tilde{K}}{dK}\right|_{K_c} = \left.\frac{df}{dt}\right|_{t_c} = \left.\frac{d\tilde{t}}{dt}\right|_{t_c} = \left.\underbrace{\cancel{\frac{4t^2}{(1+t^2)^2}}}_{= t_c}\,\frac{2(1-t^2)}{\cancel{t}(1+t^2)}\right|_{t_c} = \frac{2(1-t_c^2)}{1+t_c^2}$$

$$\left.\frac{d\tilde{t}}{dt}\right|_{t_c} \simeq 1{,}82 \cdot 0{,}91 \simeq 1{,}656$$

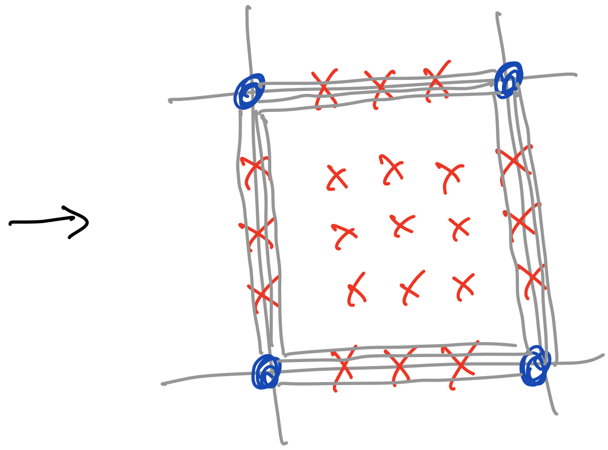
$$\boxed{\left.\frac{d\tilde{t}}{dt}\right|_{t_c} = 1{,}67857\ldots}$$

$$\frac{1}{\nu} \simeq \frac{\ln 1{,}656}{\ln 2} \simeq 1{,}33827\ldots \qquad \nu^{exact} = 1$$

4.



$b = 4$



$K \to bK$

Other stategy: Bond counting

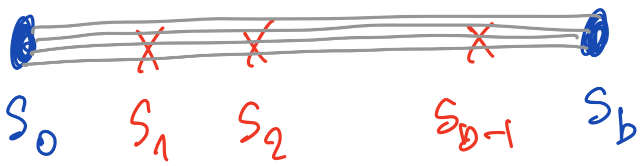
$N = \#$ of spins $= (L/a)^2$

$B = \#$ of bonds

$B = 2N$

$\tilde{N} = \#$ of • spins $= \left(L/ba\right)^2 = N/b^2$

$$B = 2\tilde{N} \cdot m \cdot b = 2\frac{N}{b^2} mb = 2N \Longrightarrow m = b$$



$$\sum_{\substack{s_1=\pm 1\\ \vdots \\ s_{b-1}=\pm 1}} e^{bK(s_0 s_1 + \dots + s_{b-1} s_b)}$$

$$= 2^{b-1}\, \mathrm{ch}^b(bK) \left(1 + \mathrm{th}^b(bK)\, s_0 s_b \right)$$

$$\boxed{\tilde{t} = \mathrm{th}^b(bK)}$$

5. $b = 1+\epsilon$

$$\tilde{K} = \mathrm{atanh}\left(\mathrm{th}^b(bK)\right)$$

$$\mathrm{th}(K+\epsilon K) \simeq \mathrm{th}(K) + \frac{\epsilon K}{\mathrm{ch}^2(K)}$$

$$\left(\mathrm{th}(K+\epsilon K)\right)^b = e^{(1+\epsilon)\ln\left[\mathrm{th}(K)\left(1 + \frac{\epsilon K}{\mathrm{sh}(K)\,\mathrm{ch}(K)}\right)\right]}$$

$$\simeq e^{(1+\epsilon)\left[\ln \mathrm{th}(K) + \frac{\epsilon K}{\mathrm{sh}(K)\,\mathrm{ch}(K)}\right]}$$

$$\simeq e^{\ln \mathrm{th}(K) + \epsilon\left(\ln \mathrm{th}(K) + \frac{K}{\mathrm{sh}(K)\,\mathrm{ch}(K)}\right)}$$

$$\left(\text{th}(K+\epsilon K)\right)^{1+\epsilon} \simeq \text{th}(K)\left(1+\epsilon\left(\ln \text{th}(K) + \frac{K}{\text{sh}(K)\,\text{ch}(K)}\right)\right)$$

$$= \text{th}(K) + \epsilon\left(\text{th}(K)\ln \text{th}(K) + \frac{K}{\text{ch}^2(K)}\right)$$

$$\text{atanh}(\text{th}(K)+\delta) \simeq \text{atanh}(\text{th}(K)) + \frac{1}{1-\text{th}^2(K)}\cdot\delta$$

$$= K + \text{ch}^2(K)\cdot\delta$$

$$\text{atanh}(\text{th}^b(bK)) \simeq K + \epsilon\left(\text{sh}(K)\,\text{ch}(K)\ln \text{th}(K) + K\right)$$

$$\boxed{\tilde{K} = K + \epsilon\left(K + \text{sh}(K)\,\text{ch}(K)\ln \text{th}(K)\right)}$$

6. $K_c^{exact} = \frac{\ln(1+\sqrt{2})}{2}$

$$K_c + \underbrace{\text{sh}(K_c)\,\text{ch}(K_c)}_{=\frac{\text{sh}(2K_c)}{2}} \ln \text{th}(K_c) = 0$$

$$e^{K_c} = \sqrt{1+\sqrt{2}}$$

$$e^{-K_c} = \frac{1}{\sqrt{1+\sqrt{2}}} = \frac{\sqrt{1+\sqrt{2}}}{1+\sqrt{2}}$$

$$\mathrm{sh}(K_c) = \frac{1}{2}\sqrt{1+\sqrt{2}}\left(1 - \frac{1}{1+\sqrt{2}}\right) = \frac{1}{2}\sqrt{1+\sqrt{2}}\ \frac{\sqrt{2}}{1+\sqrt{2}} = \frac{\sqrt{2}/2}{\sqrt{1+\sqrt{2}}}$$

$$\mathrm{ch}(K_c) = \frac{1}{2}\sqrt{1+\sqrt{2}}\left(1 + \frac{1}{1+\sqrt{2}}\right) = \frac{1}{2}\sqrt{1+\sqrt{2}}\ \frac{2+\sqrt{2}}{1+\sqrt{2}} = \frac{2+\sqrt{2}}{2\sqrt{1+\sqrt{2}}}$$

$$= \frac{\sqrt{2}(1+\sqrt{2})}{2\sqrt{1+\sqrt{2}}} = \frac{\sqrt{1+\sqrt{2}}}{\sqrt{2}}$$

$$\mathrm{th}(K_c) = \frac{1}{\cancel{\sqrt{2}}\sqrt{1+\sqrt{2}}} \cdot \frac{\cancel{\sqrt{2}}}{\sqrt{1+\sqrt{2}}} = \frac{1}{1+\sqrt{2}}$$

$$\ln\sqrt{1+\sqrt{2}} + \frac{1}{\sqrt{2}\cancel{\sqrt{1+\sqrt{2}}}}\ \frac{\cancel{\sqrt{1+\sqrt{2}}}}{\sqrt{2}}\ \ln\frac{1}{1+\sqrt{2}} = 0 \qquad \checkmark$$

$$\frac{1}{\nu} = \frac{\ln d\tilde{K}/dK|_{K_c}}{\ln b}$$

$$\frac{d\tilde{K}}{dK} = 1 + \epsilon\left[1 + \left(\mathrm{ch}^2(K) + \mathrm{sh}^2(K)\right)\ln \mathrm{th}(K) + \frac{\cancel{\mathrm{sh}(K)}\cancel{\mathrm{ch}(K)}}{\frac{\cancel{\mathrm{sh}(K)}}{\cancel{\mathrm{ch}(K)}}\ \cancel{\mathrm{ch}^2(K)}}\right]$$

$$\frac{d\tilde{K}}{dK} = 1 + \epsilon\left[2 + \left(\mathrm{ch}^2(K) + \mathrm{sh}^2(K)\right)\ln \mathrm{th}(K)\right]$$

$$\left.\frac{d\tilde{K}}{dK}\right|_{K_c} = 1+\epsilon\left[2+\left(\frac{1+\sqrt{2}}{2}+\frac{1}{2(1+\sqrt{2})}\right)\ln\frac{1}{1+\sqrt{2}}\right]$$

$$\left.\frac{d\tilde{K}}{dK}\right|_{K_c} = 1+\epsilon\left[2+\underbrace{\frac{1}{2}\,\frac{1+2+2\sqrt{2}+1}{1+\sqrt{2}}}_{\frac{2+\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{2}\,\cancel{(1+\sqrt{2})}}{\cancel{1+\sqrt{2}}}}\ln\frac{1}{1+\sqrt{2}}\right]$$

$$\left.\frac{d\tilde{K}}{dK}\right|_{K_c} = 1+\epsilon\left[2+\sqrt{2}\ln\frac{1}{1+\sqrt{2}}\right]$$

$$\frac{1}{\nu} = \frac{\ln\left(1+\epsilon\left(2+\sqrt{2}\ln\frac{1}{1+\sqrt{2}}\right)\right)}{\ln(1+\epsilon)} = 2+\sqrt{2}\ln\frac{1}{1+\sqrt{2}}$$

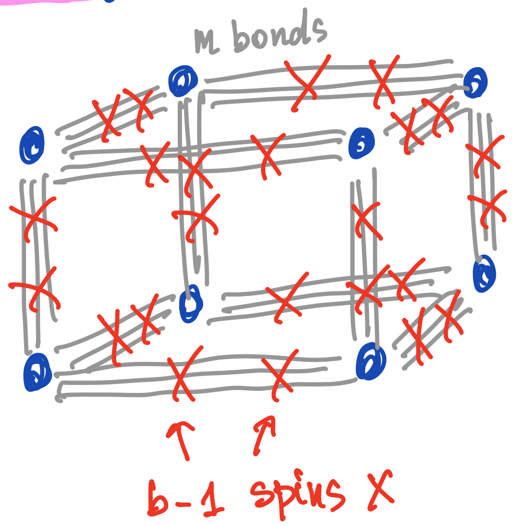
$$= 2-\sqrt{2}\ln\left(1+\sqrt{2}\right)$$

$$\nu = \frac{1}{2-\sqrt{2}\ln(1+\sqrt{2})} \simeq 1{,}327\ldots$$

$\nu^{exact} = 1$ $\qquad\qquad$ $\nu^{MF} = 1/2$

$\nu^{MK} \simeq 1{,}3\ldots$

(C) Generalization to arbitrary d



After the bond moving step

$\tilde N$ sites • kept after the decimation

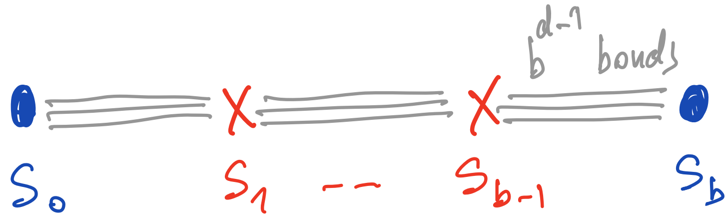
1. $b^d \tilde N = N \Rightarrow \tilde N = N/b^d$

2. $\tilde B = \tilde N \cdot d \cdot mb$

3. $\tilde B = B$

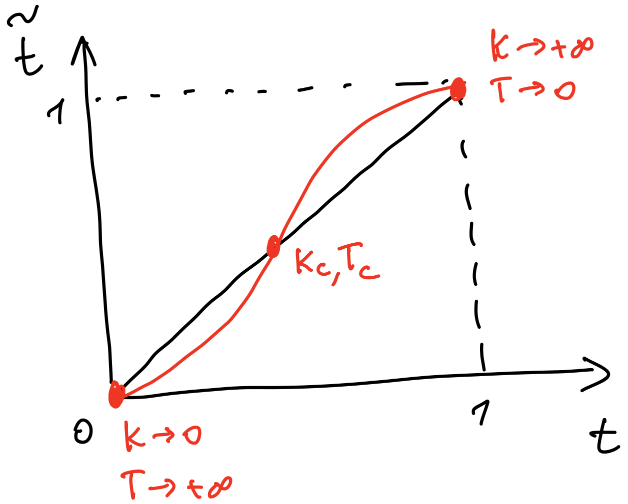
$$\tilde N d m b = N d$$

$$\frac{\cancel{N}}{b^d}\cdot \cancel{d}\, m b = \cancel{N}\cancel{d} \Rightarrow m = b^{d-1}$$



4. $$\sum_{\begin{Bmatrix} S_1=\pm1 \\ \vdots \\ S_{b-1}=\pm1 \end{Bmatrix}} e^{b^{d-1}K(S_0S_1+\dots+S_{b-1}S_b)} = 2^{b-1}\cosh^b(b^{d-1}K)\left(1+S_0S_b\tanh^b(b^{d-1}K)\right)$$

$$\boxed{\tilde{t} = \operatorname{th}^b\left(b^{d-1}K\right)}$$



5. $K \to +\infty$ $(T \to 0)$ Expand around $K \to +\infty$

$$\operatorname{th}(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} \xrightarrow[x \to +\infty]{} 1 - 2e^{-2x}$$

$$\operatorname{th}(b^{d-1}K) \simeq 1 - 2e^{-2b^{d-1}K}$$

$$\left(\operatorname{th}(b^{d-1}K)\right)^b \simeq 1 - 2b\, e^{-2b^{d-1}K}$$

$$\cancel{1} - \cancel{2}e^{-2\tilde{K}} \simeq \cancel{1} - \cancel{2}b\, e^{-2b^{d-1}K}$$

$$e^{-2\tilde{K}} \simeq b\, e^{-2b^{d-1}K}$$

$$-2\tilde{K} \simeq \ln b - 2b^{d-1}K \Rightarrow \tilde{K} \simeq b^{d-1}K - \frac{\ln b}{2}$$

$$\tilde{T} \simeq \frac{1}{\frac{b^{d-1}}{T} - \frac{\ln b}{2}} = \frac{T}{b^{d-1} - \frac{T\ln b}{2}}$$

$$\tilde{T} = \frac{T}{b^{d-1}} \frac{1}{1 - \frac{T \ln b}{2 b^{d-1}}} \simeq \frac{T}{b^{d-1}} \left(1 + \frac{T \ln b}{2 b^{d-1}} \right)$$

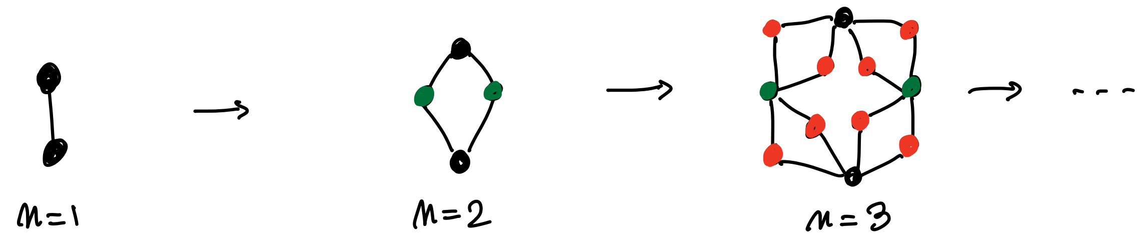
$$\left.\frac{d\tilde{T}}{dT}\right|_{T=0} = \frac{1}{b^{d-1}}$$

$d > 1 \longrightarrow$ stable F.P.

$d = 1 \longrightarrow$ marginal

(lower critical dimension)

III | Ising Model – Berker lattice



1. # of edges $\quad 1 \longrightarrow 2c \longrightarrow 2c \cdot 2c \longrightarrow \ldots$

after n steps the number of edges is $(2c)^{n-1}$

each edge is replaced by $2c$ edges with c vertices in between:

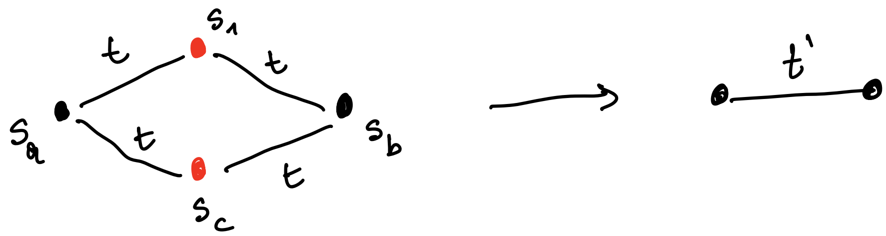
# of vertices $\quad 2 \longrightarrow 2 + c \longrightarrow 2 + c + 2c \cdot c \longrightarrow 2 + c + 2c^2 + 4c^3$

# of vertices after n steps $= 2 + \frac{1}{2}(2c + 4c^2 + 8c^3 + \ldots)$

$$= 2 + \frac{1}{2}\left(\sum_{k=1}^{n-1} (2c)^k\right) = 2 + \frac{1}{2}\left(\frac{1 - (2c)^n}{1 - 2c} - 1\right)$$

$$= 2 + \frac{1}{2}\left(\frac{1 - (2c)^n - 1 + 2c}{1 - 2c}\right) = 2 + \frac{1}{2}\,\frac{(2c)^n - 2c}{2c - 1}$$

2.



$$\sum_{\{s_1, s_2, \dots, s_c\}} e^{K s_a (s_1 + \dots + s_c) + K s_b (s_1 + \dots + s_c)}$$

$$= \prod_{i=1}^{c} \sum_{s_i = \pm 1} e^{K(s_a s_i + s_i s_b)} = \prod_{i=1}^{c} \mathrm{ch}^2(K) \sum_{s_i = \pm 1} (1 + t s_a s_i)(1 + t s_i s_b)$$

$$= \prod_{i=1}^{c} \mathrm{ch}^2(K)\, 2\, (1 + t^2 s_a s_b)$$

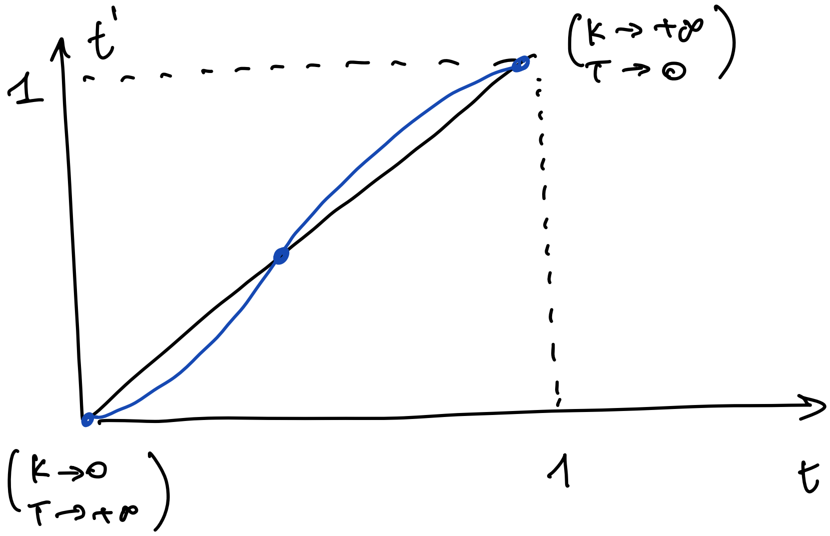
$$= \left(2\, \mathrm{ch}^2(K)\right)^c \prod_{i=1}^{c} (1 + t^2 s_a s_b)$$

$$= \left(2\, \mathrm{ch}^2(K)\right)^c \prod_{i=1}^{c} \frac{e^{\mathrm{atanh}(t^2) s_a s_b}}{\mathrm{ch}(\mathrm{atanh}(t^2))}$$

$$= \frac{\left[2\, \mathrm{ch}^2(K)\right]^c}{\prod_{i=1}^{c} \mathrm{ch}(\mathrm{atanh}(t^2))} \, e^{\sum_{i=1}^{c} \mathrm{atanh}(t^2) s_a s_b}$$

$$= \mathrm{cste} \cdot e^{K' s_a s_b}$$

$$\boxed{K' = c\ \mathrm{atanh}(\mathrm{th}^2(K))} \Leftrightarrow \boxed{t' = \mathrm{th}(c\ \mathrm{atanh}(t^2))}$$



3. $c = 1$

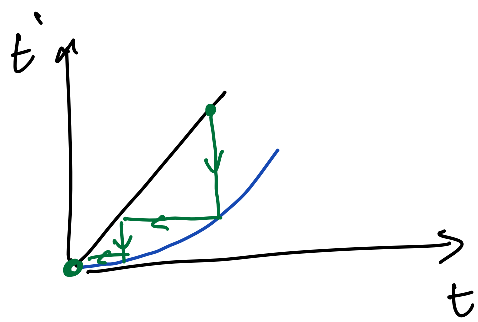
$$t' = th\left(atanh(t^2)\right) \Rightarrow t' = t^2$$

4. $T \to 0 \qquad K \to \infty \qquad t = th(K) \to 1$

$T \to \infty \qquad K \to 0 \qquad t = th(K) \to 0$

5. $$\frac{dt'}{dt} = \left(1 - th^2\left(c\, atanh(t^2)\right)\right) \frac{c}{1 - t^4} \, 2t$$

$t \to 0$
$(T \to +\infty)$ $\qquad \left.\dfrac{dt'}{dt}\right|_{t=0} = 0 \Rightarrow$ stable fixed point



$T \to 0$
$(K \to +\infty)$

$$\text{th}(K) \simeq 1 - 2e^{-2K}$$

$$t^2 \simeq 1 - 4e^{-2K}$$

$$1 - 2e^{-2K'} = 1 - 2e^{-2c\,\text{atanh}(t^2)}$$

$$-2K' = -2c\,\text{atanh}(t^2)$$

$$K' = c\,\text{atanh}(1 - 4e^{-2K})$$

$$\text{atanh}(1 - 2e^{-2K}) = K$$

$$\delta = 2e^{-2K}$$
$$\ln\delta = \ln 2 - 2K$$
$$K = \frac{\ln 2 - \ln\delta}{2}$$

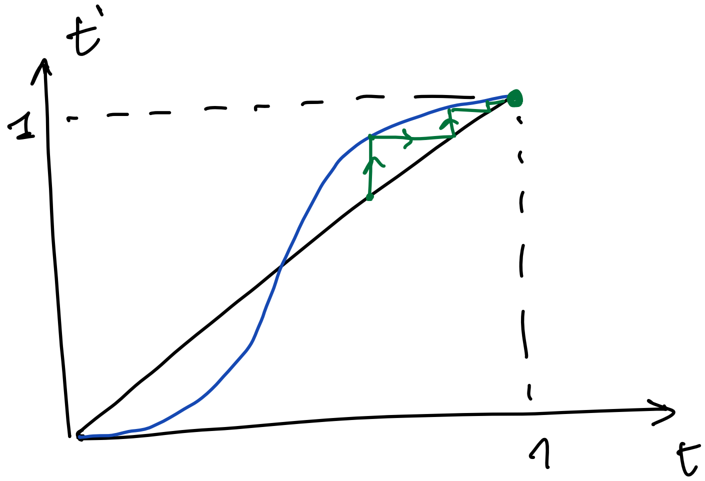
$$\text{atanh}(1-\delta) = \frac{\ln 2 - \ln\delta}{2}$$

$$K' = c\,\frac{\ln 2 - \ln(4e^{-2K})}{2} = \frac{c}{2}\left(\ln 2 - 2\ln 2 + 2K\right)$$

$$K' = -\frac{c\ln 2}{2} + cK$$

$$\frac{T'}{J} = \frac{1}{\frac{cJ}{T}\left(1 - \frac{T\ln 2}{2J}\right)} \approx \frac{T}{cJ}\left(1 + \frac{T\ln 2}{2J}\right)$$

$$\left.\frac{dT'}{dT}\right|_{T=0} = \frac{1}{c} \quad \Rightarrow \quad \text{stable fixed point for } c > 1$$



Alternative formulation of the recursion RG relation:

$$\mathrm{th}(k') = \mathrm{th}\left(c\, \mathrm{arctanh}\left(\mathrm{th}^2(k)\right)\right)$$

$$\mathrm{th}(x) = \frac{e^{2x}-1}{e^{2x}+1} \implies e^{2x}t + t = e^{2x} - 1$$

$$e^{2x}(t-1) = 1+t$$

$$e^{2x} = \frac{1+t}{1-t} \implies x = \frac{1}{2}\ln\frac{1+t}{1-t}$$

$$\mathrm{atanh}(t) = x = \frac{1}{2}\ln\frac{1+t}{1-t}$$

$$t' = \mathrm{th}\left(c\,\frac{1}{2}\ln\frac{1+t^2}{1-t^2}\right) = \mathrm{th}\left(\ln\left(\frac{1+t^2}{1-t^2}\right)^{c/2}\right)$$

$$= \frac{e^{2\ln\left(\frac{1+t^2}{1-t^2}\right)^{c/2}} - 1}{e^{2\ln\left(\frac{1+t^2}{1-t^2}\right)^{c/2}} + 1} = \frac{\left(\frac{1+t^2}{1-t^2}\right)^{c} - 1}{\left(\frac{1+t^2}{1-t^2}\right)^{c} + 1} = \frac{(1+t^2)^c - (1-t^2)^c}{(1+t^2)^c + (1-t^2)^c}$$

$$t \to 0 \qquad t' \approx \frac{1 + ct^2 - 1 + ct^2}{1 + ct^2 + 1 - ct^2} = ct^2$$

$$\left.\frac{dt'}{dt}\right|_0 = 0$$

$$t \to 1 \qquad t = 1 - \epsilon$$

$$t' = \frac{\left(1 + (1-\epsilon)^2\right)^c - \left(1 - (1-\epsilon)^2\right)^c}{\left(1 + (1-\epsilon)^2\right)^c + \left(1 - (1-\epsilon)^2\right)^c} \approx \frac{2^c(1 - c\epsilon) - 2^c\epsilon^c}{2^c(1 - c\epsilon) + 2^c\epsilon^c}$$

$$= \frac{1 - c\epsilon - \epsilon^c}{1 - c\epsilon + \epsilon^c}$$

$$(1-\epsilon)^2 \approx 1 - 2\epsilon$$

$$1 + (1-\epsilon)^2 = 2(1-\epsilon)$$

$$\left(1 + (1-\epsilon)^2\right)^c = 2^c(1 - c\epsilon)$$

$$1 - (1-\epsilon)^2 = 2\epsilon$$

$$\left(1 - (1-\epsilon)^2\right)^c = (2\epsilon)^c$$

$$\simeq (1 - c\epsilon - \epsilon^c)(1 + c\epsilon - \epsilon^c + c^2\epsilon^2 + \dots)$$

$$= 1 + \cancel{c\epsilon} - \epsilon^c - \cancel{c\epsilon} - \cancel{c^2\epsilon^2} + c\epsilon^{c+1} - \epsilon^c + \cancel{c^2\epsilon^2} + \dots$$

$$\simeq 1 - 2\epsilon^c + \dots$$

$$t' \simeq 1 - 2(1-t)^c$$

$$\frac{dt'}{dt} = 2c(1-t)^{c-1} \qquad \left.\frac{dt'}{dt}\right|_{t \to 1} = 0$$

$c = 2$

$$t' = \frac{(1+t^2)^2 - (1-t^2)^2}{(1+t^2)^2 + (1+t^2)^2} = \frac{2t^2}{1+t^4}$$

Fixed point : $t = 0$, $t = 1$, $t = t_* \simeq 0{,}5437\ldots$

$$\frac{dt'}{dt} = \frac{4t}{1+t^4} - \frac{2t^2}{(1+t^4)^2} \cdot 4t^3 = \frac{4t}{1+t^4}\left(1 - \frac{2t^4}{1+t^4}\right)$$

$$\frac{2t_*}{1+t_*^4} = 1 \Rightarrow \left.\frac{dt'}{dt}\right|_{t_*} = 2\left(1 - \frac{2t_*^4}{2t_*}\right) = 2(1-t_*^3)$$

$$\nu = \frac{\ln b}{\ln[2(1-t_*^3)]} = 1{,}338\ldots$$

$b = 2$