TD : RG Approach for the XY model

A) Phenomenological analysis

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \simeq -2JN + \frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2$$

$$\sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 = \sum_i \left[(\theta_i - \theta_{i+a\vec{e}_x})^2 + (\theta_i - \theta_{i+a\vec{e}_y})^2 \right]$$

$$\simeq \sum_i \left[a^2 (\partial_x \theta)^2 + a^2 (\partial_y \theta)^2 \right] = \sum_{x,y} (\vec{\nabla} \theta)^2 a^2 \quad \leftarrow d^2\vec{r}$$

$$\mathcal{H} \simeq -2JN + \frac{J}{2} \int d^2\vec{r} \left(\vec{\nabla} \cdot \theta(\vec{r}) \right)^2$$

1. Low energy configurations $\longrightarrow \dfrac{\delta H}{\delta \theta} = 0$

$$\frac{\delta \mathcal{H}}{\delta \theta} = 0 \longrightarrow \nabla^2 \theta(\vec{r}) = 0 \qquad \text{(Harmonic theory)}$$

$$\oint d\theta = 2\pi n \qquad n \in \mathbb{Z}$$

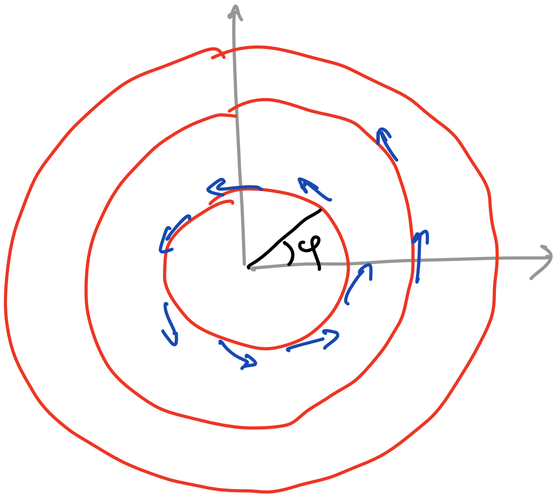
If there is no singular point $\Rightarrow \theta(\vec{r}) = \text{cst} \qquad (n=0)$

LIOUVILLE THEOREM

Hence for $n \neq 0$ there must be a singular point $\longrightarrow$ "defect"

2. $\nabla^2 \theta(\vec{r}) = 0$

$$\theta(\vec{r}) = \theta_0 + n\varphi \quad \rightarrow \text{charge}$$



$n = 1 \quad \theta_0 = \frac{\pi}{2}$

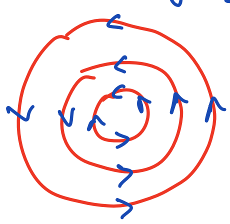
$$\nabla \theta(\vec{r}) = \left(\frac{1}{r} \partial_\varphi \vec{e}_\varphi + \partial_r \vec{e}_r \right) \theta(\vec{r}) = \frac{n}{r} \vec{u}_\varphi$$

$$\nabla^2 \theta(\vec{r}) = \nabla \left(\nabla \theta(\vec{r}) \right) = 0$$

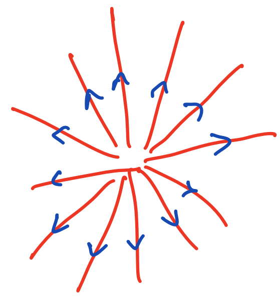
Singularity in 0

Defects cannot be killed by smooth transformations of the field

$\theta_0 = \frac{\pi}{2}$



$\theta_0 = 0$



Same topological charge $n = 1$

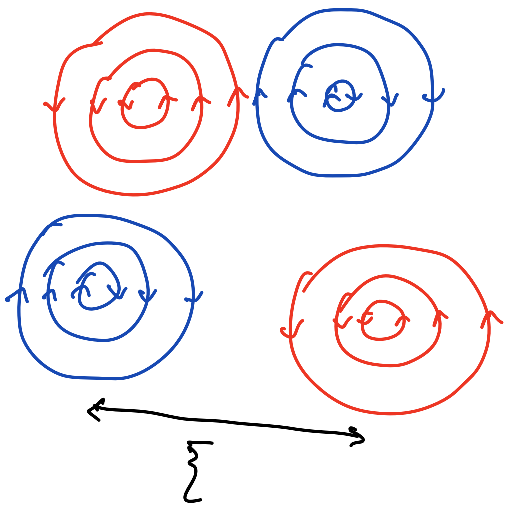
No continuous transformation of the field can change the value of n

3. $$H \simeq \frac{J}{2} \int (\vec{\nabla}\theta)^2 d^2\vec{r} = \frac{J}{2} \int \frac{n^2}{r^2} r \, dr \, d\varphi$$

$$= \pi J n^2 \int_a^L \frac{1}{r} dr = \pi J n^2 \ln\left(\frac{L}{a}\right) \rightarrow \text{Energy of the defect}$$

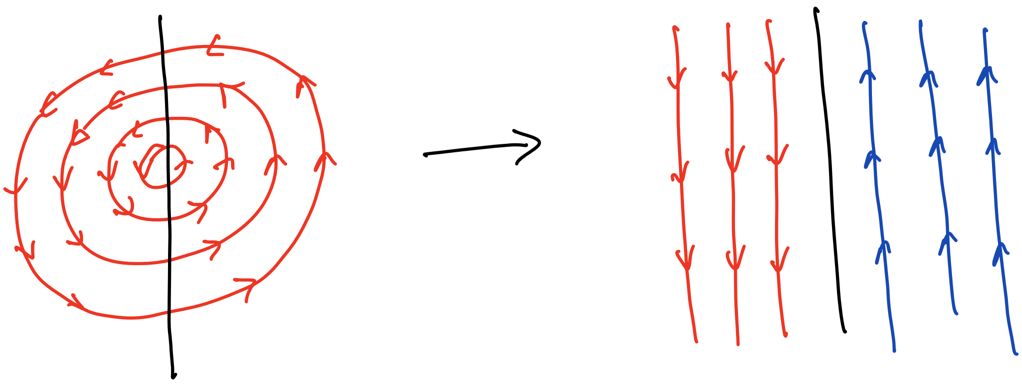
4. Properties of the topological defects

- Finite density of defects $\longrightarrow$ No power law correlations



Average distance $\xi = \left(\frac{L^2}{N_D}\right)^{1/2}$

Correlations decay exponentially

- 

Defects are energetically stable

energetic cost of the interface $\propto JL$

5. PEIERLS Argument

$$\Delta E = J\pi n^2 \ln\left(\frac{L}{a}\right) \qquad \Delta S = 2k_B \ln\left(L/a\right)$$

$$\Delta F = \left(J\pi n^2 - 2k_B T\right) \ln\left(L/a\right)$$

$$\boxed{T_c = \frac{J\pi}{2k_B}}$$

$T < T_c$ | T_c | $T > T_c$ → T

$T < T_c$:

No free vortices

(zro density of pairs of bounded vortices that form and coalesce)

power-law correlations

SPIN-WAVES Regime

$T > T_c$:

proliferation of vortices

$\Downarrow$

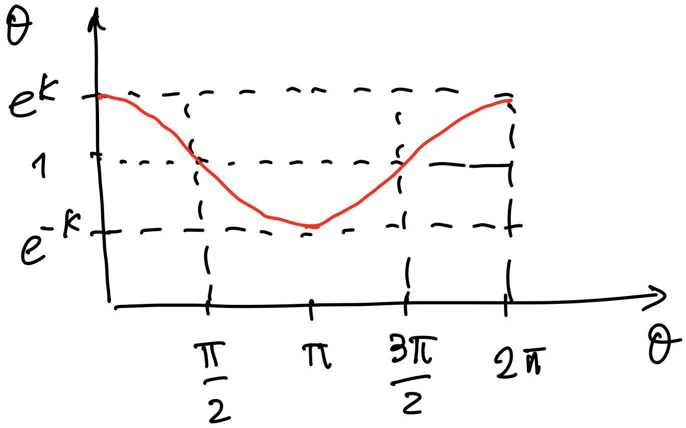
exponentiel correlations

Kosterlitz - Thouless phase transition

B) Coulomb Gas formulation – Villain approach

$$I_n(x) = \int_0^{2\pi} e^{x\cos\theta + in\theta}\,\frac{d\theta}{2\pi}$$

1. $e^{k\cos\theta}$

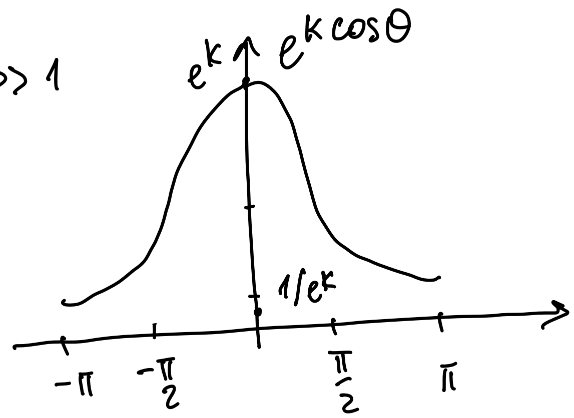


$$e^{k\cos\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta}\, c_n$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{k\cos\theta}\, e^{-in\theta}\, d\theta = I_n(k)$$

$$e^{k\cos\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta}\, I_n(k)$$

$K \gg 1$



Very peaked around $K=0$

$\cos\theta \simeq 1 - \frac{\theta^2}{2} + \dots$ (similar to spin waves)

$$I_n(K) \simeq \int_{-\pi}^{\pi} e^{K(1-\theta^2/2)} e^{in\theta} \frac{d\theta}{2\pi}$$

$$= \frac{e^K}{2\pi} \int_{-\pi}^{\pi} e^{-\frac{K}{2}\theta^2 + in\theta} d\theta$$

$$\simeq \frac{e^K}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{K}{2}\theta^2 + in\theta} d\theta = \frac{e^K}{2\pi} \sqrt{\frac{2\pi}{K}} e^{\frac{1}{2}\frac{(in)^2}{K}}$$

$$= \frac{e^K}{\sqrt{2\pi K}} e^{-n^2/2K}$$

2. $$e^{K\cos\theta} \approx \frac{e^K}{\sqrt{2\pi K}} \sum_{n=-\infty}^{+\infty} e^{in\theta - n^2/2K}$$

$K \gg 1$

$k_B T \ll J$

Preserves the LOCAL symmetry corresponding to the rotation by 2π by any θ

In this approximation $e^{K\cos\theta}$ is still periodic with period 2π, while in the SW approximation $(e^{K\cos\theta} \simeq e^{K(1-\theta^2/2)})$ the periodicity is lost

3. $\partial_\mu \theta_{\vec{r}} = \dfrac{\theta_{\vec{r}+a\vec{e}_\mu} - \theta_{\vec{r}}}{a}$ $\qquad (\mu = x, y)$

$$Z = \int D\theta \; e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \qquad \left(D\theta = \prod_i d\theta_i\right)$$

$$= \int D\theta \; e^{K \sum_i \left(\cos(\theta_i - \theta_{i-ae_x}) + \cos(\theta_i - \theta_{i-ae_y})\right)}$$

$$= \int D\theta \prod_i e^{K\cos(\underbrace{\theta_i - \theta_{i-ae_x}}_{= a\partial_x \theta_i})} \; e^{K\cos(\underbrace{\theta_i - \theta_{i-ae_y}}_{= a\partial_y \theta_r})}$$

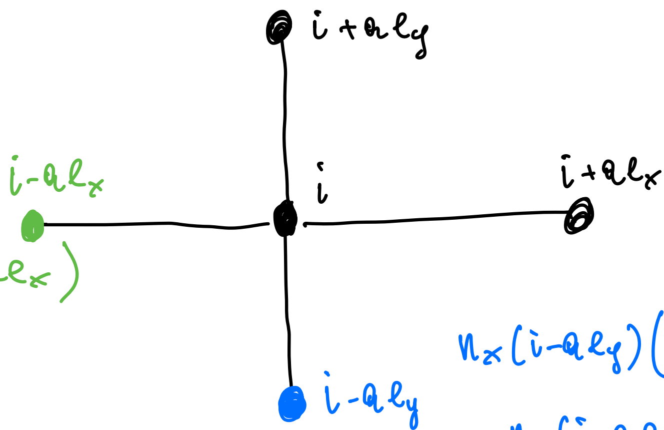
$$e^{K\cos(a\partial_\mu \theta_i)} \simeq \frac{e^K}{\sqrt{2\pi K}} \sum_{n_\mu(i)=-\infty}^{+\infty} e^{i n_\mu(i)(a\partial_\mu \theta_i) - n_\mu^2(i)/2K}$$

$$Z \simeq \int D\theta \prod_i \frac{e^{2K}}{2\pi K} \sum_{n_x(i)=-\infty}^{+\infty} \sum_{n_y(i)=-\infty}^{+\infty} e^{ia(n_x(i)\partial_x + n_y(i)\partial_y)\theta_i} \times e^{-\frac{(n_x^2(i) + n_y^2(i))}{2K}}$$

$$Z \simeq \left(\frac{e^{2K}}{2\pi K}\right)^N \sum_{\{\vec{n}(i) \in \mathbb{Z}\}} \int D\theta \; e^{\sum_i \left[ia(n_x(i)\partial_x + n_y(i)\partial_y)\theta_i - n^2(i)/2K\right]}$$

4. $n_x(i)\partial_x\theta_i + n_y(i)\partial_y\theta_i = \vec{n}(i)\cdot\vec{\nabla}\,\theta_i$

$$a\sum_i \vec{n}(i)\,\vec{\nabla}\theta_i = \sum_i\Big(n_x(i)\big(\theta_{i+a\vec{e}_x}-\theta_i\big)+n_y(i)\big(\theta_{i+a\vec{e}_y}-\theta_i\big)\Big) =$$



$$n_x(i-ae_x)\big(\theta_i-\theta_{i-ae_x}\big) + n_y(i-ae_x)\big(\theta_{i-ae_x+ae_y}-\theta_{i-ae_x}\big)$$

$$n_x(i-ae_y)\big(\theta_{i-ae_y+ae_x}-\theta_{i-ae_y}\big) + n_y(i-ae_y)\big(\theta_i-\theta_{i-ae_y}\big)$$

$$= \sum_i\Big(-n_x(i)+n_x(i-ae_y)+n_y(i-ae_y)-n_y(i)\Big)\theta_i$$

$$= -\sum_i\Big(\underbrace{n_x(i)-n_x(i-ae_y)}_{=\,a\partial_x n_x(i)} + \underbrace{n_y(i)-n_y(i-ae_y)}_{=\,a\partial_y n_y(i)}\Big)\theta_i$$

$$= -a\sum_i\Big(\partial_x n_x(i)+\partial_y n_y(i)\Big)\theta_i$$

$$a\,\partial_x n_x(i)\theta_i = \big(\underbrace{n_x(i)-n_x(i-ae_x)}_{\in\mathbb{Z}}\big)\theta_i$$

$$\int_0^{2\pi}\frac{d\theta_i}{2\pi}\, e^{-i\left[\left(n_x(i)-n_x(i-ae_x)\right)+\left(n_y(i)-n_y(i-ae_y)\right)\right]\theta_i} = \begin{cases} 1 & \text{if } \operatorname{div}\vec{n}(i)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$\int \frac{d\theta_i}{2\pi}\, e^{-ia\left(\partial_x n_x(i) + \partial_y n_y(i)\right)\theta_i} = \delta_{a\left(\partial_x n_x(i) + \partial_y n_y(i)\right),0}$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\{\vec{n}(i)\in\mathbb{Z}^2\}} e^{-\sum_i \vec{n}^2(i)/2K} \prod_i \delta_{\vec{\nabla}\cdot\vec{n}(i),0}$$

Alternative approach (directly in the continuous limit)

$$Z = \left(\frac{e^{2K}}{2\pi K}\right)^N \sum_{\{\vec{n}_i\in\mathbb{Z}^2\}} \int D\theta\, e^{\sum_i \left[ia\left(n_x(i)\partial_x + n_y(i)\partial_y\right)\theta_i - \vec{n}^2(i)/2K\right]}$$

$$n_x(i)\partial_x\theta_i + \theta_i\partial_x n_x(i) = \partial_x\left(n_x(i)\theta_i\right)$$

$$i\sum_i a\left(n_x(i)\partial_x\theta_i + n_y(i)\partial_y\theta_i\right) \simeq \frac{i}{a}\int\left(n_x(\vec{r})\partial_x\theta_{\vec{r}} + n_y(\vec{r})\partial_y\theta_{\vec{r}}\right)d^2\vec{r}$$

$$= i\int\cancel{\left[\, n_x(\vec{r})\theta_{\vec{r}} + n_x(\vec{r})\theta_{\vec{r}}\,\right]}\,d^2\vec{r} \longrightarrow \text{boundary term}$$

$$- i\int\theta_{\vec{r}}\left(\partial_x n_x(\vec{r}) + \partial_y n_y(\vec{r})\right)d^2\vec{r}$$

$$Z \simeq \left(\frac{e^{2K}}{2\pi K}\right)^N \sum_{\{\vec{n}(\vec{r})\in\mathbb{Z}^2\}} \int D\theta\, e^{-i\int\theta_{\vec{r}}\left(\operatorname{div}\vec{n}(\vec{r})\right)d^2\vec{r} - \int\frac{\vec{n}^2(\vec{r})}{2K}d^2\vec{r}}$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\substack{\{\vec{n}(\vec{r}) \in \mathbb{Z}^2\} \\ \mathrm{div}\, \vec{n}(\vec{r}) = 0}} e^{-\int \frac{n^2(\vec{r})}{2K} d^2\vec{r}}$$

5. $\vec{\nabla} \cdot \vec{n}(\vec{r}) = 0$

$$\vec{n}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

The simplest choice for $\vec{A}(\vec{r})$ is $\vec{A}(\vec{r}) = a\, p(\vec{r})\, \vec{e}_z$

$$n(\vec{r}) = a\left(\partial_y p(\vec{r})\, \vec{e}_x - \partial_x p(\vec{r})\, \vec{e}_y\right) \qquad p(\vec{r}) \in \mathbb{Z}$$

$$n_x^2 = a^2\left(\partial_y p(\vec{r})\right)^2 \qquad n_y^2 = a^2\left(\partial_x p(\vec{r})\right)^2$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\{p_z(i) \in \mathbb{Z}\}} e^{-\frac{a^2}{2K} \sum_i \underbrace{\left[\left(\partial_x p(i)\right)^2 + \left(\partial_y p(i)\right)^2\right]}_{= \left(\vec{\nabla} p(\vec{r})\right)^2}}$$

6. $$\sum_{p=-\infty}^{+\infty} f(p) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\phi\, f(\phi)\, e^{2\pi i m \phi}$$ ← Poisson summation formula

$$f(x) \longrightarrow \hat{f}(q) = \int_{-\infty}^{+\infty} f(x)\, e^{-ixq}\, d\phi$$

$$\sum_{p=-\infty}^{+\infty} f(x + pa) \equiv S(x)$$ ← At the end we will consider the particular case $x = 0$ and $a = 1$

$S(x)$ is periodic of period a

$$S(x+a) = S(x)$$

$$S(x) = \sum_{m=-\infty}^{+\infty} c_m \, e^{-i\frac{2\pi}{a}mx}$$

$$c_m = \frac{1}{a}\int_0^a e^{i\frac{2\pi}{a}mx} S(x)\,dx = \sum_{p=-\infty}^{+\infty} \frac{1}{a}\int_0^a e^{i\frac{2\pi}{a}mx} f(\underbrace{x+pa}_{=\phi})\,dx$$

$$= \sum_{p=-\infty}^{+\infty} \frac{1}{a}\int_{pa}^{(p+1)a} e^{i\frac{2\pi}{a}m(\phi-pa)} f(\phi)\,d\phi$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} e^{i\frac{2\pi}{a}m\phi} f(\phi)\,d\phi \; \underbrace{e^{-i2\pi mp}}_{=1}$$

$$S(x) = \sum_{m=-\infty}^{+\infty} e^{-i\frac{2\pi}{a}mx} \underbrace{\frac{1}{a}\int_{-\infty}^{+\infty} e^{i\frac{2\pi}{a}m\phi} f(\phi)\,d\phi}_{=\hat{f}\left(-\frac{2\pi m}{a}\right)} \qquad \square$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\{p(i)\in\mathbb{Z}\}} e^{-\frac{a^2}{2K}\sum_i\left(\vec{\nabla}\cdot p(i)\right)^2}$$

$$= \left(\frac{e^{2K}}{K}\right)^N \prod_i \sum_{p(i)=-\infty}^{+\infty} \underbrace{e^{-\frac{a^2}{2K}\left(\vec{\nabla}\cdot p(i)\right)^2}}_{f(p(i))}$$

$$\sum_{p(i)=-\infty}^{+\infty} e^{-\frac{a^2}{2K}\left(\vec{\nabla}\cdot p(i)\right)^2} = \sum_{m(i)=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i2\pi m(i)\phi(i)} \, e^{-\frac{a^2}{2K}\left(\vec{\nabla}\phi(i)\right)^2} d\phi(i)$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \prod_i \sum_{m(i)=-\infty}^{+\infty} \int d\phi(i) \, e^{i2\pi m(i)\phi(i) - \frac{a^2}{2K}\left(\vec{\nabla}\phi(i)\right)^2}$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\{m(i)\in\mathbb{Z}\}} \int D\phi \, e^{\sum_i\left[i2\pi m(i)\phi(i) - \frac{a^2}{2K}\left(\vec{\nabla}\phi(i)\right)^2\right]}$$

7. $$\int D\phi \, e^{-\frac{1}{2}\sum_{i,j}\phi_i M_{ij}\phi_j + \sum_i B_i\phi_i} = \frac{(2\pi)^N}{\sqrt{\det M}} e^{\frac{1}{2}\sum_{ij} B_i (M^{-1})_{ij} B_j}$$

$$M_{ij} = -\frac{a^2}{K}\nabla^2 \qquad \left(M^{-1}\right)_{ij} = K\,G_{|i-j|}$$

$$B_i = 2\pi i\, m(i)$$

$$Z \simeq \left(\frac{e^{2K}}{K}\right)^N \sum_{\{m(i)\in\mathbb{Z}\}} e^{-\frac{K}{2}\cdot 4\pi^2 \sum_{i,j} m(i)\, G(i-j)\, m(j)}$$

$$Z_{SW} = \int D\theta \, e^{-\frac{K}{2}\sum_i \theta_i(-a^2\nabla^2)\theta_i} = \frac{(2\pi)^N}{\sqrt{\det\left(K(-a^2\nabla^2)\right)}}$$

$$\frac{(2\pi)^N}{\sqrt{\det\left(\frac{-a\nabla^2}{K}\right)}} = Z_{SW} K^N$$

$-\beta \times$ ground state energy

$$Z \simeq e^{2NK} \sum_{\{m(i)\in\mathbb{Z}\}} e^{-2\pi^2 K \sum_{i,j} m(i) G(|i-j|) m(j)}$$

$$G(|i-j|) \simeq G_0 - \frac{1}{2\pi} \ln \frac{|i-j|}{a} - c \qquad G_0 \simeq \frac{1}{2\pi} \ln \frac{L}{a}$$

$$\sum_{i,j} m(i) G(|i-j|) m_j = \sum_{i,j} m(i) \underbrace{\left(G(|i-j|) - G_0\right)}_{\bar{G}(|i-j|)} m(j) + G_0 \left(\sum_i m(i)\right)^2$$

$$\bar{G}(|i-j|) \equiv G(|i-j|) - G_0$$

$$e^{-2\pi^2 K \frac{1}{2\pi} \ln\frac{L}{a}\left(\sum_i m(i)\right)^2} = e^{-\pi K \ln\frac{L}{a}\left(\sum_i m(i)\right)^2}$$

Configurations with $\sum_i m(i) \neq 0$ are strongly suppressed in the thermodynamic limit

$$Z \simeq e^{2NK} Z_{SW} \sum_{\substack{\{m_i \in \mathbb{Z}\} \\ \sum_i m(i) = 0}} e^{-2\pi^2 K \sum_{i,j} m(i) \bar{G}(|i-j|) m_j}$$

$$\sum_{i,j} m(i)\,\tilde{G}(|i-j|)\,m(j) = \sum_{i\neq j} m(i)\left(-\frac{1}{2\pi}\ln\frac{|i-j|}{a} - c\right) m(j)$$

$$= -\frac{1}{2\pi}\sum_{i\neq j} m(i)\ln\frac{|i-j|}{a}\,m(j) - c\sum_{i\neq j} m(i)\,m(j)$$

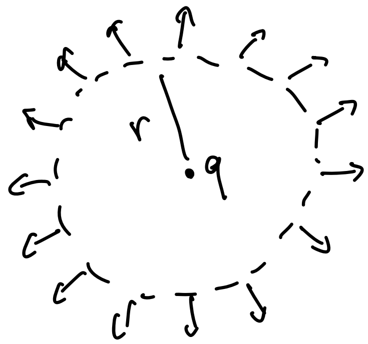
$$\sum_{i\neq j} m(i)\,m(j) = \sum_{i,j} m(i)\,m(j) - \sum_i m^2(i)$$

$$= \cancel{\left(\sum_i m(i)\right)^2} - \sum_i m^2(i)$$

$$Z \simeq e^{2NK} Z_{SW} \sum_{\substack{\{m(i)\in\mathbb{Z}\}\\ \sum_i m(i)=0}} e^{\pi K \sum_{i\neq j} m(i)\ln\frac{|i-j|}{a}\,m(j) - 2\pi^2 c K \sum_i m^2(i)}$$

c) Real-space RG

1. Electrostatic in 2D



$$2\pi r E_r(r) = \frac{q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{q}{r} \cdot \vec{e}_r$$

$$\vec{E} = -\vec{\nabla} \cdot V \Rightarrow V = -\frac{q}{2\pi\epsilon_0} \ln r$$

q_1 —— $|r_1 - r_2|$ —— q_2

$$U = -\frac{q_1 q_2}{2\pi\epsilon_0} \ln(|r_1 - r_2|)$$

Coulomb gas in 2d

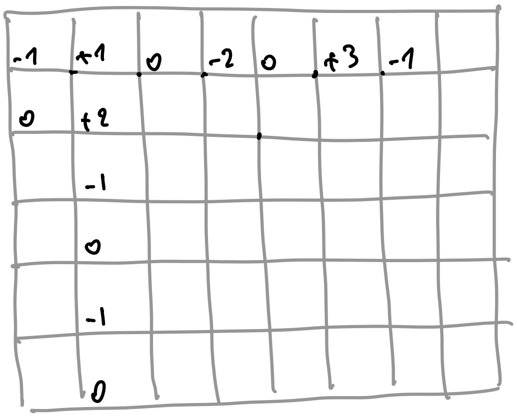
q_1, q_2, q_3, q_4

$q_i = \pm 1, \pm 2, \ldots$

$$U = -\frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{2\pi\epsilon_0} \ln(|r_i - r_j|)$$

Coulomb gas on a 2d lattice.
Instead of using $\{q_i, r_i\}$ as variables
we use $m(i) \longrightarrow$ total number of charges on the node i

$m(i) = 0, \pm 1, \pm 2, \ldots$ $\quad \{m(i)\} \in \mathbb{Z}^N$



$$U = -\frac{1}{2\pi\epsilon_0}\sum_{i,j} m(i)\, m(j) \ln\frac{|i-j|}{a}$$

$$Z = \sum_{\{m(i)\}} e^{\frac{\beta}{2\pi\epsilon_0}\sum_{i,j} m(i)\, m(j) \ln\frac{|i-j|}{a}}$$

Study the problem in the grand canonical ensemble

$$\Xi = \sum_{Q=0}^{+\infty} Z(Q)\, e^{\beta\mu Q}$$

Q = total number of particles

$$Q = \sum_i |m_i|$$

Low energy $(T \ll \pi J / 2k_B) \longrightarrow$ only a few defects

$$m(i) = 0,\ \pm 1,\ \cancel{\pm 2},\ \ldots$$

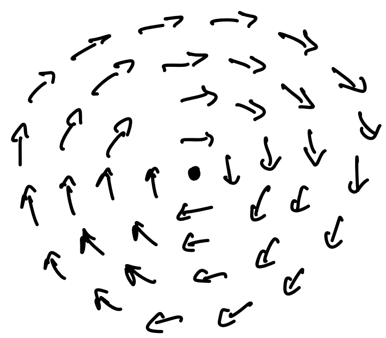
higher energy configurations less probable

If $m(i) = 0, \pm 1$ then $\sum_i |m(i)| = \sum_i m^2(i)$

$$Z = e^{2NK} Z_{SW} \sum_{\substack{\{m_i \in \mathbb{Z}\} \\ \sum_i m(i) = 0}} e^{\underbrace{\pi K \sum_{i \neq j} m(i) \ln \frac{|i-j|}{a} m(j)}_{\text{Coulomb potential}} - \underbrace{2\pi^2 cK}_{\text{Chemical potential}} \sum_i m^2(i)}$$

The charges represent the TOPOLOGICAL DEFECTS

Vortex / Anti-vortex configurations



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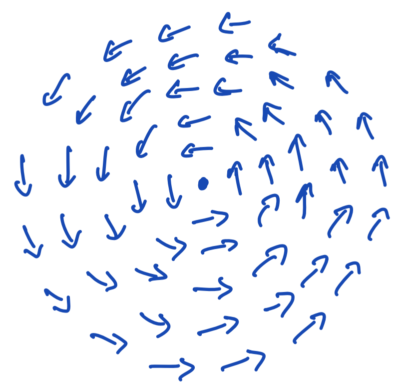

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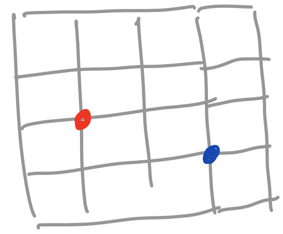
Origin of the topological defects $\longrightarrow$ low energy configurations

2. Very low $T \rightarrow$ expansion of Z keeping only the first terms

No charges or at most two opposite charges are present

$$m(i) = \{0, \dots 0, +1, 0, \dots 0, -1, 0 \dots 0\}$$

$i = r_1$, $j = r_2$ $\qquad$ $i = r_2$, $j = r_1$



$$\frac{Z}{e^{2\pi K} Z_{SW}} \simeq 1 + \sum_{\{r_1, r_2\}} e^{-2\pi K \ln \frac{|r_1 - r_2|}{a} - 4\pi^2 c K}$$

$z \equiv e^{-2\pi^2 c K}$

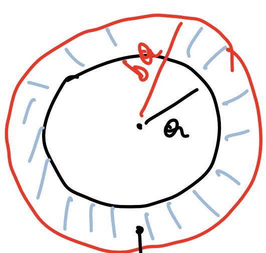
Fugacity

$[c \simeq 1/4]$

$$\tilde{Z} \simeq 1 + z^2 \sum_{\vec{r}_1} \sum_{\vec{r}_2} \left(\frac{a}{|r_1 - r_2|} \right)^{2\pi K} \frac{\Delta \vec{r}_1}{a^2} \frac{\Delta \vec{r}_2}{a^2}$$

$$\tilde{Z} \simeq 1 + \frac{z^2}{a^4} \int_{|r_1 - r_2| \geq a} d^2 r_1 \, d^2 r_2 \left(\frac{a}{|r_1 - r_2|} \right)^{2\pi K}$$

3.



$b = e^{dl}$ $\qquad$ $dl \ll 1$

$b \simeq 1 + dl$

Integrate the charges into this small shell and rescale

$$\int_{|r_1-r_2|\leq a} d^2r_1\, d^2r_2 \left(\frac{a}{|r_1-r_2|}\right)^{2\pi K} = \int_{a\leq |r_1-r_2|\leq ba} d^2r_1\, d^2r_2 \left|\frac{a}{r_1-r_2}\right|^{2\pi K} \sim \underline{L^2 2\pi a^2 d\ell}$$

$$+ \int_{|r_1-r_2|\geq ba} d^2r_1\, d^2r_2 \left|\frac{a}{r_1-r_2}\right|^{2\pi K}$$

$$r_1 = b s_1 \qquad d^2r_1 = b^2 d^2 s_1$$
$$r_2 = b s_2 \qquad d^2r_2 = b^2 d^2 s_2$$

$$|r_1 - r_2| = b|s_1 - s_2|$$

$$\int_{|r_1-r_2|\geq ba} d^2r_1\, d^2r_2 \left|\frac{a}{r_1-r_2}\right|^{2\pi K} = b^{4-2\pi K}\int_{|s_1-s_2|\geq a} d^2s_1\, d^2s_2 \left|\frac{a}{s_1-s_2}\right|^{2\pi K}$$

$$\tilde{Z} \simeq 1 + \frac{z^2}{a^4}\left(\cancel{L^2 2\pi a^2 d\ell} + b^{4-2\pi K}\int_{|s_1-s_2|\geq a} d^2s_1\, d^2s_2 \left|\frac{a}{s_1-s_2}\right|^{2\pi K}\right)$$

renormalization of the partition function

$$z^2(b) = z^2 b^{4-2\pi K}$$

$$z^2(\ell+d\ell) = z^2(\ell)\, e^{d\ell(4-2\pi K)}$$

$$\left(z(\ell) + \frac{dz}{d\ell}.d\ell\right)^2 = z^2(\ell)\left(1 + d\ell(4-2\pi K)\right)$$

$$\cancel{z^2(\ell)} + 2z(\ell)\frac{dz}{d\ell}.d\ell = \cancel{z^2(\ell)} + z^2(\ell)(4-2\pi K)\,d\ell$$

$$\boxed{\frac{dz}{d\ell} = z(\ell)(2-\pi K)}$$

$$K_{KT} = \frac{2}{\pi}$$

$$T_{KT} = \frac{\pi J}{2 k_B}$$ (Peierls Argument)

4. $C(|r_1 - r_2|) \propto |r_1 - r_2|^{-\frac{1}{2\pi K_{eff}}}$

$$\frac{1}{K_{eff}} = \frac{1}{K} + 4\pi^3 z^2 \int_a^L \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K}$$

$2\pi K - 3 > 1 \implies$ the integral converges for $L \to +\infty$

$2\pi K > 4$

$K > \frac{2}{\pi} \implies \frac{J}{k_B T} > \frac{2}{\pi} \implies T < \frac{\pi J}{2 k_B}$

For $K < 2/\pi$ $(T > T_{KT} = \pi J / 2 k_B)$ the perturbative expansion for small numbers of defects does not converge

$\implies$ PROLIFERATION OF DEFECTS

$$\int_a^L \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K} = \underbrace{\int_a^{ab} \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K}}_{=\int_1^b dx\, x^{3-2\pi K}} + \int_{ab}^L \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K}$$

$r = bs$

$r = ab \iff s = a$

$$\int_1^b dx\, x^{3-2\pi K} = \frac{x^{4-2\pi K}}{4-2\pi K}\Big|_1^b = \frac{b^{4-2\pi K} - 1}{4 - 2\pi K}$$

$$= \frac{b^{4-2\pi K}-1}{4-2\pi K} + \int_a^{L/b} \frac{ds}{a}\left(\frac{s}{a}\right)^{3-2\pi K} \cdot b^{4-2\pi K}$$

$$\frac{1}{K_{eff}} = \frac{1}{K} + 4\pi^3 z^2 \left(\frac{b^{4-2\pi K}-1}{4-2\pi K} + b^{4-2\pi K} \int_a^{L} \frac{ds}{a}\left(\frac{s}{a}\right)^{3-2\pi K} \right)$$

$$\frac{1}{K_{eff}} = \frac{1}{K} + 4\pi^3 z^2 \, \frac{b^{4-2\pi K}-1}{4-2\pi K} + 4\pi^3 z^2 b^{4-2\pi K} \int_a^{L} \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K}$$

$$\frac{1}{K_{eff}} = \frac{1}{\tilde{K}(b)} + 4\pi^3 \tilde{z}(b) \int_a^{L} \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K}$$

$$\frac{1}{K(\ell+d\ell)} = \frac{1}{K(\ell)} + \frac{4\pi^3 z^2(\ell)\,\cancel{(4-2\pi K)}\,d\ell}{\cancel{4-2\pi K}}$$

$$z^2(\ell+d\ell) = z^2(\ell)\, b^{4-2\pi K}$$

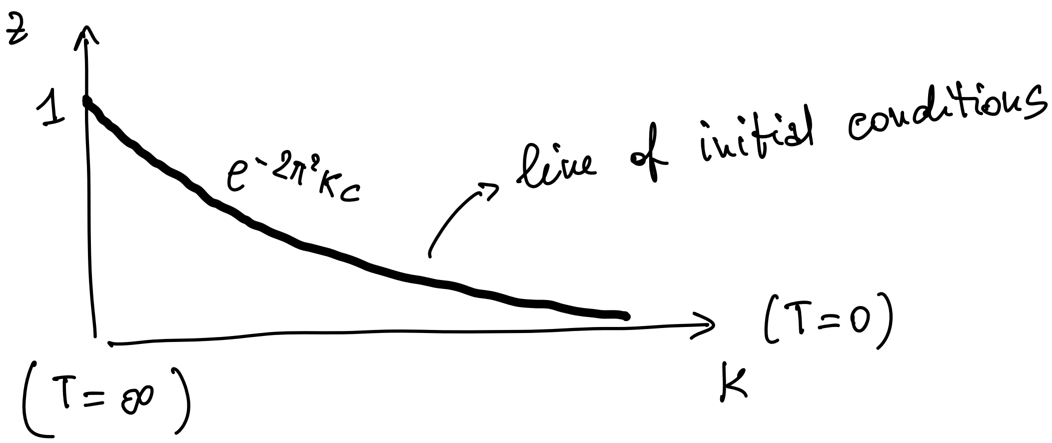
$$K(\ell+d\ell) = \frac{1}{\frac{1}{K(\ell)} + 4\pi^3 z^2(\ell)\, d\ell} = \frac{1}{\frac{1}{K(\ell)}\left(1 + 4\pi^3 z^2 K\, d\ell\right)}$$

$$k(l+dl) = k(l)\left(1 - 4\pi^3 z^2(l) K(l)\, dl\right)$$

$$k(l+dl) - k(l) = -4\pi^3 z^2(l) K^2(l)\, dl$$

$$\boxed{\frac{dk}{dl} = -4\pi^3 z^2(l) K^2(l)}$$

5. $z = e^{-2\pi^2 K c}$ $\qquad c \simeq 1/4 \qquad z \simeq e^{-\pi^2 K/2}$



6. $\dfrac{dz}{dl} = z(2-\pi K) \qquad \dfrac{dK}{dl} = -4\pi^3 z^2(l) K^2(l)$

Fixed points: $\dfrac{dz}{dl} = 0 \qquad \dfrac{dK}{dl} = 0 \iff K = 0$ or $z = 0$

$$\Updownarrow$$

$z = 0$ or $K = 2/\pi$

$K > 2/\pi \longrightarrow$ stable fixed points

$K < 2/\pi \longrightarrow$ unstable fixed points

$z = 0 \iff$ line of fixed points

7. $K = \frac{2}{\pi} + \epsilon$

$$\frac{dz}{d\ell} = z(2 - \pi K)$$

$$2z\frac{dz}{d\ell} = 2z^2(2-\pi K)$$

$$\frac{dz^2}{d\ell} = 2z^2(2-\pi K) = 2z^2\left(2 - \pi\left(\frac{2}{\pi}+\epsilon\right)\right) = -2\pi z^2\epsilon$$

$$\frac{dK}{d\ell} = \frac{d\epsilon}{d\ell} = -4\pi^3 z^2\left(\frac{2}{\pi}+\epsilon\right)^2 \simeq -16\pi z^2$$

$$2\epsilon\frac{d\epsilon}{d\ell} = -32\pi z^2\epsilon$$

$$\frac{d\epsilon^2}{d\ell} = -32\pi z^2\epsilon$$

$$16\frac{dz^2}{d\ell} - \frac{d\epsilon^2}{d\ell} = 0 \quad\Rightarrow\quad \frac{d}{d\ell}\left(16z^2 - \epsilon^2\right) = 0$$

$$16z^2 - \left(K - \frac{2}{\pi}\right)^2 = \text{cste}$$

$$\boxed{16\pi^2 z^2 - (2-\pi K)^2 = \text{cste}}$$

equation of a family of hyperboles in the $(K\text{-}z)$ plane

$$16\pi^2 z^2 = C + (2-\pi K)^2$$

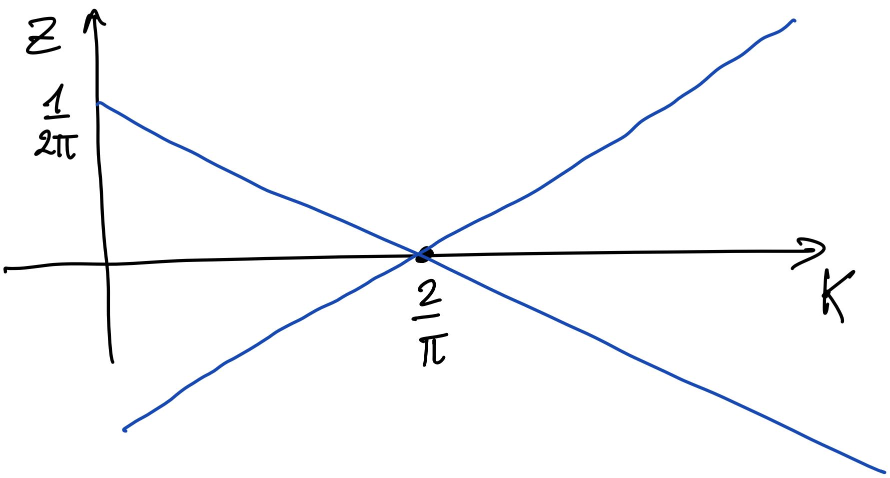
$$z = \pm\frac{1}{4\pi}\sqrt{C + (2-\pi K)^2}$$

$C = 0 \longrightarrow$ asymptotes of the hyperboles

$$z = \pm\frac{1}{4\pi}(2-\pi K) = \pm\left(\frac{1}{2\pi} - \frac{K}{4}\right)$$

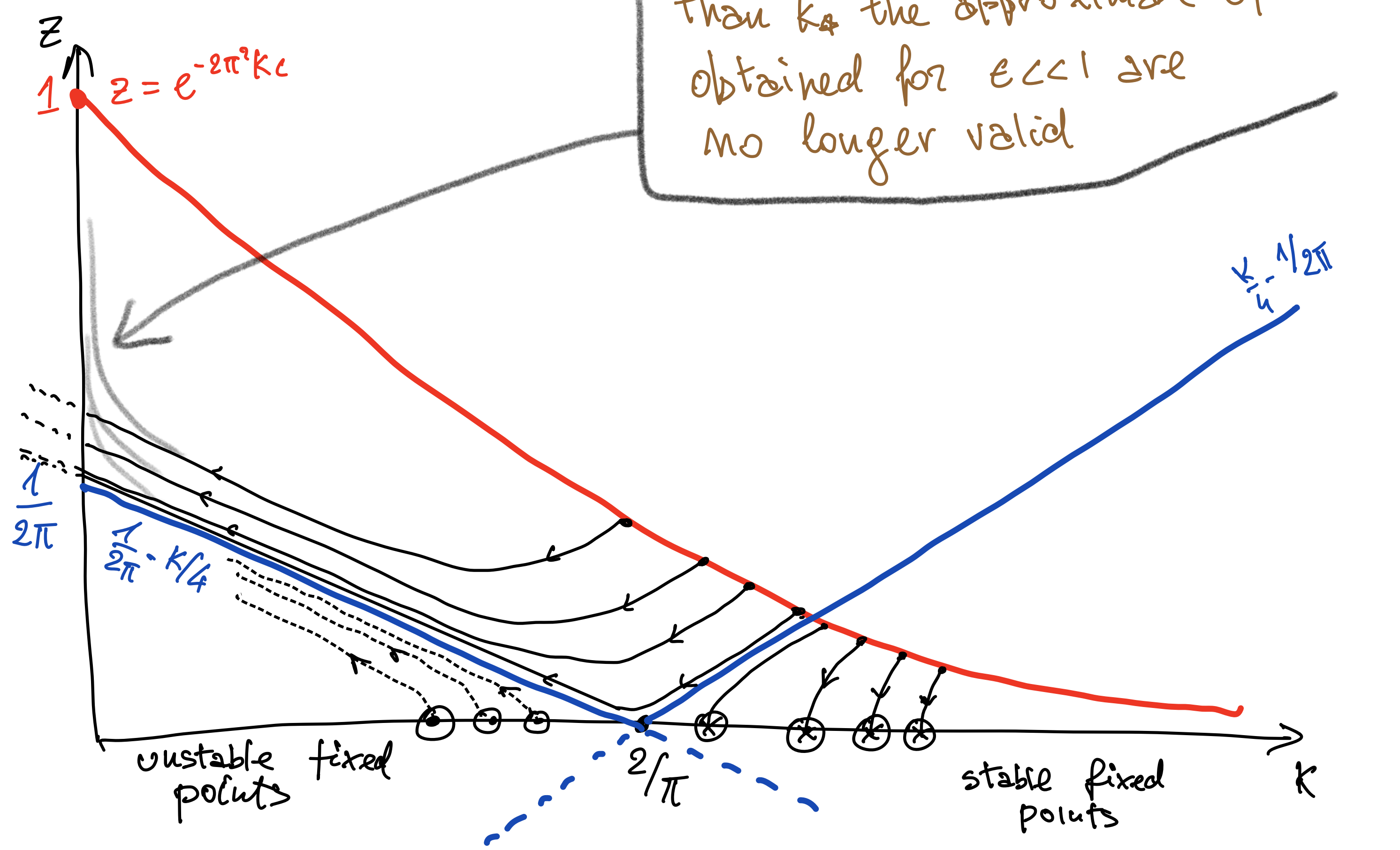
$\nearrow \frac{1}{2\pi} - K/4$

$\searrow \frac{K}{4} - 1/2\pi$



8. Phase diagram

When K becomes much smaller than K_c the approximate equations obtained for $\epsilon \ll 1$ are no longer valid



$z = 0 \longrightarrow$ no vortices $\Longrightarrow$ power-law correlations

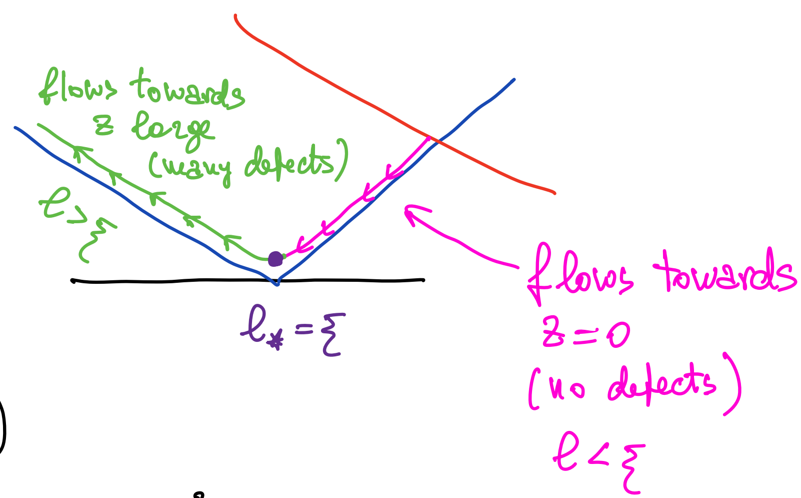
$T_c \longrightarrow$ Initial condition such that the system flows towards $z=0$

(O(1) pairs of strongly coupled vortices)

$$\boxed{\frac{K_c}{4} - \frac{1}{2\pi} = e^{-2\pi^2 K_c c}}$$

9. $T = T_c + \delta$

Initial condition very close to the hyperbole



$$16\pi^2 z^2 - (2-\pi K)^2 = a(T-T_c)$$

$$16\pi^2 z^2 = a(T-T_c) + \left(2-\pi\left(\frac{2}{\pi}+\epsilon\right)\right)^2$$

$$16\pi^2 z^2 = a(T-T_c) + \pi^2\epsilon^2 \Longrightarrow 16\pi z^2 = \frac{a}{\pi}(T-T_c) + \pi\epsilon^2$$

$$\frac{d\epsilon}{d\ell} = -16\pi z^2$$

$$-\frac{d\epsilon}{d\ell} = \pi\epsilon^2 + a(T-T_c)$$

$$\int \frac{d\epsilon}{\epsilon^2 + \frac{a}{\pi}(T-T_c)} = -\pi\int d\ell$$

$$\frac{d\,\operatorname{atan}(x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d\,\operatorname{atan}(x/\alpha)}{dx} = \frac{1/\alpha}{1+\left(\frac{x}{\alpha}\right)^2} = \frac{\alpha}{\alpha^2+x^2}$$

$$\left.\frac{\operatorname{atan}\left(\dfrac{\epsilon}{\sqrt{\frac{a}{\pi}(T-T_c)}}\right)}{\sqrt{\frac{a}{\pi}(T-T_c)}}\right|_0^{\ell} = -\pi \ell \qquad \epsilon = k - \frac{2}{\pi}$$

$$\frac{\operatorname{atan}\left(\dfrac{k(\ell)-2/\pi}{\sqrt{\frac{a}{\pi}(T-T_c)}}\right) - \operatorname{atan}\left(\dfrac{k(0)-2/\pi}{\sqrt{\frac{a}{\pi}(T-T_c)}}\right)}{\sqrt{\frac{a}{\pi}(T-T_c)}} = -\pi \ell$$

$$\ell_* \longrightarrow k(\ell_*) \simeq 2/\pi \Rightarrow \operatorname{atan}\left(\frac{k(\ell^*)-2/\pi}{\sqrt{\frac{a}{\pi}(T-T_c)}}\right) \simeq 0$$

$$\frac{k(0)-2/\pi}{\sqrt{\frac{a}{\pi}(T-T_c)}} \longrightarrow +\infty \qquad \operatorname{atan}(+\infty) = \pi/2$$

$$-\frac{\pi}{2}\frac{1}{\sqrt{\frac{a}{\pi}(T-T_c)}} = -\pi \ell_* \Longrightarrow \ell_* = \frac{1}{\sqrt{\frac{4a}{\pi}(T-T_c)}}$$

$$\xi = b(\ell_*) = e^{\ell_*} \Longrightarrow \boxed{\xi \propto e^{\frac{A}{\sqrt{T-T_c}}}}$$