Advanced Statistical Physics
TD — Real-space renormalization group

October 2024

The purpose of this set of problems is to learn how to implement the real space renormalization group ideas on simple physical systems. We will study first the percolation transition and the Ising model on the bidimensional triangular lattice (in the spirit of the so-called "majority rule"), then the Ising model on d-dimensional hyper-cubic lattices within the Migdal-Kadanoff approximation and on the hierarchical Berker lattice.

I. REAL-SPACE RENORMALIZATION GROUP APPROACH TO THE PERCOLATION TRANSITION

In statistical physics and mathematics, the percolation transition describes a geometric type of phase transition occurring when nodes (or links) are added to a given network. The probability of each node being occupied is independent of the other sites' occupancy, and is equal to p. Neighboring occupied sites form clusters. For an infinite size system, there is a critical value of p, called *percolation threshold* p_c, such that for $p < p_c$ the system does not have any clusters that span the entire system length, and for $p > p_c$ it does.

(A) One-dimensional chain: exact solution

1. Consider a $1d$ chain with sites occupied with probability p or empty with probability $1-p$. Define n_s to be the cluster number – the number of clusters containing s sites per lattice site. Find it in terms of p and s.
2. What is S, the average size of a finite cluster?
3. Calculate the correlation function $g(r)$ – the probability that a site, that is a distance r away from an occupied site, belongs to the same cluster. Rewrite it in terms of the correlation length as $g(r) = e^{-r/\xi}$.
4. What is the percolation threshold in this problem? What happens to S and ξ at p_c? The fact that the divergence of ξ at the percolation threshold can be described in general by a simple power law, $\xi \propto (p - p_c)^{-\nu}$.

(B) One-dimensional chain: RG approach

Now we will use this exactly solvable model to demonstrate how real-space renormalization works. The idea is to replace a group of sites by a coarse-grained super-site, whose linear dimension is b, with $1 < b < \xi$. The super-site is said to be occupied if there is a cluster of the original sites that spans the length of the cell. Away from the critical point, the probability p' of the super-sites will be different from p, while at the critical point $p' = p = p_c$ since ξ is infinite. The new lattice will have a new lattice constant b, and a new correlation length ξ', measured in the units of b. Since this is an exact transformation, one has that $b\xi'(p') = \xi(p)$.

1. Show that
$$\frac{1}{\nu} = \frac{\log \left.\frac{\mathrm{d}p'}{\mathrm{d}p}\right|_{p_c}}{\log b}.$$
This expression allows us to estimate the critical exponent ν via the renormalization group method.
2. Group the sites in our one-dimensional chain into cells of b sites. Find p'.
3. Values of p that stay constant under renormalization are called *fixed points*. What are the fixed points in this problem?
4. Calculate ν and compare it to the earlier result.

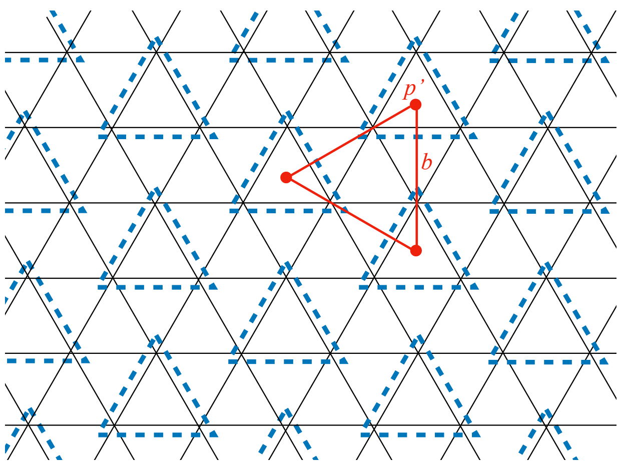

FIG. 1 – *Two-dimensional triangular lattice. Sites belonging to the same triangular plaquette (dashed blue) are grouped together.*

(C) Two-dimensional triangular lattice: RG approach

1. Consider now a two-dimensional triangular lattice. The lattice is divided in triangular plaquettes, as shown in the figure. Group together the three sites belonging to each plaquette and find p' in such a way that the plaquette supports the percolating cluster.
2. What are the fixed points of the recursion relation? Which of them is stable and which of them is unstable?
3. Show that the lattice made by the triangular plaquettes is still a triangular lattice (but rotated by $\pi/6$ with respect to the original lattice) with a renormalized lattice constant. Find b and ν.

II. REAL-SPACE RENORMALIZATION OF THE ISING MODEL ON THE BIDIMENSIONAL TRIANGULAR LATTICE WITHIN A VARIATIONAL APPROXIMATION

We consider the Ising model on a two-dimensional triangular lattice represented in the figure, in which the N spins $\underline{S} = (S_1, \ldots, S_N)$ interact according to the Hamiltonian:

$$\mathcal{H}[\underline{S}] = -J \sum_{\langle i,j \rangle} S_i S_j \,, \tag{1}$$

where $\langle i,j \rangle$ denotes the pairs of nearest neighbors on the lattice.

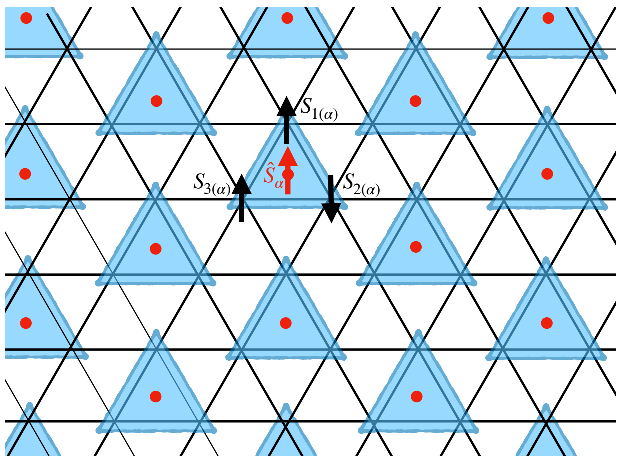

FIG. 2 – *Ising model on the two-dimensional triangular lattice. Spins belonging to the same triangular plaquette (blue) are grouped together. A new coarse-grained spin is assigned to each plaquette according to the majority rule.*

In this exercise we consider a transformation of the real-space renormalization group which consists in reducing the number of degrees of freedom of the system by introducing new spins, each of them representing the state of a block of several coarse-grained original spins. The blue triangles (blocks) of the figure are indexed by $\alpha = 1, \ldots, \hat{N}$, and we denote $1(\alpha)$, $2(\alpha)$, and $3(\alpha)$ the three sites of the original lattice at the vertices of the triangle α. A new Ising spin $\hat{S}_\alpha$ is placed at the center of each blue triangle. As sketched in the figure, to each configuration of the original spins $\underline{S} = (S_1, \ldots, S_N)$ one associates a configuration $\underline{\hat{S}} = (\hat{S}_1, \ldots, \hat{S}_{\hat{N}})$ of the new coarse-grained block spins of the new lattice according to the majority rule inside each block:

$$\hat{S}_\alpha = \mathrm{sign}(S_{1(\alpha)} + S_{2(\alpha)} + S_{3(\alpha)}) \,.$$

You can check that each spin S_i belongs to a single block.

1. What is the distance b (scale factor of the transformnation) between two nearest-neighbor spins in the new triangular lattice formed by the blocks α? What is the

number of $\hat{N}$ spins in the new lattice? We call $C(\underline{\hat{S}})$ the set of configurations $\underline{S}$ of the original spins that lead to the configuration $\underline{\hat{S}}$ of the block spins by the decimation rule. How many configurations of the original spins $S_{1(\alpha)}$,$S_{2(\alpha)}$,$S_{3(\alpha)}$ are decimated into $\hat{S}_\alpha$? What is the cardinality of $C(\underline{\hat{S}})$?

2. A renormalized Hamiltonian acting on the new spins is defined as:

$$e^{-\beta \hat{H}[\underline{\hat{S}}]} = \sum_{\underline{S} \in C(\underline{\hat{S}})} e^{-\beta H[\underline{S}]} \, .$$

   Compare the partition functions computed from $\hat{H}[\underline{\hat{S}}]$ and $H[\underline{S}]$. What is the most general form of the renormnalized Hamiltonian $\hat{H}[\underline{\hat{S}}]$?

3. In general the exact computation of the $\hat{H}[\underline{\hat{S}}]$ after the decimation is impossible (except in $1d$, as we will see in the next exercise). To bypass this difficulty we will consider an approximate Hamiltonian with the same form of $H[\underline{S}]$, inspired by a variational method.
   Consider an arbitrary approximate Hamiltonian $H_0[\underline{S}]$ on the original spins. For each configuration $\underline{\hat{S}}$ of the decimated spins one introduces an average over the configurations of the original spins using the approximate Hamiltonian as:

$$\langle \cdots \rangle_{0,\underline{\hat{S}}} = \frac{1}{Z_0[\underline{\hat{S}}]} \sum_{\underline{S} \in C(\underline{\hat{S}})} \cdots \, e^{-\beta H_0[\underline{S}]} \, , \qquad Z_0[\underline{\hat{S}}] = \sum_{\underline{S} \in C(\underline{\hat{S}})} e^{-\beta H_0[\underline{S}]} \, .$$

   Show that

$$e^{-\beta \hat{H}[\underline{\hat{S}}]} = Z_0[\underline{\hat{S}}] \langle e^{-\beta (H[\underline{S}] - H_0[\underline{S}])} \rangle_{0,\underline{\hat{S}}} \, .$$

   Using the Jensen inequality, $\langle f(x) \rangle \geq f(\langle x \rangle)$, for any convex function f and any random variable x, deduce the following upper bound for $\hat{H}[\underline{\hat{S}}]$:

$$\hat{H}[\underline{\hat{S}}] \leq -\frac{1}{\beta} \ln Z_0[\underline{\hat{S}}] + \langle H[\underline{S}] - H_0[\underline{S}] \rangle_{0,\underline{\hat{S}}} \, .$$

4. We will take as approximate Hamiltonian

$$H_0[\underline{S}] = -J \sum_{\alpha=1}^{\hat{N}} \sum_{\langle i,j \rangle \in \alpha} S_{i(\alpha)} S_{j(\alpha)} \, ,$$

   where the sum $\langle i,j \rangle \in \alpha$ involves only the three edges inside the block α. Compute $Z_0[\underline{\hat{S}}]$. Compute $\langle S_i \rangle_{0,\underline{\hat{S}}}$ and express it in terms of the configuration of the new spins $\underline{\hat{S}}$. We will introduce the notation $\alpha(i)$ denoting the block α to which the site i belongs. Compute $\langle S_i S_j \rangle_{0,\underline{\hat{S}}}$ if i and j belongs to two *different* blocks.

5. Deduce then the value of $\langle H[\underline{S}] - H_0[\underline{S}] \rangle$ and show that

$$\hat{H}[\underline{\hat{S}}] \leq c - J' \sum_{\langle \alpha,\beta \rangle} \hat{S}_\alpha \hat{S}_\beta \, ,$$

   where $\langle \alpha,\beta \rangle$ denote the pairs of nearest neighbors blocks in the new lattice. Find the new coupling constant J'.

6. What are the fixed points J_c of this transformation? Study their stability by computing $\partial J'/\partial J|_{J_c}$.
7. Compute the critical exponent ν at the non-trivial fixed point and compare it to its exact value $\nu = 1$ in $d = 2$.

III. REAL-SPACE RENORMALIZATION OF THE ISING MODEL À LA MIGDAL-KADANOFF

We consider the Ising model on a d-dimensional hyper-cubic lattice with spins $S_i = \pm 1$, described by the Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j \,, \tag{2}$$

where $\langle i,j \rangle$ denotes the pairs of nearest neighbors on the lattice. We introduce the dimensionless coupling constant $t = \tanh(K)$ with $K = \beta J > 0$.

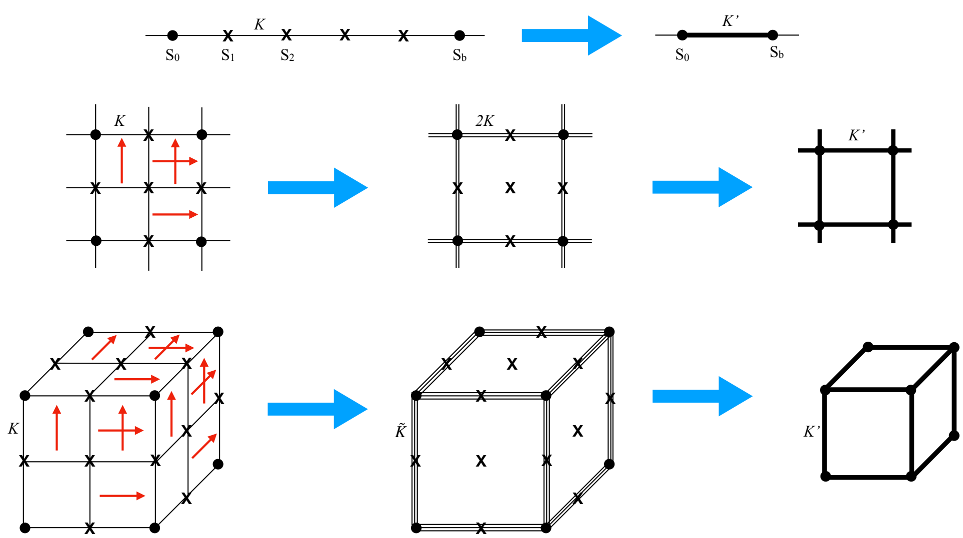

FIG. 3 – *Top: Decimation of $b-1$ spins along a one-dimensional chain. Middle: Migdal-Kadanoff procedure on a two-dimensional square lattice for $b = 2$. Bottom: Migdal-Kadanoff procedure on a three-dimensional cubic lattice for $b = 2$. Bonds are moved along the directions of the arrows. Spins marked with circles are kept and spins marked with crosses are integrated out.*

(A) Decimation rule in one dimension

We start by the case $d = 1$ in which the decimation can be carried out exactly. We study the renormalization procedure illustrated in the figure by which $b - 1$ spins are integrated out (crosses) and the other spins are kept (circles). The thick bonds indicate the new coupling t' obtained upon decimation.

1. Show that $e^{KS_iS_{i+1}}$ can always be written as $A + BS_iS_{i+1}$, where the constants A and B depend on K and must be determined.

2. Consider three subsequent spins on the chain, S_1, S_2, and S_3. For S_1 and S_3 being fixed, integrate out S_2 and compute:

$$\sum_{S_2=\pm 1} e^{K(S_1 S_2 + S_2 S_3)} .$$

3. Determine the recursion relation for the coupling constant when $b-1$ consecutive spins on the chain are integrated out.

(B) **The two-dimensonal model on the square lattice**

Unfortunately, a simple decimation in d larger than 1 cannot be performed exactly. To circumvent this problem approximations have to be made. We follow here an idea put forward by Migdal (1975) and Kadanoff (1976). We consider a two-step decimation procedure on a square lattice sketched in the figure. We start by the case $b=2$. The approximation consists in *moving* the bonds that are not connected to the spins that are kept (circles). These spins are then connected by bonds of strength $2K$ instead of K.

1. Spins marked by crosses now are decimated. What is the new coupling constant t' after decimation?

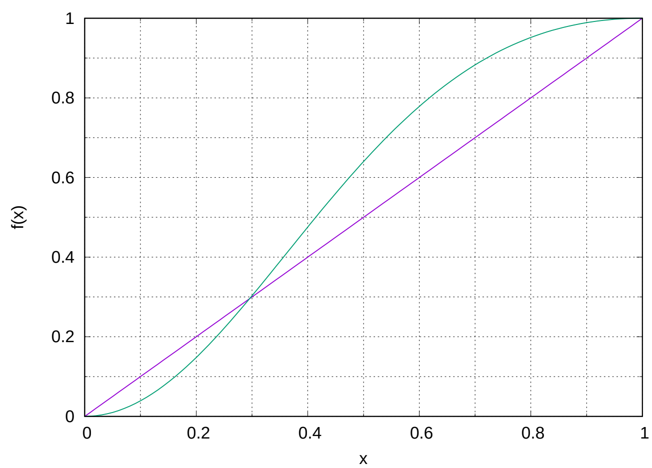

FIG. 4 – *Plot of the function* $f(x) = 4x^2/(1+x^2)^2$ *for* $x \in (0,1)$.

2. Discuss the fixed points of the recursion relation (we can use here the fact that $\tanh(2x) = 2\tanh(x)/(1+\tanh^2(x)))$. Using the figure above provide an approximate value for the fixed point K_c. The exact value, computed by Onsager, is $K_c^{\text{exact}} = \log(1+\sqrt{2})/2 \approx 0.44$.
3. The correlation length (measured in units of the lattice constant) of the model depends only on K. Imposing that ξ measured in units of the original lattice spacing is the same before and after the decimation, and imposing that in the vicinity of K_c the correlation length diverges as $\xi \propto (K-K_c)^{-\nu}$, compute the critical exponent ν within the Migdal-Kadanoff approximation.

4. Optional: We now generalize the procedure described above to any b. We move the bonds that are not connected to the spins that are kept along the x and y directions. The spins marked with a cross are then integrated out by one-dimensional decimation. What is the value of the new coupling constant?
5. Optional: We consider the analytic continuation limit in which an infinitesimal fraction of spins is renormalized at each step (and an infinitesimal fraction of bonds is moved). We denote $b = 1 + \epsilon$, with $\epsilon \ll 1$. Show that:

$$K' \approx K + \epsilon \left[K + f(K) \log \tanh(K) \right] ,$$

with $f(K)$ a function to be determined.
6. Optional: The exact value $K_c^{\text{exact}} = \log(1 + \sqrt{2})/2 \approx 0.44$ is now a fixed point of the recursion relation. Compute the value of the critical exponent ν.

(C) **Optional exercise: Generalizations to arbitrary dimensions**

We now consider the Ising model on the d-dimensional hyper-cubic lattice and consider the Migdal-Kadanoff decimation procedure for arbitrary b.

1. Compute N', the number of sites kept after the decimation, as a function of N and b.
2. Compute B', the number of bonds that survive after the bond moving step, as a function of N' and d.
3. Imposing that the total number of couplings is the same before and after the bond moving step, determine the value of the coupling constant $\tilde{K}$ (see figure) as a function of b and d.
4. Determine the recursion relation for any arbitrary value of b and d.
5. Discuss the stabiliy of the low-temperature ($K \to \infty$) fixed point and connect the result to the lower critical dimension of the model.

IV. REAL-SPACE RENORMALIZATION OF THE ISING MODEL ON THE HIERARCHICAL BERKER LATTICE

The Berker diamond lattice is a hierarchical lattice that can be constructed iteratively as explained below and illustrated in the figure: Starting from two vertices and a single edge, one recursively replace each edge by $2c$ edges, inserting c vertices in between, where $c \geq 1$ is an integer parameter. The first steps of this procedure are represented on the figure for $c = 2$. To mimic Euclidean d-dimensional lattices, c is taken equal to 2^{d-1}.

We place the Ising spins on the vertices of the graph obtained after a certain number of steps of this recursive procedure, and consider the Hamiltonian (2), where the sum is over the $\langle i,j \rangle$ edges of the graph. We will study this model by the decimation method, eliminating the spins in the reverse order of their introduction.

1. How many edges and how many vertices are present in the graph after n steps?
2. Setting, as usual, $t = \tanh(\beta J)$, find the recursion relation giving the new dimensionless coupling constant t' as a function of the coupling constant t at the previous step of the decimation procedure.

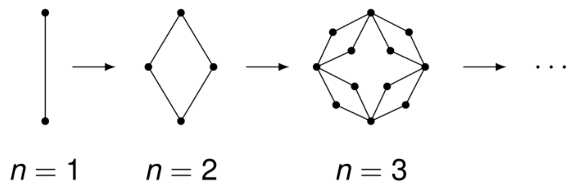

FIG. 5 – *Iterative construction of the hierarchical diamond lattice with $c = 2$.*

3. Show that for $c = 1$ one recovers the recursion relation of the one-dimensional Ising model.
4. What are the values of t corresponding to the high and low temperature limits? Check that these correspond to fixed points of the renormalization transformation.
5. Study the behavior of the RG transformation around these trivial fixed points and discuss their stability for $c > 1$.
6. The distance between the boundary sites of the lattice is equal to 2^n edges after n steps: this naturally fixes the length scale after n iterations as 2^n. For $c = 2$ the fixed point of the RG transformation is found at $t_\star \simeq 0.5437$. Compute the critical exponent ν.